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Research Article

# $T^n_M$-Coherent Modules and $T^n_M$-Flat Modules

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**Abstract.** In this paper, with respect to a tilting module $T$, the notions of $T^n_M$-coherence and $T^n_M$-flatness are introduced, for every module $M$ and every nonnegative integer $n$. Some characterizations of $T^n_M$-coherent modules are proved. We show that an $R$-module $F$ is $T^n_M$-flat (injective) if and only if $F$ is $T^n_{Rm}$-flat (injective), for any $m \in M$. Also, some sufficient conditions under which any direct product (direct limit) of $T^n_M$-flat ($T^n_M$-injective) modules is $T^n_M$-flat ($T^n_M$-injective) are given. Among other results, $T^n_M$-coherent rings are studied.

**Keywords.** Coherent module; Flat module; Tilting module

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## 1. Definitions and Notations

Throughout this paper, $R$ is an associative ring with non-zero identity, all modules are unitary left $R$-modules and $T$ is a tilting module. We denote by $\operatorname{Add} T$ (resp. $\operatorname{FAdd} T$), the class of modules isomorphic to direct summands of direct sum of copies (resp. finitely many copies) of $T$. Following [2], a module $T$ is called tilting (1-tilting) if it satisfies the following conditions:

(1) $pd(T) \le 1$, where $pd(T)$ denotes the *projective dimension of* $T$.

(2) $\operatorname{Ext}^i(T, T^{(\lambda)}) = 0$, for each $i > 0$ and for every cardinal $\lambda$.

(3) There exists the exact sequence $0 \to R \to T_0 \to T_1 \to 0$, where $T_0, T_1 \in \operatorname{Add} T$.

Also, by $\mathrm{Pres}^n T$ (resp. $\mathrm{FPres}^n T$) and $\mathrm{Pres}^\infty T$ (resp. $\mathrm{FPres}^\infty T$) the set of all modules $M$ such that there exists exact sequences

$$T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

and

$$\cdots \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0,$$

respectively, where $T_i \in \mathrm{Add}\,T$ (resp. $\mathrm{FAdd}T$), for every $i \geq 0$. A module $M$ is said to be *generated* (resp. *cogenerated*) by $T$, denoted by $M \in \mathrm{Gen}\,T$ (resp. $M \in \mathrm{Cogen}\,T$) if there exists an exact sequence $T^n \to M \to 0$ (resp. $0 \to M \to T^n$), for some positive integer $n$. Let $\mathcal{C}$ be a class of modules and $M$ be a module. A $\mathcal{C}$-resolution of $M$ is a long exact sequence $\cdots \to C_1 \to C_0 \to M \to 0$, where $C_i \in \mathcal{C}$, for all $i \geq 0$. Let $M \in \mathrm{Gen}\,T$. Since $T$ is tilting, [2, Theorem 3.11] implies that $T$ is a 1-star module (see [9, Definition 3.1]) and $\mathrm{Gen}\,T = \mathrm{Pres}^\infty T$. This shows that any module generated by $T$ has an $\mathrm{Add}\,T$-resolution, see also [5, Proposition 2.1].

For any module $M$, $M^* = \mathrm{Hom}_{\mathbb{Z}}\left(M, \frac{\mathbb{Q}}{\mathbb{Z}}\right)$ denotes the *character module* of $M$. For any homomorphism $f$, we denote by $\ker f$ and $\mathrm{im}\, f$, the kernel and image of $f$, respectively. Let $B$ and $M \in \mathrm{Gen}\,T$ be two modules. We define the functors

$$\Gamma^T_n(M,B) := \frac{\ker(\delta_n \otimes 1_B)}{\mathrm{im}(\delta_{n+1} \otimes 1_B)}; \quad \mathcal{E}^n_T(M,B) := \frac{\ker \delta^n_*}{\mathrm{im}\,\delta^{n-1}_*},$$

where

$$\cdots \longrightarrow T_2 \xrightarrow{\delta_2} T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

is an $\mathrm{Add}\,T$-resolution of $M$ and $\delta^n_* = \mathrm{Hom}(\delta_n, \mathrm{id}_B)$, for every $i \geq 0$, see [5,8] for more details.

**Definition 1.1.** Let $T$ be a tilting module and $n$ be a nonnegative integer.

(1) A module $F$ is called $T^n_M$-flat if $\Gamma^T_{n+1}\left(\frac{M}{K}, F\right) = 0$, for every submodule $K$ of $M$.

(2) A module $F$ is called $T^n_M$-injective if $\mathcal{E}^{n+1}_T\left(\frac{M}{K}, F\right) = 0$, for every submodule $K$ of $M$.

Let $M \in \mathrm{Gen}\,T$ and $N$ be two modules. A similar proof to that of [6, Lemma 2.11] shows that $\mathcal{E}^0_T(M,N) \cong \mathrm{Hom}(M,N)$. Similarly, it is seen that $\Gamma^0_T(M,N) \cong M \otimes N$. Moreover, $\mathcal{E}^1_T(M,-) = 0$ implies that $M \in \mathrm{Add}\,T$. We say that $M$ has *$T$-projective dimension $n$* (briefly, $\mathrm{T.p.dim}(M) = n$) if $n$ is the least non-negative integer such that there exists a long exact sequence

$$0 \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with $T_i \in \mathrm{Add}\,T$, for each $i \geq 0$. It is clear that $\mathrm{T.p.dim}(M) = n$ if and only $n$ is the least non-negative integer such that $\mathcal{E}^n_T(M,B) = 0$, for any module $B$, see [5, Remark 2.2] for more details. Also, we say that $M$ has *$T$-flat dimension $n$* (briefly, $\mathrm{T.f.dim}(M) = n$) if $n$ is the least non-negative integer such that $\Gamma^T_n(M,B) = 0$, for any module $B$, see [5, Definition 2.2]. We denote by $\mathcal{TP}_n$ and $\mathcal{TF}_n$, the class of modules with $T$-projective dimension at most $n$ and the class of modules with $T$-flat dimension at most $n$, respectively.

A similar proof to that of [6, Proposition 2.3] shows that the definition of $\Gamma^T_n(M,B)$ (resp. $\mathcal{E}^n_T(C,M)$) is independent from the choice of left $\operatorname{Add}T$-resolutions. For unexplained concepts and notations in this area, we refer the reader to [1,3,5,7].

## 2. Relative Coherence with Respect to a Tilting Module

We start with two useful lemmas which will be used in the proof of the main results of this paper.

**Lemma 2.1.** *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

*be an exact sequence. Then*

(1) *If* $A \in \operatorname{Pres}^{n+1} T$ *and* $C \in \operatorname{Pres}^{n+1} T$*, then* $B \in \operatorname{Pres}^{n+1} T$.

(2) *If* $A \in \operatorname{Pres}^{n} T$ *and* $B \in \operatorname{Pres}^{n+1} T$*, then* $C \in \operatorname{Pres}^{n+1} T$.

(3) *If* $B \in \operatorname{Pres}^{n} T$ *and* $C \in \operatorname{Pres}^{n+1} T$*, then* $A \in \operatorname{Pres}^{n} T$.

*Proof.* (1): We prove the assertion by induction on $n$. If $n = 0$, then the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & T_0' & \xrightarrow{i_0} & T_0'\oplus T_0'' & \xrightarrow{\pi_0} & T_0'' & \longrightarrow & 0\\ & & \downarrow h_0' & & \downarrow h_0 & & \downarrow h_0'' & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

exists, where $T_0', T_0'' \in \operatorname{Add}T$, $i_0$ is the inclusion map, $\pi_0$ is a canonical epimorphism and $h_0 = f h_0'$ is epimorphism, by Five Lemma. Let $K_1' = \ker h_0'$, $K_1 = \ker h_0$ and $K_1'' = \ker h_0''$. It is clear that $K_1', K_1'' \in \operatorname{Pres}^n T$; so, the induction implies that $K_1 \in \operatorname{Pres}^n T$. Hence $B \in \operatorname{Pres}^{n+1} T$.

(2): First assume that $n = 0$. If $B \in \operatorname{Pres}^1 T$ and $A \in \operatorname{Pres}^0 T$, then the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & T_0' & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \gamma & & \downarrow f & & \\ T_1 & \xrightarrow{\alpha_2} & T_0 & \xrightarrow{\alpha_1} & B & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \\ T_0'\oplus T_1 & \xrightarrow{h} & T_0 & \xrightarrow{g\alpha_1} & C & \longrightarrow & 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

in which the existence of $\gamma$ follows from the exactness of the sequence $\mathrm{Hom}(T'_0, T_0) \to \mathrm{Hom}(T'_0, B) \to 0$. Also, $h$ is defined by $h(t'_0, t_1) = \gamma(t'_0) + \alpha(t_1)$. Therefore, we deduce that $C \in \mathrm{Pres}^1 T$. For $n > 0$, the assertion follows from induction.

(3): This is proved similarly. □

**Remark 2.1.** If $T$ is finitely presented, then every finite direct sum of copies of $T$ is finitely presented. Thus every module in $\mathrm{FAdd}T$ is finitely presented and so all modules in $\mathrm{FPres}^n T$ are finitely presented.

**Lemma 2.2.** *If $T$ is finitely presented and $F \in \mathrm{FPres}^{n+2} T$, then $\Gamma^T_{n+1}(F, M^I) \cong \Gamma^T_{n+1}(F, M)^I$, for every cardinal $I$.*

*Proof.* Since $F \in \mathrm{FPres}^{n+2} T$, the exact sequence

$$T_{n+2} \longrightarrow T_{n+1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow F \longrightarrow 0$$

exists, where $T_i \in \mathrm{FAdd}T$ for every $i \geq 0$. Setting $K_n = \ker(T_n \to T_{n-1})$, it is clear that $K_n \in \mathrm{FPres}^1 T$. Thus for any cardinal $I$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma^T_1(K_{n-1}, M^I) & \longrightarrow & K_n \otimes M^I & \longrightarrow & T_n \otimes M^I & \longrightarrow & \cdots \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & \Gamma^T_1(K_{n-1}, M)^I & \longrightarrow & (K_n \otimes M)^I & \longrightarrow & (T_n \otimes M)^I & \longrightarrow & \cdots
\end{array}$$

By Remark 2.1, $K_n$ and $T_n$ are finitely presented, so $g$ and $h$ are isomorphisms by [4, Theorem 2.1.5]. Hence $f$ is an isomorphism. Therefore, by [5, Proposition 2.2],

$$\begin{aligned}
\Gamma^T_{n+1}(F, M^I) &\cong \Gamma^T_1(K_{n-1}, M^I) \\
&\cong \Gamma^T_1(K_{n-1}, M)^I \\
&\cong \Gamma^T_{n+1}(F, M)^I.
\end{aligned}$$

□

We denote by $\Omega_M(N)$, the set of all factor modules of $N$, say $\frac{B}{A}(A \leq B \leq N)$, such that there exists an element $m \in M$ with $\frac{B}{A} \hookrightarrow Rm$. In particular, $\Omega_M(R)$ consists the set of all modules of the form $\frac{L}{m^\perp}$ for any $m \in M$, where $m^\perp = \{r \in R \mid rm = 0\} \subseteq L \leq R$.

**Definition 2.1.** A module $N$ is called $T^n_M$-coherent if $\Omega_M(N) \cap \mathrm{FPres}^n T \subseteq \mathrm{FPres}^{n+1} T$. A ring $R$ is called $T^n_M$-coherent if it is $T^n_M$-coherent as an module.

In the following theorem, some characterizations of $T^n_M$-coherent modules are given.

**Theorem 2.1.** *Let $T$, $M$ and $N$ be modules. If $T$ is finitely presented and $x \in N$, then the following statements are equivalent:*

(1) *$N$ is $T^n_M$-coherent;*

(2) *If $R \in \mathrm{FPres}^{n+1} T$ and $0 \leq A < B \leq N$, then $\frac{B}{A} \in \mathrm{FPres}^n T$ and $\frac{B+xR}{A} \in \Omega_M(N) \cap \mathrm{FPres}^n T$ implies that $x^{-1}B \in \mathrm{FPres}^n T$;*

(3) *If* $R \in \mathrm{FPres}^{n+1} T$ *and* $0 \le A \le N$*, then* $\frac{A+xR}{A} \in \Omega_M(N) \cap \mathrm{FPres}^n T$ *implies that* $x^{-1}A \in \mathrm{FPres}^n T$*. And for any* $0 \le A < B \le N$*, and* $0 \le A < C \le N$*,* $\frac{B}{A}, \frac{C}{A} \in \Omega_M(N) \cap \mathrm{FPres}^n T$ *implies* $\frac{(B\cap C)}{A} \in \mathrm{FPres}^n T$.

*Proof.* (1)$\Longrightarrow$(3): We have that $\frac{(A+xR)}{A} \cong (x+A)R \cong \frac{R}{x^{-1}A}$, and also, $\frac{R}{x^{-1}A} \in \mathrm{FPres}^n T$. So, $\frac{R}{x^{-1}A} \in \mathrm{FPres}^{n+1} T$ by (1), and by Lemma 2.1(3), $x^{-1}A \in \mathrm{FPres}^n T$. Note that we have $\frac{B+C}{A}, \frac{B}{A} \oplus \frac{C}{A} \in \mathrm{FPres}^n T$. Thus $\frac{B+C}{A} \in \mathrm{FPres}^{n+1} T$, by (1). Therefore by Lemma 2.1(3), the exactness of the sequence

$$0 \longrightarrow \frac{(B\cap C)}{A} \longrightarrow \frac{B}{A} \oplus \frac{C}{A} \longrightarrow \frac{B+C}{A} \longrightarrow 0$$

implies that $\frac{(B\cap C)}{A} \in \mathrm{FPres}^n T$.

(3)$\Longrightarrow$(1): We need to show that for any $Y = \frac{B}{A} \in \Omega_M(N) \cap \mathrm{FPres}^n T$ implies that $Y \in \mathrm{FPres}^{n+1} T$. From Lemma 2.1(2), we deduce that $\frac{A+xR}{A} \cong \frac{R}{x^{-1}A} \in \mathrm{FPres}^{n+1} T$. Also, by Remark 2.1, $Y = \frac{B}{A}$ is finitely generated. So, assume by induction that any $(n-1)$-generated submodule $\frac{B}{A}$ belong to $\mathrm{FPres}^{n+1} T$. Now, every $n$-generated submodule, which is isomorphic to a subquotient module of $N$, is of the form $\frac{(B+xR)}{A}$ for some $x \in N$. Consider the following exact sequence:

$$0 \longrightarrow \frac{B\cap(A+xR)}{A} \longrightarrow \frac{B}{A} \oplus \frac{A+xR}{A} \longrightarrow \frac{B+xR}{A} \longrightarrow 0$$

The first term belong to $\mathrm{FPres}^n T$ by (3). Hence, by Lemma 2.1(2), the last term belong to $\mathrm{FPres}^{n+1} T$. Thus (1) holds.

(1)$\Longrightarrow$(2): By hypothesis, the exact sequence

$$0 \longrightarrow \frac{B}{A} \longrightarrow \frac{(B+xR)}{A} \longrightarrow \frac{R}{x^{-1}B} \longrightarrow 0$$

exists, where $\frac{(B+xR)}{A} \in \mathrm{FPres}^{n+1} T$, by (1). So, $\frac{R}{x^{-1}B} \in \mathrm{FPres}^{n+1} T$; therefore, by Lemma 2.1(3) $x^{-1}B \in \mathrm{FPres}^n T$.

(2)$\Longrightarrow$(1): This is similar to (3)$\Longrightarrow$(1). □

## 3. Relative Flatness and Relative Injectivity

First, we study the concepts of relative flatness and relative injectivity, with respect to the tilting module $T$ in short exact sequences.

**Theorem 3.1.** *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of modules.*

(1) *If* $F$ *is* $T^n_{M_2}$*-flat, then* $F$ *is* $T^n_{M_1}$*-flat and* $T^n_{M_3}$*-flat.*

(2) *If* $F$ *is* $T^n_{M_2}$*-injective, then* $F$ *is* $T^n_{M_1}$*-injective and* $T^n_{M_3}$*-injective.*

*Proof.* (1): It is clear that for every submodule $K_3$ of $M_3$, there exists a submodule $K_2$ of $M_2$ such that $K_3 \cong \frac{K_2}{M_1}$. Thus we have $\Gamma^T_{n+1}\left(\frac{M_3}{K_3}, F\right) \cong \Gamma^T_{n+1}\left(\frac{M_2}{K_2}, F\right) = 0$, by hypothesis, and so $F$ is $T^n_{M_3}$-flat. Now, choose a submodule $0 < K_1 \le M_1$. Then there exists an exact sequence

$0\to\frac{M_1}{K_1}\to\frac{M_2}{K_1}\to\frac{M_2}{M_1}\to 0$ which induces the long exact sequence

$$\cdots\longrightarrow\Gamma^T_{n+2}\left(\frac{M_2}{M_1},F\right)\longrightarrow\Gamma^T_{n+1}\left(\frac{M_1}{K_1},F\right)\longrightarrow\Gamma^T_{n+1}\left(\frac{M_2}{K_1},F\right)\longrightarrow\cdots$$

Therefore, $T^n_{M_2}$-flatness of $F$ implies that $\Gamma^T_{n+1}\left(\frac{M_1}{K_1},F\right)=0$, and this proves (1).
The proof of (2) is similar to that of (1). □

By using similar proofs to those of [7, Propositions 7.6 and 7.21], one can obtain the isomorphisms $\Gamma^T_{n+1}(\bigoplus_{i\in I}M_i,P)\cong\bigoplus_{i\in I}\Gamma^T_{n+1}(M_i,P)$ and $\mathcal{E}^{n+1}_T(\bigoplus_{i\in I}M_i,P)\cong\prod_{i\in I}\mathcal{E}^{n+1}_T(M_i,P)$. So, we have the following lemma.

**Lemma 3.1.** *Let $\{M_i\}_{i\in I}$ be a family of modules. Then the following statements hold.*

(1) *A module $F$ is $T^n_{M_i}$-flat, for every $i\in I$, if and only if $F$ is $T^n_{\oplus M_i}$-flat.*

(2) *A module $F$ is $T^n_{M_i}$-injective, for every $i\in I$, if and only if $F$ is $T^n_{\oplus M_i}$-injective.*

For any module $M$, we denote by $\sigma[M]$, the full subcategory of modules whose objects are isomorphic to $\frac{Y}{X}$, where $X\le Y\le M^{(I)}$, for some index set $I$.

Many ring and module theoretic concepts have been reformulated for the full subcategory $\sigma[M]$ of $R$-modules subgenerated by a given $R$-module $M$ (see [10]). Here, it will be shown how $\sigma[M]$ can be used as a tool in the category of $R$-modules, which is totally outside of $\sigma[M]$. For any subcategory of $R$-modules, such as $\sigma[M]$ there is always an associated concept of flatness. A module $F$ is $T^n_{\sigma[M]}$-flat if $\Gamma^T_{n+1}\left(\frac{Y}{X},F\right)=0$, for every submodule $X\le Y\in\sigma[M]$. It will be shown that $T^n_{\sigma[M]}$-flatness is equivalent to a simpler definition $T^n_M$-flatness. Also, it will be shown that $\sigma[M]\subseteq\mathcal{TP}_n$ if and only if every module is $T^n_M$-injective, and $\sigma[M]\subseteq\mathcal{TF}_n$ if and only if every module is $T^n_M$-flat.

**Proposition 3.1.** *For any module $F$, the following statements are true.*

(1) *A module $F$ is $T^n_M$-flat if and only if $\Gamma^T_{n+1}\left(\frac{B}{A},F\right)=0$ for any $A\le B\in\sigma[M]$.*

(2) *A module $F$ is $T^n_M$-injective if and only if $\mathcal{E}^{n+1}_T\left(\frac{B}{A},F\right)=0$ for any $A\le B\in\sigma[M]$.*

*Proof.* (1)($\Longleftarrow$): This follows immediately by taking $B=M$.
($\Longrightarrow$): It suffices to show that $F$ is $T^n_B$-flat. Let $B=\frac{X}{Y}\le\frac{M^{(I)}}{Y}$ for some $Y<X\le M^{(I)}$. By Lemma 3.1(1), $F$ is a $T^n_{M^{(I)}}$-flat module. Thus Theorem 3.1(1) implies that $F$ is $T^n_{\frac{M^{(I)}}{Y}}$-flat, for any $Y\le M^{(I)}$. Hence for any $B=\frac{X}{Y}\le\frac{M^{(I)}}{Y}$, $F$ is $T^n_B$-flat, again by Theorem 3.1(1).

The proof of (2) is similar to that of (1). □

The next theorem extends Proposition 3.1 to a larger category $\pi[M]\supseteq\sigma[M]$, where $\pi[M]$ is the full subcategory of modules whose objects are of the form $\frac{B}{A}\le\frac{M^I}{A}$, for some cardinal $I$ and some modules $A\le B\le M^I$.

**Theorem 3.2.** *Let $T$ be finitely presented. Then $F \in \mathrm{FPres}^{n+2}T$ is $T^n_M$-flat if and only if $\Gamma^T_{n+1}\left(\frac{Y}{X}, F\right) = 0$, for any $X < Y \in \pi[M]$.*

*Proof.* ($\Longleftarrow$): This is the special case when $Y = M$.

($\Longrightarrow$): It suffices to show that $F$ is $T^n_{M^I}$-flat for any cardinal $I$. To prove this, we use the induction on $n$. If $n = 0$, we need to show that $\Gamma^T_1\left(\frac{M^I}{K}, F\right) = 0$, for any submodule $K < M^I$. On the other hand, the map $\alpha \otimes 1_F : K \otimes F \to M^I \otimes F$ is monomorphism. Let $\pi : M^I \to M$ be the projection on the first component. By hypotheses $\alpha_i \otimes 1_F : \pi K \otimes F \to M_i \otimes F$ is monomorphism. By Remark 2.1, $F$ is finitely presented. So by Lemma 2.2, the natural map $\beta : M^I \otimes F \to (M \otimes F)^I$ is an isomorphism and also, the map $\rho_i : (M \otimes F)^I \to M_i \otimes F$ is the projection. From a commutative diagram we have that $(\alpha_i \otimes 1_F)(\pi \otimes 1_F) = \rho_i \beta(\alpha \otimes 1_F) = (\pi \otimes 1_F)(\alpha \otimes 1_F)$. Therefore $(\alpha \otimes 1_F)x = 0 \Longrightarrow (\pi \otimes 1_F)x = 0 \Rightarrow x = 0$. Hence, the map $\alpha \otimes 1_F$ is monomorphism. Assume that $n \geq 1$. The exact sequence $0 \to N \to T_0 \to F \to 0$ induces that $\Gamma^T_{n+1}\left(\frac{M^I}{K}, F\right) \cong \Gamma^T_n\left(\frac{M^I}{K}, N\right)$. It suffices to show that $N$ is $T^{n-1}_M$-flat. For any submodule $D$ of $M$, we have that $\Gamma^T_{n+1}\left(\frac{M}{D}, F\right) \cong \Gamma^T_n\left(\frac{M}{D}, N\right)$. Since $F$ is $T^n_M$-flat, $\Gamma^T_n\left(\frac{M}{D}, N\right) = 0$ implies that $N$ is $T^{n-1}_M$-flat. So, by hypothesis induction, $N$ is $T^{n-1}_{M^I}$-flat. Hence $\Gamma^T_{n+1}\left(\frac{M^I}{K}, F\right) = 0$ and this completes the proof. □

**Proposition 3.2.** *A module $F$ is $T^n_M$-flat if and only if the character module of $F$ is $T^n_M$-injective.*

*Proof.* We only need to show that an isomorphism $\mathcal{E}^m_T\left(\frac{M}{K}, F^*\right) \cong \Gamma^T_m\left(\frac{M}{K}, F\right)^*$ exists, for every submodule $K$ of $M$ and for every integer $m \geq 0$. First suppose that $m = 0$. Then from [7, Theorem 2.75], we deduce that

$$\mathcal{E}^0_T\left(\frac{M}{K}, F^*\right) \cong \mathrm{Hom}\left(\frac{M}{K}, F^*\right) \cong \left(\frac{M}{K} \otimes F\right)^* \cong \Gamma^T_0\left(\frac{M}{K}, F\right)^*.$$

If $m > 0$, then the assertion follows from [5, Proposition 2.2] and induction. □

**Example 3.1.** Let $R$ be a 1-Gorenstein ring and $0 \to R \to E_0 \to E_1 \to 0$ be the minimal injective resolution of $R$. Then by [3], $T = E_0 \oplus E_1$ is a tilting module. Hence, for any submodule $T'$ of $T$, the exact sequence $0 \to E_0 \to T \to E_1 \to 0$ implies that $\mathcal{E}^{n+1}_T\left(\frac{T}{T'}, T\right) = 0$ for any $n \geq 0$. So, $T$ is $T^n_T$-injective. Moreover, from the exact sequence $0 \to T' \to T \to \frac{T}{T'} \to 0$, we deduce that $\Gamma^T_{n+1}\left(\frac{T}{T'}, R\right) = 0$ for any $n \geq 0$; therefore, $R$ is a $T^n_T$-flat module.

In the following theorem, some characterizations of the modules with finite $T$-projective dimension and modules with finite $T$-flat dimension are given.

**Theorem 3.3.** *For any module $M$, the following statements hold:*

1. *$\sigma[M] \subseteq \mathcal{TP}_n$ if and only if every module is $T^n_M$-injective.*
2. *$\sigma[M] \subseteq \mathcal{TF}_n$ if and only if every module is $T^n_M$-flat.*
3. *If $\sigma[M] \subseteq \mathcal{TP}_n$, then $\sigma[M] \subseteq \mathcal{TF}_n$.*

(4) *$\sigma[M]\subseteq \mathcal{TP}_{n+1}$ if and only if every factor module of an $T^n_M$-injective module is $T^n_M$-injective.*

(5) *$\sigma[M]\subseteq \mathcal{TF}_{n+1}$ if and only if every submodule of an $T^n_M$-flat module is $T^n_M$-flat.*

*Proof.* (1): Choose $B\in\sigma[M]\subseteq\mathcal{TP}_n$. Then there exists a submodule $Y<M\le M^{(I)}$ such that $B=\frac{M}{Y}\le\frac{M^{(I)}}{Y}$. Thus $\mathcal{E}_T^{n+1}\left(\frac{M}{Y},F\right)=0$, for every module $F$. So, every module is $T^n_M$-injective. Conversely, assume that any module is $T^n_M$-injective. Then by Lemma 3.1 (2), all modules are $T^n_{M^{(I)}}$-injective. So by Theorem 3.1 (2), for any $A\le M^{(I)}$, all modules are $T^n_{\frac{M^{(I)}}{A}}$-injective; therefore, $\frac{M^{(I)}}{A}\in\mathcal{TP}_n$. Now, let $X=\frac{B}{C}\in\sigma[M]$. Then there exists an exact sequence $0\to\frac{B}{C}\to\frac{M^{(I)}}{C}\to\frac{M^{(I)}}{B}\to 0$ which induces the exact sequence

$$0=\mathcal{E}_T^{n+1}\left(\frac{M^{(I)}}{C},F\right)\longrightarrow\mathcal{E}_T^{n+1}\left(\frac{B}{C},F\right)\longrightarrow\mathcal{E}_T^{n+2}\left(\frac{M^{(I)}}{B},F\right)=0.$$

So, $X=\frac{B}{C}\subseteq\mathcal{TP}_n$, as desired.

(2): This is similar to (1).

(3): Assume that $\sigma[M]\subseteq\mathcal{TP}_n$. Then by (1), every module is $T^n_M$-injective.
Hence, Proposition 3.2 implies that every module is $T^n_M$-flat. So, the assertion follows from (2).

(4): Let $A$ be a submodule of the $T^n_M$-injective module $B$. By hypothesis, for every $Y<M$, the exact sequence

$$0=\mathcal{E}_T^{n+1}\left(\frac{M}{Y},B\right)\longrightarrow\mathcal{E}_T^{n+1}\left(\frac{M}{Y},\frac{B}{A}\right)\longrightarrow\mathcal{E}_T^{n+2}\left(\frac{M}{Y},A\right)=0$$

exists. Thus $\mathcal{E}_T^{n+1}(\frac{M}{Y},\frac{B}{A})=0$ and so, $\frac{B}{A}$ is $T^n_M$-injective. Conversely, for any module $X$, there exists an exact sequence $0\to X\to E\to N\to 0$ with $E$ injective. So by hypothesis, for every $Y<M$, the sequence

$$0=\mathcal{E}_T^{n+1}\left(\frac{M}{Y},N\right)\longrightarrow\mathcal{E}_T^{n+2}\left(\frac{M}{Y},X\right)\longrightarrow\mathcal{E}_T^{n+2}\left(\frac{M}{Y},E\right)=0$$

is exact, and we have that $\mathcal{E}_T^{n+2}\left(\frac{M}{Y},X\right)=0$. Thus $X$ is $T^{n+1}_M$-injective, and $\sigma[M]\subseteq\mathcal{TP}_{n+1}$ by (1).

(5): This is similar to (4). □

**Proposition 3.3.** *For any module $F$, the following statements are equivalent:*

(1) *$F$ is $T^n_M$-flat;*

(2) *$\Gamma^T_{n+1}\left(\frac{R}{L},F\right)=0$, for any $m\in M$ and $m^\perp\subseteq L\le R$;*

(3) *$F$ is $T^n_{Rm}$-flat, for all $m\in M$.*

*Proof.* (2)$\Longleftrightarrow$(3): Let $X=mL<Rm$, where $L=m^{-1}X$. Then we have $m^\perp\subseteq L$. But $X\cong\frac{L}{m^\perp}$, while $Rm\cong\frac{R}{m^\perp}$.

(1)$\Longrightarrow$(2): It is clear that $\frac{R}{m^\perp}\cong Rm\in\sigma[M]$, so Proposition 3.1(1) implies that $\Gamma^T_{n+1}\left(\frac{R}{L},F\right)=0$.

(2)$\Longrightarrow$(1): There is an exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus\{Rm \mid m \in M\} \longrightarrow M \longrightarrow 0,$$

where the last map is the natural sum map and $K$ is its kernel. Since $Rm \cong \frac{R}{m^\perp}$, $F$ is $T^n_{Rm}$-flat. Therefore, Lemma 3.1 and Theorem 3.1 complete the proof. □

The next theorem provides some sufficient conditions under which any direct product (direct limit) of $T^n_M$-flat ($T^n_M$-injective) modules is $T^n_M$-flat ($T^n_M$-injective).

**Theorem 3.4.** *Let $T$ be finitely presented and $\Omega_M(R) \subseteq \mathrm{FPres}^{n+1}T$. Then the following statements hold.*

(1) *If $R$ is an $T^{n+1}_M$-coherent, then any direct product of $T^n_M$-flat modules is $T^n_M$-flat.*

(2) *If $R$ is an $T^{n+1}_M$-coherent, then every direct limit of $T^n_M$-injective modules is $T^n_M$-injective.*

*Proof.* (1): By hypothesis $\frac{L}{m^\perp}, \frac{R}{m^\perp} \in \mathrm{FPres}^{n+1}T$. Since $R$ is $T^{n+1}_M$-coherent, $\frac{L}{m^\perp}, \frac{R}{m^\perp} \in \mathrm{FPres}^{n+2}T$. So by Lemma 2.1, the sequence $0 \to \frac{L}{m^\perp} \to \frac{R}{m^\perp} \to \frac{R}{L} \to 0$ implies that $\frac{R}{L} \in \mathrm{FPres}^{n+2}T$. Therefore by Lemma 2.2,

$$\Gamma^T_{n+1}\left(\frac{R}{L}, \prod_{i\in I} F_i\right) \cong \prod_{i\in I} \Gamma^T_{n+1}\left(\frac{R}{L}, F_i\right) = 0.$$

Hence by Proposition 3.3, $\prod\limits_{i\in I} F_i$ is $T^n_M$-flat.

(2): This is similar to (1). □

**Proposition 3.4.** *For any module $F$, the following are equivalent:*

(1) *$F$ is $T^n_M$-injective;*

(2) *$\mathcal{E}^{n+1}_T(\frac{R}{L}, F) = 0$, for any $m \in M$ and $m^\perp \subseteq L \leq R$;*

(3) *$F$ is $T^n_{Rm}$-injective, for all $m \in M$.*

*Proof.* This is similar to Proposition 3.3. □

## 4. Conclusion

Let $n$ be a nonnegative integer, $T$ be a tilting $R$-module and $M$ be a fixed $R$-module. From the results proved in this paper, we conclude that:

- The $T^n_M$-coherence of a module is equivalent to $R \in \mathrm{Fpres}^n T$ and some conditions on the factor modules of $N$.
- The relative flatness (resp. injectivity) of modules with respect to the elements of any short exact sequence can be compared.
- A module $F$ is $T^n_M$-flat if and only if $\Gamma^T_{n+1}(\frac{B}{A}, F) = 0$ for any $A \leq B \in \sigma[M]$.
- A module $F$ is $T^n_M$-injective if and only if $\mathcal{E}^{n+1}_T(\frac{B}{A}, F) = 0$ for any $A \leq B \in \sigma[M]$.

- If $T$ is finitely presented, then the $T^n_M$-flatness of a module in $\mathrm{FPres}^{n+2}T$ is equivalent to the vanishing of the functor $\Gamma^T_{n+1}(-,F)$, on the factor modules in $\pi[M]$.
- The $T^n_M$-flatness of any module is equivalent to the $T^n_M$-injectivity of its character module.
- The relative flatness with respect to any $R$-module $M$ is equivalent to the relative flatness with respect to cyclic submodules of $M$.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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