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Research Article

Functions and Intuitionistic Fuzzy Volterra Spaces

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Abstract. The focus of this paper is to investigate results of functions that preserve intuitionistic fuzzy Volterra spaces in the context of images and preimages.

Keywords. Intuitionistic fuzzy G_δ set; Somewhat intuitionistic fuzzy Volterra function; Intuitionistic fuzzy dense; Somewhat intuitionistic fuzzy continuous; Somewhat intuitionistic fuzzy open

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1. Introduction

L.A. Zadeh [12] introduced the concept of fuzzy sets and fuzzy set operations in 1965. Thereafter, C.L. Chang [4] paved the way for the subsequent tremendous growth of numerous fuzzy topological concepts, after which much attention has been paid to generalise the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The concepts of Volterra spaces have been studied extensively in classical topology [3, 6, 7, 8, 9, 10]. The concept of Volterra spaces in intuitionistic fuzzy setting was introduced and studied by the authors [11]. In this paper some results concerning functions that preserve intuitionistic fuzzy Volterra spaces in the context of images and preimages are obtained.

2. Preliminaries

Definition 2.1 ([1]). An intuitionistic fuzzy set (IFS, in short) A in X is an object having the form $A = \{x, \mu_A(x), \upsilon_A(x)/x \in X\}$, where the functions $\mu_A : X \to I$ and $\upsilon_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\upsilon_A(x)$) of each element $x \in X$ to the set A on a nonempty set X and $0 \le \mu_A(x) + \upsilon_A(x) \le 1$ for each $x \in X$. Obviously every fuzzy set A on a nonempty set X is an IFS's A and B be in the form $A = \{x, \mu_A(x), 1 - \mu_A(x)/x \in X\}$.

Definition 2.2 ([1]). Let X be a nonempty set and the IFS's A and B be in the form $A = \{x, \mu_A(x), \upsilon_A(x)/x \in X\}$, $B = \{x, \mu_B(x), \upsilon_B(x)/x \in X\}$ and let $A = \{A_j : j \in J\}$ be an arbitrary family of IFS's in X. Then, we define

(i) $A \subseteq B$if and only if $\mu_A(x) \le \mu_B(x)$ and $\upsilon_A(x) \ge \upsilon_B(x)$ for all $x \in X$.

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(iii) $\overline{A} = \{x, \upsilon_A(x), \mu_A(x)/x \in X\}$.

(iv) $A \bigcap B = \{x, \mu_A(x) \bigcap \mu_B(x), \upsilon_A(x) \bigcup \upsilon_B(x)/x \in X\}$.

(v) $A \bigcup B = \{x, \mu_A(x) \bigcup \mu_B(x), \upsilon_A(x) \bigcap \upsilon_B(x)/x \in X\}$.

(vi) $1_\sim = \{\langle x, 1, 0\rangle, x \in X\}$ and $0_\sim = \{\langle x, 0, 1\rangle, x \in X\}$.

Definition 2.3 ([7]). Let X and Y be two non-empty sets and $f : X \to Y$ be a function.

(a) If $B = \{\langle y, \mu_B(y), \upsilon_B(y)\rangle/y \in Y\}$ is an IFS in Y, then the pre-image of B under f is denoted and defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B(x)), f^{-1}(\upsilon_B(x)\rangle/x \in X\}.$$

Since $\mu_B(x), \upsilon_B(x)$ are fuzzy sets, we explain that $f^{-1}(\mu_B(x)) = \mu_B(x)(f(x))$, $f^{-1}(\upsilon_B(x)) = \upsilon_B(x)(f(x))$.

(b) If $A = \{\langle y, \mu_A(y), \upsilon_A(y)\rangle/x \in X\}$ is an IFS in X, then the image of A under f is denoted and defined by $f(A) = \{\langle y, f(\mu_A(y)), f(\upsilon_A(y))\rangle/y \in Y\}$ where $f(\upsilon_A) = 1 - f(1 - \upsilon_A)$.

Corollary 2.4 ([7]). *Let A, A_i's ($i \in J$) be IFS's in X,B,B_j's ($j \in K$) IFS's in Y and $f : X \to Y$ a function. Then*

(i) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$.

(ii) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

(iii) $A \subseteq f^{-1}(f(A))$ *(if f is injective, then $A = f^{-1}(f(A))$).*

(iv) $f(f^{-1}(B)) \subseteq B$ *(if f is surjective, then $f(f^{-1}(B)) = B$).*

(v) $f^{-1}(\bigcup B_j)) = \bigcup f^{-1}(B_j)$.

(vi) $f^{-1}(\bigcap B_j)) = \bigcap f^{-1}(B_j)$.

(vii) $f(\bigcup A_i)) = \bigcup f(A_i)$.

(viii) $f(\bigcap A_i)) \subseteq \bigcap f(A_i)$ (*if f is injective then* $f(\bigcap A_i)) = \bigcap f(A_i)$).

(ix) $f^{-1}(1_\sim) = 1_\sim$.

(x) $f^{-1}(0_\sim) = 0_\sim$.

(xi) *If f is surjective, then* $f(1_\sim) = 1_\sim$, $f(0_\sim) = 0_\sim$.

(xii) *If f is surjective, then* $\overline{f(A)} \subseteq f(\overline{A})$ (*further more, if f is injective, then* $\overline{f(A)} = f(\overline{A})$).

(xiii) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Definition 2.5 ([7])**.** An intuitionistic fuzzy topology (IFT, in short) on a nonempty set X is a family τ of an intuitionistic fuzzy set (IFS, in short) in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$.

(ii) $A_1 \bigcap A_2 \in \tau$ for any $A_1, A_2 \in \tau$.

(iii) $\bigcup A_j \in \tau$ for any $A_j : j \in J \subseteq \tau$.

The complement $\overline{A}$ of intuitionistic fuzzy open set (IFOS, in short) in intuitionistic fuzzy topological space (IFTS, in short) (X,τ) is called an intuitionistic fuzzy closed set (IFCS, in short).

Definition 2.6 ([7])**.** Let (X,τ) be an IFTS and $A = \{x, \mu_A(x), \upsilon_A(x)\}$ be an IFS in X. Then, the fuzzy interior and closure of A are denoted by

(i) $\mathrm{cl}(A) = \bigcap\{K : K$ is an IFCS in X and $A \subseteq K\}$.

(ii) $\mathrm{int}(A) = \bigcup\{G : G$ is an IFOS in X and $G \subseteq A\}$.

Note that, for any IFS A in (X,τ), we have $\mathrm{cl}(\overline{A}) = \overline{\mathrm{int}(A)}$ and $\mathrm{int}(\overline{A}) = \overline{\mathrm{cl}(A)}$.

Definition 2.7 ([6])**.** An IFS A in IFTS (X,τ) is called IF dense if there exists no IFCS B in (X,τ) such that $A \subseteq B \subseteq 1_\sim$. Then is, $\mathrm{IFcl}(A) = 1_\sim$.

Definition 2.8 ([6])**.** An IFS A in an IFTS (X,τ) is called IF nowhere dense if there exists no non-zero IFOS B in (X,τ) such that $B \subseteq \mathrm{IFcl}(A)$. Then is, $\mathrm{IFint}(\mathrm{IFcl}(A)) = 0_\sim$.

Let A be an intuitionistic fuzzy set in (X,τ). If A is an IFCS in $(X,\tau)$with $\mathrm{IFint}(A) = 0_\sim$, then A is an IF nowhere dense set in (X,τ).

Definition 2.9 ([11])**.** An intuitionistic fuzzy topological space (X,τ) is called intuitionistic fuzzy volterra space, if $\mathrm{IFcl}\left(\bigcap_{i=1}^{N} A_i\right) = 1_\sim$, where A_i's are intuitionistic fuzzy dense and intuitionistic fuzzy G_δ-sets in (X,τ).

Definition 2.10 ([6])**.** An IFTS (X,τ) is called intuitionistic fuzzy first category if $\bigcup_{i=1}^{\infty} A_i = 1_\sim$, where A_i's are IF nowhere dense sets in (X,τ). An IFTS which is not IF first category is said to be IF second category.

Definition 2.11 ([6])**.** Let (X,τ) and (Y,κ) be any two IFTSs. A function $f : (X,\tau) \to (Y,\kappa)$ is called somewhat IF continuous if $A \in (Y,\kappa)$ and $f^{-1}(A) \neq 0_\sim$, then there exists a $B \in (X,\tau)$, such that $B \neq 0_\sim$ and $B \subseteq f^{-1}(A)$.

Definition 2.12 ([6]). Let (X,τ) and (Y,κ) be any two IFTSs. A function $f:(X,\tau)\to(Y,\kappa)$ is called somewhat IFopen if $A\in(X,\tau)$ and $A\neq 0_\sim$, then there exists a $B\in(Y,\kappa)$, such that $B\neq 0_\sim$ and $B\subseteq f(A)$.

3. Intuitionistic Fuzzy Volterra Spaces and Functions

Proposition 3.1. *If a function $f:(X,\tau)\to(Y,\kappa)$ from an IFTS (X,τ) into another IFTS (Y,κ) is an IF continuous, one-one and if A is an IF dense set in (X,τ), then $f(A)$ is an IF dense set in (Y,κ).*

Proof. Suppose $f(A)$ is not an IF dense set in (Y,κ). Then, there exists an IFCS B in (Y,κ) such that $f(A)<B<1$. Then, $f^{-1}(f(A))<f^{-1}(B)<f^{-1}(1)$.

Since f is one-one, $f^{-1}(f(A))=A$. Hence, we have $A<f^{-1}(B)<1$. Since f is an IF continuous and B is an IFCS in (Y,κ), $f^{-1}(B)$ is an IFCS in (X,τ). Then, $\text{IFcl}(A)\neq 1_\sim$, which is a contradiction to A being an IFdense set in (X,τ). Therefore, $f(A)$ is an IFdense set in (Y,κ). □

Proposition 3.2. *If a function $f:(X,\tau)\to(Y,\kappa)$ from an IFTs (X,τ) onto another IFTS (Y,κ) is IFopen and if A is an IFdense set in (Y,κ), then $f^{-1}(A)$ is an IFdense set in (X,τ).*

Proof. Let A be an IF dense set in (Y,κ), then we have $\text{IFcl}(A)=1_\sim$. Since f is an IFopen function, then $f^{-1}(\text{IFcl}(A))\leq \text{IFcl}(f^{-1}(A))$. Then, $f^{-1}(1_\sim)\leq \text{IFcl}(f^{-1}(A))$ which implies that $1_\sim\leq \text{IFcl}(f^{-1}(A))$. That is, $\text{IFcl}(f^{-1}(A))=1_\sim$. Hence $f^{-1}(A)$ is an IF dense set in (X,τ). □

Proposition 3.3. *If a function $f:(X,\tau)\to(Y,\kappa)$ from an IFTs (X,τ) onto another IFTS (Y,κ) is IF continuous, one-one and IFopen function and if (X,τ) is an IF Volterra space, then (Y,κ) is an IF Volterra space.*

Proof. Let (X,τ) be an IF Volterra space and A_i's, $i=1,2,\dots,N$ be IFdense and IFG_δ sets in (Y,κ). Then, $\text{IFcl}(A_i)=1_\sim$ and $A_i=\bigcap_{j=1}^{\infty}(A_{ij})$, where A_{ij}'s are IFopen sets in (Y,κ). Now $f^{-1}(A_i)=f^{-1}\left(\bigcap_{j=1}^{\infty}(A_{ij})\right)=\bigcap_{j=1}^{\infty}(f^{-1}(A_{ij}))$.

Since f is an IF continuous and A_{ij}'s are IFopen sets in (Y,κ), $f^{-1}(A_{ij})$'s are IFopen sets in (X,τ). Hence $\bigcap_{j=1}^{\infty}(f^{-1}(A_{ij}))$ is an IFG_δ set in (X,τ).

Then, $f^{-1}(A_i)$ is an IFG_δ set in (X,τ) $(i=1,2,\dots,N)$.

Now A_i is an IFdense set in (Y,κ). Since f is an IFopen function from (X,τ) onto (Y,κ), by Proposition 3.2, $f^{-1}(A_i)$ is an IFdense set in (X,τ). Since (X,τ) is an IF Volterra space, we have

$$\text{IFcl}\left(\bigcap_{j=1}^{\infty}(f^{-1}(A_i))\right)=1_\sim,$$

where $f^{-1}(A_i)$'s are IF dense and IFG_δ sets in (X,τ). Then, $\text{IFcl}\left(f^{-1}\left(\bigcap_{j=1}^{\infty}(f^{-1}(A_i))\right)\right)=1_\sim$.

That is, $f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)$ is an IFdense in (X,τ). Since f is an IFcontinuous and one-one, by Proposition 3.1, $f(f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)$ is an IFdense set in (Y,κ). That is,

$$\text{IFcl}\left(f\left(f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)\right)\right)=1_\sim\ . \tag{3.1}$$

Since f is onto, $f(f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)=\bigcap_{j=1}^{\infty}(A_i)$. Then, from (3.1), we have $\text{IFcl}\left(\bigcap_{j=1}^{\infty}(A_i)\right)=1_\sim$. Therefore (Y,κ) is an IF Volterra space. □

Theorem 3.4. *Suppose (X,τ) and (Y,κ) be IFTSs. Let $f:(X,\tau)\to(Y,\kappa)$ be an onto function. Then, the following conditions are equivalent.*

(i) *f is somewhat fuzzy open*

(ii) *If A is an IFdense set in (Y,κ), then $f^{-1}(A)$ is an IF dense set in (X,τ).*

Proof. (i)⇒(ii): Let A be an IFdense set in (Y,κ). Suppose $f^{-1}(A)$ is not an IF dense set in (X,τ). Then, there exists a non-zero IFCS B on (X,τ) such that $f^{-1}(A)<B<1_\sim$. Then, $1-B<1-f^{-1}(A)=f^{-1}(1-A)$. Now $1-B$ is an IFOS in (X,τ). Since $B<1_\sim$, $1-B\neq 0_\sim$. Since f is somewhat IF open function, there exists an IFOS $C\neq 0_\sim$ somewhat IFopen in (Y,κ) such that $C\le f(1-B)$ and hence $C\le f(f^{-1}(1-B))\le 1-B$. That is, $A<1-C<1_\sim$ and $1-C$ is an IFCS in (Y,κ), implies that A is not an IF dense set in (X,τ), which is a contradiction to the assumption on A. Thus (i)⇒(ii) is proved.

(ii)⇒(i): Let A be a non-zero IFS and $f(A)\neq 0_\sim$. Suppose that there exists no IFOS $B\neq 0_\sim$ in (Y,κ) such that $B\le f(A)$. That is, $\text{IFint}(f(A))=0_\sim$. Then, $1-\text{IFint}(f(A))=1_\sim$. This will imply that $\text{IFcl}(1-f(A))=1_\sim$. Now $f^{-1}(1-f(A))=1-f^{-1}(f(A))\le 1-A<1_\sim$ (since $A\neq 0_\sim$). That is, $f^{-1}(1-f(A))<1_\sim$. Then, $\text{IFcl}(f^{-1}(1-f(A)))<\text{IFcl}(1_\sim)=1_\sim$. This will imply that $\text{IFcl}(f^{-1}(1-f(A)))\neq 1_\sim$, a contradiction. Hence we must have $\text{IFint}(f(A))\neq 0_\sim$. Therefore f is somewhat IFopen function from (X,τ) into (Y,κ). □

Proposition 3.5. *If $f:(X,\tau)\to(Y,\kappa)$ is an onto function in an IFTS is IFcontinuous, one-one and somewhat IF open function, then (X,τ) is an IF Volterra space if and only if (Y,κ) is an IF Volterra space.*

Proof. Let (X,τ) be an IF Volterra space and A_i's, $i=1,2,\dots,N$ be IFdense and IFG_δ sets in (Y,κ). Then, $\text{IFcl}(A_i)=1_\sim$ and $A_i=\bigcap_{j=1}^{\infty}(A_{ij})$, where A_{ij}'s are IFopen sets in (Y,κ). Now

$$f^{-1}(A_i)=f^{-1}\left(\bigcap_{j=1}^{\infty}(A_{ij})\right)=\bigcap_{j=1}^{\infty}(f^{-1}(A_{ij})).$$

Since f is an IF continuous and A_{ij}'s are IFopen sets in (Y,κ), $f^{-1}(A_{ij})$'s are IFopen sets in (X,τ). Hence $\bigcap_{j=1}^{\infty}(f^{-1}(A_{ij}))$ is an IF$G_\delta$set in (X,τ).

Then, $f^{-1}(A_i)$ is an IFG$_\delta$ set in (X,τ) $(i=1,2,\ldots,N)$. Now A_i is an IFdense set in (Y,κ). Since f is an IFopen function from (X,τ) onto (Y,κ), by Proposition 3.2, $f^{-1}(A_i)$ is an IFdense set in (X,τ). Since (X,τ) is an IF Volterra space, we have

$$\text{IFcl}\left(\bigcap_{j=1}^{\infty}(f^{-1}(A_i)\right)=1_\sim,$$

where $f^{-1}(A_i)$'s are IF dense and IFG$_\delta$ sets in (X,τ). Then, IFcl$(f^{-1}\left(\bigcap_{j=1}^{\infty}(f^{-1}(A_i))\right)=1_\sim$.

That is, $f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)$ is an IFdense in (X,τ). Since f is an IFcontinuous and one-one, by Proposition 3.1, $f\left(f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)\right)$ is an IFdense set in (Y,κ). That is,

$$\text{IFcl}\left(f\left(f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)\right)\right)=1_\sim \tag{3.2}$$

Since f is onto, $f\left(f^{-1}\left(\bigcap_{j=1}^{\infty}(A_i)\right)\right)=\bigcap_{j=1}^{\infty}(A_i)$. Then, from (3.2), we have IFcl$(\bigcap_{j=1}^{\infty}(A_i))=1_\sim$. Therefore (Y,κ) is an IF Volterra space.

Conversely, let (Y,κ) be an IF Volterra space and A_i's, $i=1,2,\ldots,N$ be IFdense and IFG$_\delta$ sets in (Y,κ). Then, IFcl$(A_i)=1_\sim$ and $A_i=\bigcap_{j=1}^{\infty}(A_{ij})$, where A_{ij}'s are IFopen sets in (Y,κ). Now $f^{-1}(A_i)=f^{-1}\left(\bigcap_{j=1}^{\infty}(A_{ij})\right)=\bigcap_{j=1}^{\infty}(f^{-1}(A_{ij}))$.

Since f is an IF continuous and A_{ij}'s are IFopen sets in (Y,κ), $f^{-1}(A_{ij})$'s are IFopen sets in (X,τ). Hence $\bigcap_{j=1}^{\infty}(f^{-1}(A_{ij}))$ is an IFG$_\delta$ set in (X,τ).

Then, $f^{-1}(A_i)$ is an IFG$_\delta$ set in (X,τ) $(i=1,2,\ldots,N)$.

Now A_i is an IFdense set in (Y,κ). Since f is an IFopen function from (X,τ) onto (Y,κ), by Proposition 3.2, $f^{-1}(A_i)$ is an IFdense set in (X,τ). Since (X,τ) is an IF Volterra space, we have

$$\text{IFcl}\left(\bigcap_{j=1}^{\infty}(f^{-1}(A_i))\right)=1_\sim,$$

where $f^{-1}(A_i)$'s are IF dense and IFG$_\delta$ sets in (X,τ).

Now we claim that IFcl$\left(\bigcap_{j=1}^{N}(f^{-1}(A_i))\right)=1_\sim$. Suppose that IFcl$\left(\bigcap_{j=1}^{N}(f^{-1}(A_i))\right)\neq 1_\sim$.

Then, $1-$IFcl$\left(\bigcap_{j=1}^{N}(f^{-1}(A_i))\right)\neq 0_\sim$ which implies that

$$\text{IFint}\left(\bigcup_{j=1}^{N}(1-f^{-1}(A_i))\right)=\text{IFint}\left(\bigcup_{j=1}^{N}(f^{-1}(1-A_i))\right)\neq 0_\sim.$$

Then, there will be a non-zero IFOS B$_i$ in (X,τ) such that $B_i\leq\bigcup_{j=1}^{N}(f(1-A_i))$.

Then, $f(B_i) \leq f\left(\bigcup_{j=1}^{N}(f(1-A_i))\right) \leq \bigcup_{j=1}^{N}(ff^{-1}(1-A_i))$. Since f is onto, $ff^{-1}(1-A_i) = 1-A_i$. Hence $f(B_i) \leq \bigcup_{j=1}^{N}(1-A_i) = 1-\left(\bigcap_{j=1}^{N}(A_i)\right) = 1-1 = 0_\sim$ (since (Y,κ) is an IF Volterra space $\mathrm{IFcl}\left(\bigcap_{j=1}^{N}(f^{-1}(A_i))\right) = 1_\sim$). That is, $\mathrm{IFint}(f(B_i)) \neq 0_\sim$. Therefore (X,τ) is an IF Volterra space. □

4. Conclusion

In this paper the relativization of intuitionistic fuzzy volterra spaces and intuitionistic functions are studied. It paves way for forthcoming researchers to find the inter-relation between spaces and functions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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