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Linear Jaco Graphs: A Critical Review

Survey Article

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Abstract. The concept of linear Jaco graphs was introduced by Kok et al. [19, 20]. Linear Jaco graphs are a family of finite directed graphs which are derived from an infinite directed graph, called the $f(x)$-root digraph. The incidence function is a linear function $f(x) = mx + c$, $x \in \mathbb{N}$, $m, c \in \mathbb{N}_0$. Much research has been done for the case $f(x) = x$. Many interesting open problems remain for the case $f(x) = x$ and certainly for the general case $f(x) = mx + c$, $m, c > 0$. Despite an elegant, almost simple definition of these graphs it remains hard and predictably impossible in some cases to derive closed formula for a number of well-known invariants. Interesting to note, is the ever so often appearance of Fibonacci and Lucas numbers as well as the Golden Ratio in some results. These observations suggest that a sound number theoretic approach might resolve some of the mystery surrounding Jaco graphs.

Keywords. Linear Jaco graph; Hope graph; Jaconian vertex; Jaconian set; Fisher algorithm; Bettina's theorem.

MSC. 05C07; 05C12; 05C20; 05C38; 05C69; 05C75; 05C85; 11B39

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1. Introduction

For a general reference to notation and concepts of graph theory see [8]. The concept of linear Jaco graphs[1] was introduced by Kok et al. [19, 20]. In the initial studies the concepts of *order* 1 and *order* a Jaco graphs, denoted $J_n(1)$, $J_n(a)$ respectively, were reported on. The aforesaid notation was used in subsequent papers [17, 18, 21, 22, 23, 24, 25, 26, 27, 28]. In a more recent study (see [29]) a unifying definition was adopted and the generalised family called linear Jaco graphs was defined. In terms of linear Jaco graphs the earlier notation namely, $J_n(1)$, $J_n(a)$ have a different meaning which complicates comparative reading somewhat. Linear Jaco graphs

[1]Dedicated to the author's father, late Pieter Jaco Kok.

are a family of finite directed graphs which are derived from an infinite directed graph, called the $f(x)$-root digraph. The incidence function is a linear function $f(x) = mx + c$, $x \in \mathbb{N}$, $m, c \in \mathbb{N}_0$. The $f(x)$-root graph is denoted by $J_\infty(f(x))$. The finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and arcs to vertices) v_t, $t > n$. Hence, trivially $d(v_i) \leq f(i)$ for $i \in \mathbb{N}$. Much research has been done for the case $f(x) = x$ hence, in respect of the linear Jaco graph $J_n(x)$, $n, x \in \mathbb{N}$. In this review the incidence function remains linear so for brevity, linear Jaco graphs are called *Jaco graphs*.

2. Finite Jaco Graphs, $\{J_n(x) : n, x \in \mathbb{N}\}$

The family of *trivial* finite Jaco Graphs was introduced by Kok et al. [19]. These directed graphs are derived from the infinite Jaco Graph called, the x-root digraph. Initially this family of linear Jaco graphs was denoted $J_n(1)$, but the latter is now substituted by the case $f(x) = x$ hence, $m = 1$, $c = 0$. Within the new context the notation $f_n(1)$ will represent the case $m = 0$, $c = 1$. Of importance is to reflect on the definitions to ensure an understanding of the basic framework of the new family of directed graphs. Note that the underlying graph will be denoted $J_n^*(x)$ and if the context is clear, both the directed and undirected graph are referred to as a *Jaco graph*. Similarly the difference between *arc* and *edge* and *degree*, $d_{J_n(x)}(v)$ and $d_{J_n^*(x)}(v)$ will be understood.

Definition 2.1 ([29])**.** The infinite Jaco Graph $J_\infty(x)$, $x \in \mathbb{N}$ is defined by $V(J_\infty(x)) = \{v_i : i \in \mathbb{N}\}$, $A(J_\infty(x)) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(x))$ if and only if $2i - d^-(v_i) \geq j$.

Definition 2.2 ([29])**.** The family of finite Jaco Graphs is defined by $\{J_n(x) \subseteq J_\infty(x) : n, x \in \mathbb{N}\}$. A member of the family is referred to as the Jaco Graph, $J_n(x)$.

It follows through immediate induction that Definition 2.1 and Definition 2.2 are well-defined.

Definition 2.3 ([29])**.** The set of vertices attaining degree $\Delta(J_n(x))$ is called the set of Jaconian vertices; the Jaconian vertices or the Jaconian set of the Jaco Graph $J_n(x)$, and denoted, $\mathbb{J}(J_n(x))$ or, $\mathbb{J}_n(x)$ for brevity.

Definition 2.4 ([29])**.** The lowest numbered (subscripted) Jaconian vertex is called the prime Jaconian vertex of a Jaco Graph.

Definition 2.5 ([29])**.** If v_i is the prime Jaconian vertex of a Jaco Graph $J_n(x)$, the complete subgraph on vertices $v_{i+1}, v_{i+2}, \cdots, v_n$ is called the Hope subgraph[2] or Hope graph of a Jaco Graph and denoted, $\mathbb{H}(J_n(x))$ or, $\mathbb{H}_n(x)$ for brevity.

Definition 2.6 ([29])**.** If, in applying Definition 2.1 to vertex v_i (not necessarily exhaustively), or for logical method of proof we have the arc (v_i, v_k) linked in a Jaco Graph $J_n(x)$, then the degree vertex v_i attains at v_k is called the, "*at degree of* v_i at v_k", and is denoted, $d^*(v_i)@v_k$.

[2] Named after the author's loving mother, Hope Kok.

Definition 2.7 ([29])**.** In $J_\infty(x)$, $x \in \mathbb{N}$ we have, $n = d^+_{J_\infty(n)}(v_n) + d^-_{J_\infty(n)}(v_n) = d^+(v_n) + d^-(v_n)$ whilst in $J_n(x)$ we have, $d(v_i) = d^+_{J_n}(v_n) + d^-_{J_\infty(n)}(v_n) = \lceil d^+(v_i) \rceil + d^-(v_i), i \le n$.

The x-root digraph has four fundamental properties which are:

(i) $V(J_\infty(x)) = \{v_i : i \in \mathbb{N}\}$, and

(ii) if v_j is the head of an arc then the tail is always a vertex v_i, $i < j$, and

(iii) if v_k, for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices v_ℓ, $k < \ell < j$ are tails of arcs to v_j, and finally

(iv) the degree of vertex v_k is $d(v_k) = k$.

The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and arcs to vertices) v_t, $t > n$. Hence, trivially $d(v_i) \le i$ for $i \in \mathbb{N}$. The notion of the "*at degree of* v_i at v_k" denoted by, $d^*(v_i)@v_k$ is an interesting concept utilised in the method of proof of some results on Jaco graphs. In the earlier literature [19, 20] graphical embodiment is not found to illustrate a member of the trivial family of finite Jaco graphs. Figure 1 depicts $J_{10}(x)$ (see [29]).

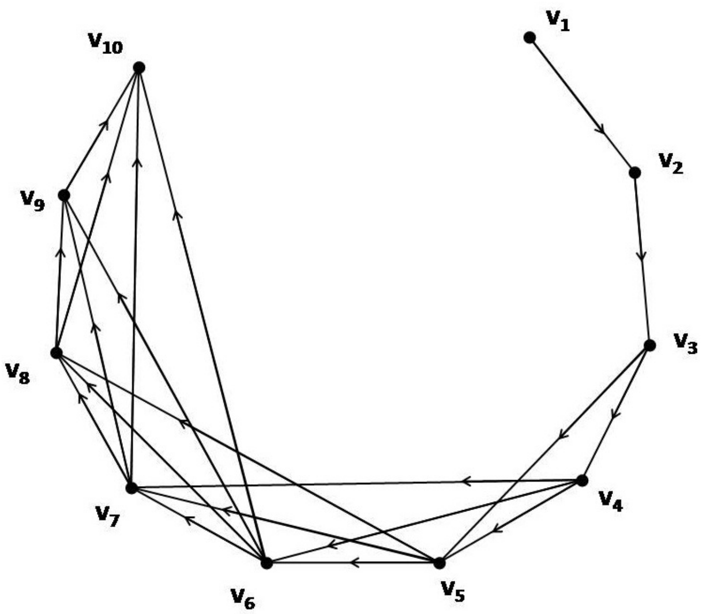

Figure 1. Jaco graph $J_{10}(x)$ [29]

Property 2.1 ([29])**.** From the definition of a Jaco Graph $J_n(x)$, it follows that for the prime Jaconian vertex v_i, we have $d(v_m) = m$ for all $m \in \{1, 2, 3, \ldots, i\}$.

Property 2.2 ([29])**.** From the definition of a Jaco Graph $J_n(x)$, it follows that $\Delta(J_k(x)) \le \Delta(J_n(x))$ for all $k \le n$.

Property 2.3 ([29])**.** The $d^-(v_k)$ for any vertex v_k of a Jaco Graph $J_n(x)$, $n \ge k$ is equal to $d(v_k)$ in the underlying Jaco Graph $J^*_k(x)$.

The importance of Property 2.3 is that it implies that $d^-(v_k)$ in $J_k(x)$ remains constant throughout $J_{k+i}(x)$, $i \geq 0$. An obvious similar property holds for $d^+(v_k)$, that is, if $t = k + d^+(v_k)$, then $d^+(v_k)$ remains constant throughout $J_{(k+d^+(v_k))+i}(x)$, $i \geq 0$ hence, $d(v_k)$ remains constant under the same condition. The following results have been derived.

Lemma 2.1 ([29]). *If in a Jaco Graph $J_n(x)$, $n,x \in \mathbb{N}$ and for smallest i with $d(v_i) = i$, the arc (v_i, v_n) is defined, then v_i is the prime Jaconian vertex of $J_n(x)$.*

Lemma 2.2 ([29]). *For all Jaco Graphs $J_n(x)$, $n,x \in \mathbb{N}$, $n \geq 2$ and, $v_i, v_{i-1} \in V(J_n(x))$ we have that in the underlying graph $J_n^*(x)$, $|(d(v_i) - d(v_{i-1})| \leq 1$.*

Corollary 2.3 ([29]). *For a Jaco Graph $J_n(x)$, $n,x \in \mathbb{N}$ the maximum degree $\Delta(J_n(x))$ might repeat itself as n increases to $n+1$, (i.e. $\Delta J_n(x) = \Delta J_{n+1}(x)$) but on an increase, $\Delta J_{n+1}(x) = \Delta(J_n(x)) + 1$.*

Note Corollary 2.3 applies to the increase in, in-degrees and out-degrees as n increases. It is observed that if for vertex v_i, the ordered in- and out-degree pair is denoted $d^{\pm}(v_i) = (d^-(v_i), d^+(v_i))$ then $d^{\pm}(v_{i+1}) =$ either, $(d^-(v_i)+1, d^+(v_i))$ or, $(d^-(v_i), d^+(v_i)+1)$. It forms the basis for many proof of results. For further analysis the very important *Fisher algorithm*[3] was derived and found to be well-defined.

2.1 The Fisher Algorithm for $\{J_i(x) : i \in \{4,5,6,\ldots,s \in \mathbb{N},\ x \in \mathbb{N}\}\}$

The family of finite Jaco Graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and arcs to vertices) v_t, $t > n$. Hence, trivially $d(v_i) \leq i$ for $i \in \mathbb{N}$.

We generally refer to the entries in a row i as: $ent_{1i} = i$, $ent_{2i} = d^-(v_i)$, $ent_{3i} = d^+(v_i)$, $ent_{4i} = \mathbb{J}(J_i(x))$, $ent_{5i} = \Delta(J_i(x))$, $ent_{6i} = d_{J_i(x)}(v_1, v_i)$ as interchangeable.

The Fisher algorithm has been described in [20]. Note that rows 1, 2 and 3 follow easily from Definition 2.1.

Step 0: Set $j = 4$, then set $i = j$ and $s \geq 4$.

Step 1: Set $ent_{1i} = i$.

Step 2: Set $ent_{2i} = ent_{1(i-1)} - ent_{5(i-1)}$ (note that $d^-(v_i) = v(\mathbb{H}_{i-1}(x)) = (i-1) - \Delta(J_{i-1}(x))$).

Step 3: Set $ent_{3i} = ent_{1i} - ent_{2i}$ (note that $d^+(v_i) = i - d^-(v_i)$).

Step 4: Consider $ent_{4(i-1)}$. If $ent_{4(i-1)} = \{v_k\}$, set $t = k$, else set $t = k+1$.

Step 5: Set the prime Jaconian vertex as v_t so $\mathbb{J}(J_i(x)) = \{v_t\}$ to begin with. Let $l = t+1, t+2, \ldots, i-1$ and recursively calculate $i - ent_{1l} + ent_{2l}$. If $i - ent_{1l} + ent_{2l} = t$, add v_l to the set of Jaconian vertices, else go to Step 6.

Step 6: Set $ent_{5i} = t$. (Note that if $\mathbb{J}_i(x) = \{v_t, v_{t+1}, \ldots v_\ell\}$, then, $\Delta(J_i(x)) = t$).

Step 7: Select smallest k such that, $k + ent_{3k} \geq i$ then set $ent_{6i} = ent_{6k} + 1$.

[3] Named after Dr Paul Fisher, Department of Mathematics, University of Botswana, who described the Fisher algorithm informally to the author in personal correspondence.

Step 8: Set $j = i+1$, then set $i = j$. If $i \leq s$, go to Step 1, else go to Step 9.

Step 9: Exit.

The following table shows the results for the application of the Fisher algorithm for $n \leq 50$.

Table 1

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i) = \nu(\mathbb{H}_{i-1})$	$d^+(v_i) = i - d^-(v_i)$	$\mathbb{J}(J_i(x))$	$\Delta(J_i(x))$
$1 = f_2$	0	1	$\{v_1\}$	0
$2 = f_3$	1	1	$\{v_1, v_2\}$	1
$3 = f_4$	1	2	$\{v_2\}$	2
4	1	3	$\{v_2, v_3\}$	2
$5 = f_5$	2	3	$\{v_3\}$	3
6	2	4	$\{v_3, v_4, v_5\}$	3
7	3	4	$\{v_4, v_5\}$	4
$8 = f_6$	3	5	$\{v_5\}$	5
9	3	6	$\{v_5, v_6, v_7\}$	5
10	4	6	$\{v_6, v_7\}$	6
11	4	7	$\{v_7\}$	7
12	4	8	$\{v_7, v_8\}$	7
$13 = f_7$	5	8	$\{v_8\}$	8
14	5	9	$\{v_8, v_9, v_{10}\}$	8
15	6	9	$\{v_9, v_{10}\}$	9
16	6	10	$\{v_{10}\}$	10
17	6	11	$\{v_{10}, v_{11}\}$	10
18	7	11	$\{v_{11}\}$	11
19	7	12	$\{v_{11}, v_{12}, v_{13}\}$	11
20	8	12	$\{v_{12}, v_{13}\}$	12
$21 = f_8$	8	13	$\{v_{13}\}$	13
22	8	14	$\{v_{13}, v_{14}, v_{15}\}$	13
23	9	14	$\{v_{14}, v_{15}\}$	14
24	9	15	$\{v_{15}\}$	15
25	9	16	$\{v_{15}, v_{16}\}$	15
26	10	16	$\{v_{16}\}$	16
27	10	17	$\{v_{16}, v_{17}, v_{18}\}$	16
28	11	17	$\{v_{17}, v_{18}\}$	17
29	11	18	$\{v_{18}\}$	18
30	11	19	$\{v_{18}, v_{19}, v_{20}\}$	18

(*Contd.*)

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i) = \nu(\mathbb{H}_{i-1})$	$d^+(v_i) = i - d^-(v_i)$	$\mathbb{J}(J_i(x))$	$\Delta(J_i(x))$
31	12	19	$\{v_{19}, v_{20}\}$	19
32	12	20	$\{v_{20}\}$	20
33	12	21	$\{v_{20}, v_{21}\}$	20
$34 = f_9$	13	21	$\{v_{21}\}$	21
35	13	22	$\{v_{21}, v_{22}, v_{23}\}$	21
36	14	22	$\{v_{22}, v_{23}\}$	22
37	14	23	$\{v_{23}\}$	23
38	14	24	$\{v_{23}, v_{24}\}$	23
39	15	24	$\{v_{24}\}$	24
40	15	25	$\{v_{24}, v_{25}, v_{26}\}$	24
41	16	25	$\{v_{25}, v_{26}\}$	25
42	16	26	$\{v_{26}\}$	26
43	16	27	$\{v_{26}, v_{27}, v_{28}\}$	26
44	17	27	$\{v_{27}, v_{28}\}$	27
45	17	28	$\{v_{28}\}$	28
46	17	29	$\{v_{28}, v_{29}\}$	28
47	18	29	$\{v_{29}\}$	29
48	18	30	$\{v_{29}, v_{30}, v_{31}\}$	29
49	19	30	$\{v_{30}, v_{31}\}$	30
50	19	31	$\{v_{31}\}$	31

From the definition of the Jaco graph and the Fisher algorithm a number of results were derived.

Proposition 2.4 ([29]). *Consider the Jaco Graph $J_i(x)$, $i,x \in \mathbb{N}$, $i \geq 4$. If the Jaconian vertex of $J_{i-1}(x)$ is unique say, v_k then $k + d^+(v_k) < i$ and $(k+1) + d^+(v_{k+1}) > i$.*

Conjecture 2.4.1 ([29]). *If for $n \in \mathbb{N}$ the out-degree $d^+(v_n) = \ell$ is non-repetitive (meaning $d^+(v_{n-1}) < d^+(v_n) < d^+(v_{n+1})$) then, $\mathbb{J}(J_n(x)) = \{v_\ell\}$.*

Theorem 2.5 (Morrie's Theorem, [29][4]). *If a Jaco Graph $J_n(x)$, $n,x \in \mathbb{N}$, $n \geq 2$ has the prime Jaconian vertex v_k then:*

(a) *$d^-(v_k) = d^-(v_{k+1})$ and $d^-(v_{k+2}) = d^-(v_{k+1}) + 1$ if and only if $\mathbb{J}(J_n(x)) = \{v_k\}$ and $\mathbb{J}(J_{n+1}(x)) = \{v_k, v_{k+1}, v_{k+2}\}$,*

(b) *$d^-(v_k) = d^-(v_{k+1}) = d^-(v_{k+2})$ if and only if $\mathbb{J}(J_n(x)) = \{v_k\}$ and $\mathbb{J}(J_{n+1}(x)) = \{v_k, v_{k+1}\}$.*

Proposition 2.6 ([29]). *For all Jaco Graphs $J_n(x)$, $n,x \in \mathbb{N}$ we have Card $\mathbb{J}(J_n(x)) \leq 3$.*

[4] In memory of young Morrie Gilbert who had to park his soul somewhere in space well before reaching maturity in his mathematical thinking.

Corollary 2.7 ([29]). *From Proposition 2.4 it follows that if and only if the Jaconian vertex of $J_{i-1}(x)$, $i \geq 2$ is unique say, v_k then $\mathbb{J}(J_i(x)) =$ either $\{v_k, v_{k+1}\}$ or $\{v_k, v_{k+1}, v_{k+2}\}$.*

Corollary 2.8 ([29]). *If $k + d^+(v_k) = i$ and $(k+1) + d^+(v_{k+1}) > i+1$ then v_k is the unique Jaconian vertex of $J_i(x)$.*

Proposition 2.9 ([29]). *If $d^-(v_{k-1}) = d^-(v_k) = d^-(v_{k+1})$ then v_k is the unique Jaconian vertex of $J_l(x)$, $l = 2k - d^-(v_k)$.*

Proposition 2.10 ([29]). $\mathbb{J}(J_{k-1}(x)) = \{v_{l-1}\}$ *if and only if* $d^+(v_k) = d^+(v_{k+1}) = l$.

Theorem 2.11 ([29]). *Let $m = n + \Delta(J_n(x))$, then $\Delta(J_m(x)) =$ either n or $n-1$.*

Conjecture 2.4.2 ([29]). *For the Jaco Graphs $J_n(x)$, $J_m(x)$, $n, m, x \in \mathbb{N}$, with $n \geq 3$, $m \geq 3$, $n \neq m$ we have:*

$$\Delta(J_{n+m}(x)) = \begin{cases} \Delta(J_n(x)) + \Delta(J_m(x)), & \text{if } J_n(x) \text{ or } J_m(x) \text{ has a unique Jaconian vertex,} \\ \Delta(J_n(x)) + \Delta(J_m(x)) + 1, & \text{otherwise.} \end{cases}$$

Theorem 2.12 ([29]). *If the Jaco Graph $J_n(x)$, $n, x \in \mathbb{N}$ has a unique Jaconian vertex (prime Jaconian vertex only) at v_i, then:*

(a) *arc (v_i, v_n) exists, and*

(b) $\Delta(J_n(x)) + d(v_n) = n$.

Note that $\Delta(J_n(x)) + d(v_n) = n \not\Rightarrow$ uniqueness of the Jaconian vertex.

Theorem 2.13 ([29]). *Consider the Jaco Graph $J_n(x)$, $n, x \in \mathbb{N}$. For $m < i < k \leq n$, the arc (v_m, v_i) can only exist if the arc (v_m, v_{i-1}) exists. Furthermore, if the arc (v_i, v_k) exists then the arcs $(v_{i+1}, v_k), \ldots, (v_{k-1}, v_k)$ exist.*

Lemma 2.14 ([29]). *The vertex v_i is the prime Jaconian vertex of a Jaco Graph $J_n(x)$, if and only $d(v_l) \leq d(v_i) = i$ for $l = i+1, i+2, \ldots, n$.*

Theorem 2.15 ([29]). *If for the Jaco Graph $J_n(x)$, $n, x \in \mathbb{N}$ we have $\Delta(J_n(x)) = k$, then the out-degrees of the vertices $v_{k+1}, v_{k+2}, v_{k+3}, \ldots, v_n$ are respectively, $\lceil d^+(v_{k+1}) \rceil = (n-k-1), \lceil d^+(v_{k+2}) \rceil = (n-k-2), \cdots, \lceil d^+(v_{n-1}) \rceil = 1$ and $\lceil d^+(v_n) \rceil = 0$.*

Theorem 2.16 ([29]). *If for the Jaco Graph $J_n(x)$, $n, x \in \mathbb{N}$ we can express $n = 7 + 3k$, $k \in \{0, 1, 2, \ldots\}$, then $\Delta(J_n(x)) \leq n - (3+k)$ and, arc $(v_{\Delta(J_n(x))}, v_n)$ exists.*

It is noted from the Fisher table that not all $n \in \mathbb{N}$ are a unique prime Jaconian vertex to a Jaco graph $J_n(x)$. Stemming from this observation the next conjecture remains open.

Conjecture 2.16.1 ([29]). *For the series defined by $p_0 = 1$, $p_i =$ either $(p_{i-1} + 3)$ or $(p_{i-1} + 5)$ the Jaconian set as determined by Step 4 and 5 of the Fisher algorithm is $\{\alpha_1, \alpha_1 + 1, \alpha_1 + 2\}$ and $\alpha_1 + 1$ is not a unique prime vertex of a Jaco graph, $J_n(x)$.*

It is further observed that "+5" repeats at most twice, whilst "+3" does not repeat hence, is immediately followed by "+5".

2.2 Fibonaccian Result related to Jaco Graphs

Unless stated otherwise the next results are sourced from [20, 29].

Lemma 2.17 ([20, 29]). *For $a = 1$ and the series $(a_n)_{n\in\mathbb{N}}$ defined by*

$$a_0 = 0,\ a_1 = 1,\ a_n = \min\{k < n \mid k + a_k \geq n\} \quad (n \geq 2).$$

it holds that:

(a) $d^+(v_n) + d^-(v_n) = n$.

(b) $d^-(v_{n+1}) \in \{d^-(v_n), d^-(v_n)+1\}$.

(c) *If* $(v_i, v_k) \in A(J_\infty(x))$ *and* $i < j < k$, *then* $(v_j, v_k) \in A(J_\infty(x))$.

(d) $d^+(v_n) = a_n$.

Corollary 2.18 ([20, 29]). *Note that* (a) *and* (c) *above entail that* $d^+(v_{n+1}) = n + 1 - d^-(v_{n+1}) \in \{n - d^-(v_n), n - d^-(v_n) + 1\}$ *and that* (d) *then implies that the series* (a_n) *is well defined and ascending, more specifically,* $a_{n+1} \in \{a_n, a_n + 1\}$, $(n \in \mathbb{N}_0)$.

Lemma 2.19 ([20, 29]). *Let* $i \in \mathbb{N}$. *Then* $d^+(v_{i+d^+(v_i)}) = i = d^+(v_{i+d^+(v_{i+d^+(v_{i-1})})})$.

Theorem 2.20 ([20, 29], Bettina's Theorem[5]). *Let* $\mathbb{F} = \{f_0, f_1, f_2, f_3, \ldots\}$ *be the set of Fibonacci numbers and let* $n = f_{i_1} + f_{i_2} + \cdots + f_{i_r}$, $n \in \mathbb{N}$ *be the Zeckendorf representation of* n (*see* [42]). *Then*

$$d^+(v_n) = f_{i_1-1} + f_{i_2-1} + \cdots + f_{i_r-1}.$$

Proposition 2.21 ([20, 29]). *For Fibonacci numbers* $t = f_{n-1}$, $h = f_n$ *and* $l = f_m$, $t \geq 3$, $h \geq 3$, $l \geq 3$, *we have:*

(a) $\Delta(J_h(x)) = t = f_{n-1}$,

(b) $\mathbb{J}(J_h(x)) = \{v_t\}$,

(c) $\Delta(J_{h+l}(x)) = \Delta(J_h(x)) + \Delta(J_l(x))$,

(d) $\mathbb{J}(J_{h+l}(x)) = \{v_{\Delta(J_h(x))+\Delta(J_l(x))}\}$.

2.3 Total Irregularity of Jaco Graphs

Total irregularity of a simple undirected graph G is generally defined to be $irr_t(G) = \frac{1}{2}\sum\limits_{u,v\in V(G)} |d(u) - d(v)|$ (see Abdo and Dimitrov [2, 3]). If the vertices of a simple undirected graph G on n vertices are labelled v_i, $i = 1, 2, 3, \ldots, n$ then the definition may be $irr_t(G) = \frac{1}{2}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} |d(v_i) - d(v_j)| = \sum\limits_{i=1}^{n}\sum\limits_{j=i+1}^{n} |d(v_i) - d(v_j)|$ or $\sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} |d(v_i) - d(v_j)|$. For a simple graph on a singular vertex (1-*empty* graph), the default definition $irr_t(G) = 0$ has been adopted.

[5]This theorem is dedicated to Dr. Bettina Wilkens, Department of Mathematics, University of Botswana. Dr. Wilkens proposed this result with proof thereof during the early part of the initial research project.

For illustration the adapted table below follows from the Fisher algorithm [17, 19] for $J_n(x)$, $n \leq 12$. The degree sequence of $J_n^*(x)$ is denoted $\mathbb{D}(J_n^*(x))$. Note that the values $irr_t(J_n^*(x))$ have been calculated manually, as it is not provided for in the Fisher algorithm.

Table 2

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i) = i - d^-(v_i)$	$\mathbb{D}(J_i^*(x))$	$irr_t(J_i^*(x))$
1	0	1	(0)	0
2	1	1	$(1,1)$	0
3	1	2	$(1,2,1)$	2
4	1	3	$(1,2,2,1)$	4
5	2	3	$(1,2,3,2,2)$	8
6	2	4	$(1,2,3,3,3,2)$	14
7	3	4	$(1,2,3,4,4,3,3)$	26
8	3	5	$(1,2,3,4,5,4,4,3)$	42
9	3	6	$(1,2,3,4,5,5,5,4,3)$	60
10	4	6	$(1,2,3,4,5,6,6,5,4,4)$	86
11	4	7	$(1,2,3,4,5,6,7,6,5,5,4)$	116
12	4	8	$(1,2,3,4,5,6,7,7,6,6,5,4)$	149

Note that the Fisher algorithm determines $d^+(v_i)$ on the assumption that the Jaco Graph is always sufficiently large, so at least $J_n(x)$, $n \geq i + d^+(v_i)$. For a smaller graph the degree of vertex v_i is given by $d(v_i) = d^-(v_i) + (n-i)$. In [19, 20], Bettina's theorem describes an arguably, closed formula to determine $d^+(v_i)$. Since $d^-(v_i) = n - d^+(v_i)$ it is then easy to determine $d(v_i)$ in a smaller graph $J_n(x)$, $n < i + d^+(v_i)$.

The next result presents a *partially* recursive formula to determine $irr_t(J_{n+1}^*(x))$ if $irr_t(J_n^*(x))$, $n \geq 1$ is known.

Theorem 2.22 ([17]). *Consider the Jaco Graph, $J_n^*(x)$, $n, x \in \mathbb{N}$ with $\Delta(J_n^*(x)) = k$ and $irr_t(J_n^*(x))$ known. Let $d(v_i)$, $d^*(v_i)$ denote the degree of vertex v_i in $J_n^*(x)$ and $J_{n+1}^*(x)$, respectively. Then for the graph $J_{n+1}^*(x)$ we have that:*

$$irr_t(J_{n+1}^*(x)) = irr_t(J_n^*(x)) + \sum_{i=1}^{\ell_1} i - \sum_{i=1}^{\ell_2} i + \sum_{i=1}^{n} |(n-k) - d^*(v_i)|,$$

with ℓ_1 the number of vertices v_i with $d(v_i) \leq d(v_{k+j})$, $j \in \{1, 2, \dots, n-k\}$, and ℓ_2 the number of vertices v_i with $d(v_i) > d(v_{k+j})$, $j \in \{1, 2, \dots, n-k\}$.

2.4 f_t-Irregularity of Jaco Graphs

Let $\mathbb{F} = \{f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, \dots, f_n = f_{n-1} + f_{n-2}, \dots\}$ be the set of Fibonacci numbers.

Allocate the *Fibonacci weight*, f_i to a vertex v_j of a simple undirected graph G, if and only if $d(v_j) = i$. Define the *total fibonaccian irregularity* as, $firr_t(G) = \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} |f_i - f_j|$ (see [17]). For a simple graph on a singular vertex (1-*null* graph), define $firr_t(G) = 0$. It follows that a graph is *f-regular* if it is a regular graph.

For illustration the adapted table below follows from the Fisher algorithm for $J_n(x)$, $n \le 12$. The f_i-sequence of $J_n^*(x)$ is denoted $\mathbb{F}(J_n^*(x))$. Note that the values $firr_t(J_n^*(x))$ have been calculated manually, as it is not provided for in the Fisher algorithm. Also note that in [17] the table depicting the $irr_t(J_i^*(x))$ values (see Table 2) was erroneously duplicated. Table 3 depicts the correct $firr_t(J_i^*(x))$ values.

Table 3

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i) = i - d^-(v_i)$	$\mathbb{F}(J_i^*(x))$	$firr_t(J_i^*(x))$
1	0	1	(0)	0
2	1	1	$(1,1)$	0
3	1	2	$(1,1,1)$	0
4	1	3	$(1,1,1,1)$	0
5	2	3	$(1,1,2,1,1)$	4
6	2	4	$(1,1,2,2,2,1)$	9
7	3	4	$(1,1,2,3,3,2,2)$	20
8	3	5	$(1,1,2,3,5,3,3,2)$	42
9	3	6	$(1,1,2,3,5,5,5,3,2)$	70
10	4	6	$(1,1,2,3,5,8,8,5,3,3)$	133
11	4	7	$(1,1,2,3,5,8,13,8,5,5,3)$	224
12	4	8	$(1,1,2,3,5,8,13,13,8,8,5,3)$	322

The next result presents a *partially* recursive formula to determine $firr_t(J_{n+1}^*(x))$ if $firr_t(J_n^*(x))$, $n \ge 1$ is known.

Theorem 2.23 ([17])**.** *Consider the Jaco Graph, $J_n^*(x)$, $n, x \in \mathbb{N}$ with $\Delta(J_n^*(x)) = k$ and $firr_t(J_n^*(x))$ known. Let $d(v_i)$, $d^*(v_i)$ denote the degree of vertex v_i in $J_n^*(x)$ and $J_{n+1}^*(x)$, respectively. Then for the graph $J_{n+1}^*(x)$ we have that:*

$$firr_t(J_{n+1}^*(x)) = firr_t(J_n^*(x)) + \sum_{i=1}^{n} |f_{n-k} - f_{d^*(v_i)}| + \sum_{i \in \{k+1,k+2,\dots,n\}} \ell_{(1,i)} |f_{d(v_i)+1} - f_{d(v_i)}|$$
$$- \sum_{i \in \{k+1,k+2,\dots,n\}} \ell_{(2,i)} |f_{d(v_i)+1} - f_{d(v_i)}| + \sum_{i=k+1}^{n-1} \sum_{j=i+1}^{n} \big||f_{d(v_i)} - f_{d(v_j)}| - |f_{d(v_i)+1} - f_{d(v_j)+1}|\big|,$$

with $\ell_{(1,i)}$ the number of vertices v_j, $j \in \{1,2,3,\dots,k\}$, with $d^*(v_i) > d(v_j)$, $i \in \{k+1,k+2,\dots,n\}$ and $\ell_{(2,i)}$ the number of vertices v_j, $j \in \{1,2,3,\dots,k\}$, with $d^*(v_i) \le d(v_j)$, $i \in \{k+1,k+2,\dots,n\}$.

2.5 $firr_t$ Resulting from Edge-joint between Jaco Graphs

Abdo and Dimitrov [3] observed that $irr_t(G \cup H) \geq irr(t(G) + irr_t(H))$. A result for $irr_t(J^*_n(x) \cup J^*_m(x))$ followed by a corollary in respect of $firr_t$ were presented in [17].

Theorem 2.24 ([17]). *For the Jaco Graphs $J^*_n(x)$ and $J^*_m(x)$, $n, m, x \in \mathbb{N}$ we have that:*

$$irr_t(J^*_n(x) \cup J^*_m(x)) \begin{cases} \leq 2(irr_t(J^*_n(x) + irr_t(J^*_m(x))) + \sum\limits_{i=\ell+1}^{n} \sum\limits_{j=n+(\ell+1)}^{m} |d(v_i) - d(v_j)|, & \text{if } n > m, \\ = 4(irr_t(J^*_n(x))), & \text{if } n = m, \end{cases}$$

with $\ell = \Delta J_m(x)$.

Corollary 2.25 ([17]). *For the Jaco Graphs $J^*_n(x)$ and $J^*_m(x)$, we have that:*

$$firr_t(J^*_n(x) \cup J^*_m(x)) \begin{cases} \leq 2(firr_t(J^*_n(x) + firr_t(J^*_m(x))) + \sum\limits_{i=\ell+1}^{n} \sum\limits_{j=n+(\ell+1)}^{m} |f_{d(v_i)} - f_{d(v_j)}|, & \text{if } n > m, \\ = 4(firr_t(J^*_n(x))), & \text{if } n = m. \end{cases}$$

Definition 2.8 ([17]). *The edge-joint of two simple undirected graphs G and H is the graph obtained by linking the edge $vu, v \in V(G), u \in V(H)$ and denoted $G \rightsquigarrow_{vu} H$.*

In [27] an abbreviation was prosed for families (classes) of graphs such as paths P_n, cycles C_n, complete graphs K_n, Jaco graphs $J_n(f(x))$, etc. The notation is abbreviated as $P_n \rightsquigarrow_{vu} P_m = P^{\rightsquigarrow vu}_{n,m}$ and $J^*_n(f(x)) \rightsquigarrow_{v_iu_j} J^*_m(f(x)) = J^{\rightsquigarrow v_iu_j}_{n,m}(f(x))$, etc.

Lemma 2.26 ([17]). *Consider the graphs $J^*_n(x)$ and $J^*_m(x)$ on the vertices $v_1, v_2, v_3, \dots, v_n$ and $u_1, u_2, u_3, \dots, u_m$, respectively, then $firr_t(J^*_n(x) \cup J^*_m(x)) = firr_t(J^{\rightsquigarrow v_1u_1}_{n,m}(x))$.*

Theorem 2.27 ([17]). *Consider the graphs $J^*_n(x)$, $n \geq 3$ and $J^*_m(x)$, $m \geq 1$ on the vertices $v_1, v_2, v_3, \dots, v_n$ and $u_1, u_2, u_3, \dots, u_m$, respectively. Without loss of generality choose any vertex v_i, $i \neq 1$ from $V(J^*_n(x))$. Let $V_1 = \{v_x : f_{d(v_x)} \leq f_{d(v_i)}\}$, $|V_1| = a$; $V_2 = \{v_y : f_{d(v_y)} > f_{d(v_i)}\}$, $|V_2| = b$; $V_3 = \{u_x : f_{d(u_x)} \leq f_{d(v_i)}\}$, $|V_3| = a^*$ and $V_4 = \{u_y : f_{d(u_y)} > f_{d(v_i)}\}$, $|V_4| = b^*$. For the simple connected graph $G' = J^{\rightsquigarrow v_iu_1}_{n,m}(x)$ we have that:*

$$\begin{aligned} firr_t(G') &= firr_t(J^*_n(x)) + firr_t(J^*_m(x)) + \sum_{j=1}^{n} \sum_{k=1}^{m} |f_{d(v_j)} - f_{d(u_k)}|_{v_j \in V(J^*_n(x)), u_k \in V(J^*_m(x))} \\ &+ \sum_{j=1}^{a} |f_{d(v_i)} - f_{d(v_j)}|_{v_j \in V_1} - \sum_{j=1}^{b} |f_{d(v_i)} - f_{d(v_j)}|_{v_j \in V_2} \\ &+ \sum_{j=1}^{a^*} |f_{d(v_i)} - f_{d(v_j)}|_{v_j \in V_3} - \sum_{j=1}^{b^*} |f_{d(v_i)} - f_{d(v_j)}|_{v_j \in V_4}. \end{aligned}$$

2.6 $\pm$Fibonacci Weights, $f^{\pm}$-Zagreb Indices of Jaco Graphs

The topological graph indices $irr(G)$ related to the *first Zagreb index*, $M_1(G) = \sum\limits_{v\in V(G)} d^2(v) = \sum\limits_{vu\in E(G)} (d(v)+d(u))$, and the *second Zagreb index*, $M_2(G) = \sum\limits_{vu\in E(G)} d(v)d(u)$ are of the oldest irregularity measures researched. Alberton [5] introduced the *irregularity* of G as $irr(G) = \sum\limits_{e\in E(G)} imb(e), imb(e) = |d(v)-d(u)|_{e=vu}$. In the paper of Fath-Tabar [12], Alberton's indice was named the *third Zagreb indice* to conform with the terminology of chemical graph theory. Recently Ado et al. [2] introduced the topological indice called *total irregularity* and defined it, $irr_t(G) = \frac{1}{2}\sum\limits_{u,v\in V(G)} |d(u)-d(v)|$. The latter could be called the *fourth Zagreb indice*.

If the vertices of a simple undirected graph G on n vertices are labeled v_i, $i = 1,2,3,\ldots,n$ then the respective definitions may be:

$$M_1(G) = \sum_{i=1}^{n} d^2(v_i) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(d(v_i)+d(v_j))_{v_iu_j\in E(G)},$$

$$M_2(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} d(v_i)d(v_j)_{v_iu_j\in E(G)},$$

$$M_3(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n} |d(v_i)-d(v_j)|_{v_iu_j\in E(G)}, \quad \text{and}$$

$$M_4(G) = irr_t(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|d(v_i)-d(v_j)| = \sum_{i=1}^{n}\sum_{j=i+1}^{n}|d(v_i)-d(v_j)| \ \text{ or } \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|d(v_i)-d(v_j)|.$$

For a simple graph on a singular vertex (1-*empty* graph), the default values $M_1(G) = M_2(G) = M_3(G) = M_4(G) = 0$ apply.

In [23] a new derivative of the Zagreb indices were introduced. The $\pm$*Fibonacci weight*, $f_i^{\pm}$ of a vertex v_i was defined to be $-f_{d(v_i)}$, if $d(v_i) = i$ is odd and, $f_{d(v_i)}$, if $d(v_i)$ is even. The $f^{\pm}$-Zagreb indices was then defined as:

$$f^{\pm}Z_1(G) = \sum_{i=1}^{n}(f_i^{\pm})^2 = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(|f_i^{\pm}|+|f_j^{\pm}|)_{v_iu_j\in E(G)},$$

$$f^{\pm}Z_2(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(f_i^{\pm}\cdot f_j^{\pm})_{v_iu_j\in E(G)},$$

$$f^{\pm}Z_3(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}|f_i^{\pm}-f_j^{\pm}|_{v_iu_j\in E(G)}, \quad \text{and}$$

$$f^{\pm}Z_4(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|f_i^{\pm}-f_j^{\pm}| = \sum_{i=1}^{n}\sum_{j=i+1}^{n}|f_i^{\pm}-f_j^{\pm}| \ \text{ or } \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|f_i^{\pm}-f_j^{\pm}|.$$

For a simple graph on a singular vertex (1-*empty* graph), define

$$f^{\pm}Z_1(G) = f^{\pm}Z_2(G) = f^{\pm}Z_3(G) = f^{\pm}Z_4(G) = 0.$$

Note. The degree of vertex v_i, denoted $d(v_i)$ refers to the degree in $J^*_\infty(x)$ hence $d(v_i) = i$. In the finite Jaco Graph the degree of vertex v_i is denoted $d(v_i)_{J^*_n(x)}$. The degree sequence is denoted $\mathbb{D}_n = (d(v_1)_{J^*_n(x)}, d(v_2)_{J^*_n(x)}, \dots, d(v_n)_{J^*_n(x)})$. By convention $\mathbb{D}_{i+1} = \mathbb{D}_i \cup d(v_{i+1})_{J^*_n(x)}$.

2.6.1 Algorithm to determine the degree sequence of a finite Jaco Graph, $J^*_n(x)$, $n, x \in \mathbb{N}$ [23]

Consider a finite Jaco Graph $J_n(x), n \in \mathbb{N}$ and label the vertices $v_1, v_2, v_3, \dots, v_n$.

Step 0: Set $n = n$. Let $i = j = 1$. If $j = n = 1$, let $\mathbb{D}_i = (0)$ and go to Step 6, else set $\mathbb{D}_i = \emptyset$ and go to Step 1.

Step 1: Determine the j^{th} Zeckendorf representation say, $j = f_{i_1} + f_{i_2} + \cdots + f_{i_r}$, and go to Step 2.

Step 2: Calculate $d^+(v_j) = f_{i_1-1} + f_{i_2-1} + \cdots + f_{i_r-1}$, then go to Step 3.

Step 3: Calculate $d^-(v_j) = j - d^+(v_j)$, and let $d(v_j) = d^+(v_j) + d^-(v_j)$, then go to Step 4.

Step 4: If $d(v_j) \leq n$, set $d(v_j)_{J^*_n(x)} = d(v_j)$ else, set $d(v_j)_{J^*_n(1)} = d^-(v_j) + (n - j)$ and set $\mathbb{D}_j = \mathbb{D}_i \cup d(v_j)_{J^*_n(x)}$ and go to Step 5.

Step 5: If $j = n$ go to Step 6 else, set $i = i + 1$ and $j = i$ and go to Step 1.

Step 6: Exit.

2.6.2 Tabled values of $\mathbb{F}^\pm(J_n(x))$, for finite Jaco Graphs, $J^*_n(x)$, $n \leq 12$ [23]

For illustration the adapted table below follows from the Fisher algorithm [19] for $J_n(x)$, $n \leq 12$. The $f^\pm_i$-sequence of $J^*_n(x)$ is denoted $\mathbb{F}^\pm(J^*_n(x))$.

Table 4

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i) = i - d^-(v_n)$	$\mathbb{F}^\pm(J^*_i(x))$
1	0	1	(0)
2	1	1	$(-1,-1)$
3	1	2	$(-1,1,-1)$
4	1	3	$(-1,1,1,-1)$
5	2	3	$(-1,1,-2,1,1)$
6	2	4	$(-1,1,-2,-2,-2,1)$
7	3	4	$(-1,1,-2,3,3,-2,-2)$
8	3	5	$(-1,1,-2,3,-5,3,3,-2)$
9	3	6	$(-1,1,-2,3,-5,-5,-5,3,-2)$
10	4	6	$(-1,1,-2,3,-5,8,8,-5,3,3)$
11	4	7	$(-1,1,-2,3,-5,8,-13,8,-5,-5,3)$
12	4	8	$(-1,1,-2,3,-5,8,-13,-13,8,8,-5,3)$

Since it is known that a sequence $(d_1, d_2, d_3, \dots, d_n)$ of non-negative integers is a degree sequence of some graph G *if and only if* $\sum\limits_{i+i}^{n} d_i$ is even. It implies that a degree sequence has an even number of *odd entries*. Hence, the $f_i^{\pm}$-sequence of $J_n^*(x)$ denoted, $\mathbb{F}^{\pm}(J_n^*(x))$, $n \in \mathbb{N}$ has an *even number* of, $-f_{d(v_i)}$ entries. Following from Table 4, the table below depicts the values $f^{\pm}Z_1(J_n^*(x))$, $f^{\pm}Z_2(J_n^*(x))$, $f^{\pm}Z_3(J_n^*(x))$ and $f^{\pm}Z_4(J_n^*(x))$ for $J_n^*(x)$, $n \leq 12$.

Table 5

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i)$	$f^{\pm}Z_1(J_i^*(x))$	$f^{\pm}Z_2(J_i^*(x))$	$f^{\pm}Z_3(J_i^*(x))$	$f^{\pm}Z_4(J_i^*(x))$
1	0	1	0	0	0	0
2	1	1	2	1	0	0
3	1	2	3	−2	4	4
4	1	3	4	−1	4	8
5	2	3	8	−6	11	16
6	2	4	15	5	11	25
7	3	4	32	−26	35	56
8	3	5	62	−19	50	98
9	3	6	103	0	72	138
10	4	6	211	38	119	251
11	4	7	396	−238	210	402
12	4	8	604	−158	273	566

2.7 Number of Arcs of Jaco Graphs

In earlier work [19, 20] it was accepted that finding a closed formula for the number of arcs of a finite Jaco Graph $J_n(x)$ will remain a challenging open problem. The algorithms discussed in Ahlbach et al. [1] suggest, finding such formula might not be possible. Three easy to apply alternative, *formula* to determine the number of arcs (edges) were reported. The next lemma is needed to adapt the Fisher algorithm accordingly.

Lemma 2.28 ([19, 20])**.** $\epsilon(J_n(x)) = \epsilon(J_{n-1}(x)) + d^-(v_n)$.

Adapted Fisher algorithm to table number of arcs

Note that rows 1, 2 and 3 follow easily from the definition of a Jaco graph.

Step 0: Set $j = 4$, then set $i = j$ and $s \geq 4$.

Step 1: Set $ent_{1i} = i$.

Step 2: Set $ent_{2i} = ent_{1(i-1)} - ent_{4(i-1)}$. (Note that $d^-(v_i) = v(\mathbb{H}_{i-1}(x)) = (i-1) - \Delta(J_{i-1}(x))$).

Step 3: Set $ent_{3i} = ent_{1i} - ent_{2i}$. (Note that $d^+(v_i) = i - d^-(v_i)$).

Step 4: Set $ent_{5i} = ent_{5(i-1)} + ent_{2i}$. (Lemma 2.28)

Step 5: Set $j = i + 1$, then set $i = j$. If $i \leq s$, go to Step 1, else go to Step 6.

Step 6: Exit.

2.27.1. First Recursive Formula. Note that Lemma 2.28 provides the *first* recursive formula to determine the number of arcs of $J_n(x)$. Using the adapted Fisher algorithm together with Lemma 2.28 the table below follows easily.

Table 6

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i) = \nu(\mathbb{H}_{i-1})$	$d^+(v_i) = i - d^-(v_i)$	$\epsilon(J_i(x)) = \epsilon(J_{i-1}(x)) + d^-(v_i)$
$1 = f_2$	0	1	0
$2 = f_3$	1	1	1
$3 = f_4$	1	2	2
4	1	3	3
$5 = f_5$	2	3	5
6	2	4	7
7	3	4	10
$8 = f_6$	3	5	13
9	3	6	16
10	4	6	20
11	4	7	24
12	4	8	28
$13 = f_7$	5	8	33
14	5	9	38
15	6	9	44
16	6	10	50
17	6	11	56
18	7	11	63
19	7	12	70
20	8	12	78
$21 = f_8$	8	13	86
22	8	14	94
23	9	14	103
24	9	15	112
25	9	16	121

2.27.2. Second Formula. It is well known that $\epsilon(J_n(x)) = \sum\limits_{i=1}^{n} d^-(v_i)$. Since $d^-(v_n) = n - d^+(v_n)$, the number of arcs is also given by $\epsilon(J_n(x)) = \frac{1}{2}n(n-1) - \sum\limits_{i=1}^{n} d^+(v_i)$. Furthermore, for $n \geq 2$ we have $d^+(v_1) = 1$, so rather consider $\epsilon(J_n(x)) = (\frac{1}{2}(n(n-1) - 1) - \sum\limits_{i=2}^{n} d^+(v_i)$.

Bettina's Theorem [19] provides for a method to determine $d^+(v_i)$, $\forall\ i \in \mathbb{N}$.

Example 2.1. For the Jaco Graph $J_{15}(x)$ it follows that:

$$\epsilon(J_{15}(x)) = \frac{1}{2}\cdot 15\cdot(15+1) - 1 - \sum_{i=2}^{15} = 119 - \sum_{i=2}^{15} d^+(v_i).$$

Now, $1 = f_2$, $2 = f_3$, $3 = f_4$, $4 = f_4 + f_2$, $5 = f_5$, $6 = f_5 + f_2$, $7 = f_5 + f_3$, $8 = f_6$, $9 = f_6 + f_2$, $10 = f_6 + f_3$, $11 = f_6 + f_4$, $12 = f_6 + f_4 + f_2$, $13 = f_7$, $14 = f_7 + f_2$, and $15 = f_7 + f_3$.

From Bettina's theorem it follows that:

$$\begin{aligned}\sum_{i=2}^{15} d^+(v_i) &= f_2 + f_3 + (f_3 + f_1) + f_4 + (f_4 + f_1) + (f_4 + f_2) + f_5 + (f_5 + f_1)\\ &\quad + (f_5 + f_2) + (f_5 + f_3) + (f_5 + f_3 + f_1) + f_6 + (f_6 + f_1) + (f_6 + f_2)\\ &= 5f_1 + 4f_2 + 4f_3 + 3f_4 + 5f_5 + 3f_6 = f_1 + 5f_3 + 5f_5 + 3f_7\\ &= 75.\end{aligned}$$

So, $\epsilon(J_{15}(x)) = 119 - 75 = 44$.

2.27.3. Third Formula. The third formula follows from Proposition 2.29 and 2.30.

Proposition 2.29. *If for the finite Jaco Graph $J_n(x)$, $n \in \mathbb{N}$, n can be expressed as $n = m_{=d^+(v_n)} + d^+(v_m)$ then:*

$$\epsilon(J_n(x)) = \frac{1}{2}\left(\sum_{i=1}^{m} i + \sum_{i=0}^{j_{\max}} (d^+(v_{m-i}) - i)_{d^+(v_{m-j_{\max}})-j_{\max}\geq 1} + d^+(v_m)(d^+(v_m)-1)\right).$$

Example 2.2. Determine $\epsilon(J_{31}(x))$. Now, $31 = 19 + d^+(v_{19}) = 19 + 12$. So it follows that:

$$\begin{aligned}\epsilon(J_{31}(x)) &= \frac{1}{2}\left(\sum_{i=1}^{19} i + \sum_{i=0}^{j_{\max}} (d^+(v_{19-i}) - i) + d^+(v_{19})(d^+(v_{19})-1)\right)\\ &= \frac{1}{2}\Big(\frac{1}{2}(20.19) + [(12-0) + (11-1) + (11-2) + (10-3) + (9-4) + (9-5)\\ &\qquad + (8-6) + (8-7)]_{=50} + [12.11]_{=132}\Big)\\ &= \frac{1}{2}(190 + 50 + 132) = 186.\end{aligned}$$

Note that not all $n \in \mathbb{N}$ can uniquely be written as $n = m + d^+(v_m)$ for $m \in \mathbb{N}$. From Corollary 3.5 in [20] the next proposition follows.

Proposition 2.30. *If, for the Jaco Graph $J_{n-1}(x)$ the integer $(n-1)$ cannot be expressed as $n-1 = m_{=d^+(v_{n-1})} + d^+(v_m)$, then $n = m_{=d^+(v_{n-1})} + d^+(v_m) = m_{=d^+(v_n)} + d^+(v_m)$, and $\epsilon(J_{n-1}(x)) = \epsilon(J_n(x)) - d^+(v_m)$.*

Example 2.3. Determine $\epsilon(J_{17}(x))$. Note that $d^+(v_{17})+d^+(v_{d^+(v_{17})})=18\neq 17$. However, $d^+(v_{18})+d^+(v_{d^+(v_{18})})=18$, so $\epsilon(J_{17}(x))=\epsilon(J_{18}(x))-d^+(v_{11})$. Hence $\epsilon(J_{17}(x))=63-7=56$.

Recently a closed formula for $d^-(v_n)$, $n\geq 1$ and by implication an explicit formula for the number of arcs (edges) of a Jaco graph $J_n(x)$), $(J_n^*(x))$) were identified by Stephan Wagner[6] via `www.oeis.org`, *Sequence A183137*. The results are: $d^-(v_n)=\left\lfloor\frac{4(n+1)}{(1+\sqrt{5})^2}\right\rfloor=\left\lfloor\frac{2(n+1)}{3+\sqrt{5}}\right\rfloor$ and $\epsilon(J_n^*(x))=\sum\limits_{i=1}^{n}\left\lfloor\frac{4(i+1)}{(1+\sqrt{5})^2}\right\rfloor=\sum\limits_{i=1}^{n}\left\lfloor\frac{2(i+1)}{3+\sqrt{5}}\right\rfloor$. A good approximation dependent on n only is given by $\epsilon(J_n^*(x))=\frac{3-\sqrt{5}}{4(n+1)^2}+\frac{1-\sqrt{5}}{4(n+1)}$. These findings will assist to bring a number of recursive formula to closure. An example of such closure is that Bettina's theorem (Theorem 2.20) can now be expressed as: $d^+(v_n)=n-\left\lfloor\frac{4(n+1)}{(1+\sqrt{5})^2}\right\rfloor$.

2.8 On Certain Invariants of Jaco Graphs

Let $\mu(G)$ be an arbitrary invariant of the simple connected undirected graph G. The *μ-stability number* of G is conventionally, the minimum number of vertices, of which their removal changes $\mu(G)$. If the removal of the minimum vertices results in a decrease of the invariant the result is conventionally denoted, $\mu^-(G)$ and if the change is to the contrary the change is denoted $\mu^+(G)$. It is known that the *domination number*, $\gamma(G')$, of a subgraph G' of G can be larger or smaller than $\gamma(G)$. Note that a subgraph may result from the removal of vertices and/or edges from G. Furthermore, the removal of edges only from the graph G to obtain G' can only result in $\gamma(G')\geq\gamma(G)$. The minimum number of edges, of which their removal from G results in a graph G' with $\gamma(G')>\gamma(G)$, is called the *bondage number* $b(G)$, of G. From the definition of a Jaco Graph it follows that all Jaco Graphs on $n\geq 2$ has at least one leaf (vertex with degree = 1). Hence, the *bondage number* is $b(J_n^*(x))_{n\geq 2}=1$.

2.8.1 Independence number

Consider the underlying graph, $J_n^*(x)$, $n,x\in\mathbb{N}$. Obviously the graph has vertices $v_1,v_2,v_3,\dots,v_n$. Because the independence number is defined to be the number of vertices in a maximum independent set [8], it is optimal to choose non-adjacent vertices recursively, each of minimum subscript. This observation leads to the next theorem. Observe that $v_{i,j}=v_i$ as calculated on the *j-th step* of the recursive technique applied in the proof of the next result (see [24]).

Theorem 2.31 ([24]). *The cardinality of the set* $\mathbb{I}=\{v_{i,j}: v_1=v_{1,1}\in\mathbb{I}$ *and* $v_i=v_{i,j}=v_{(d^+(v_{m,(j-1)})+m+1)}\}$*, derived from the Jaco Graph* $J_n^*(x)$*,* $n\in\mathbb{N}$ *is equal to the independence number,* $\alpha(J_n^*(x))$.

Corollary 2.32 ([24]). *It follows that the* covering number*,* $\beta(J_n^*(x))=n-\alpha(J_n^*(x))$.

2.8.2 Chromatic number and b-Chromatic number

From the definitions provided in [19] the Hope Graph of the Jaco Graph, $J_n(x)$ is the complete graph on the vertices $v_{i+1},v_{i+2},\dots,v_n$ if and only if v_i is the prime Jaconian vertex of $J_n(x)$.

[6]Department of Mathematical Sciences, Stellenbosch University, Republic of South Africa

Hence, $\mathbb{H}_n^*(x) \simeq K_{n-i}$. The reader is reminded that a *t-colouring* of a graph G is a map $\lambda : V(G) \to [c] := \{1,2,3,\dots,c, c \geq 0\}$ such that $\lambda(u) \neq \lambda(v)$ whenever $u,v \in V(G)$ are adjacent in G. The *chromatic number* of G denoted $\chi(G)$ is the minimum c such that G is *c-colourable*. The following theorem was settled in [24].

Theorem 2.33 ([24])**.** *For the Jaco Graph, $J_n(x)$, $n \in \mathbb{N}$ with the prime Jaconian vertex v_i we have that the chromatic number, $\chi(J_n^*(x))$ is given by:*

$$\chi(J_n^*(x)) = \begin{cases} (n-i)+1, & \text{if and only if the edge } v_iv_n \text{ exists,} \\ n-i, & \text{otherwise.} \end{cases}$$

Recall that the b-chromatic number of a graph is the maximum number k of colours that can be used to colour the vertices of G, such that we obtain a proper colouring and each colour i, with $1 \leq i \leq k$, has at least one representant x_i adjacent to a vertex of every colour j, $1 \leq j \neq i \leq k$. The studies on b-chromatic number attracted much interest since its introduction. The next result holds for Jaco graphs [31].

Theorem 2.34 ([31])**.** *For $n \geq 2$, the b-chromatic number of a linear Jaco graph $J_n^*(f(x))$, $n,x \in \mathbb{N}$ with prime Jaconian vertex v_i is given by $\varphi(J_{n\geq 2}^*(f(x))) = (n-i)+1$.*

2.8.3 Murtage number

Note that if vertices u and v are not adjacent in G, then $\gamma(G+uv) \leq \gamma(G)$. The significance of this concept becomes apparent in the application of domination theory. In a situation where a γ-set of a graph is to represent costly facilities in a network N, it may be preferable to establish additional links (edges) between vertices of N rather than constructing facilities at all vertices of a γ-set. The aforesaid application motivated the notion of the *murtage number*[7] of a graph.

Definition 2.9 ([24])**.** The murtage number, $m(G)$, of a simple connected graph G is the minimum number of edges that has to be added to G such that the resulting graph G' has $\gamma(G') < \gamma(G)$.

It follows from the definition that $m(G) = 0$ if and only if $\gamma(G) = 1$. In order to calculate the murtage number of a graph the concept of a d_{om}-*sequence* of a γ-set, X_i of a graph was introduced. Label the vertices of X_i such that $V(G)$ can be partitioned into sets $D_{1,i}, D_{2,i}, \dots, D_{\gamma(G),i}$ such that $D_{j,i}$ contains the vertex $v_j \in X_i$ and vertices in $V(G) - X_i$ which are adjacent to v_j and such that, $|D_{1,i}| \leq |D_{2,i}| \leq \cdots, \leq |D_{\gamma(G),i}|$ and $|D_{1,i}|$ is a minimum. Define a d_{om}-*sequence* of the γ-set X_i as $(|D_{1,i}|, |D_{2,i}|, \dots, |D_{\gamma(G),i}|)$. Clearly a γ-set can have more than one d_{om}-*sequence*. Assume G has k γ-sets namely $X_1, X_2, \dots, X_k$. Let $\theta = absolute(\min|D_{1,j}|)$ for some X_j. All γ-sets, X_ℓ for which firstly, $|D_{1,\ell}| = \theta$ (*primary condition*) and secondly, $d(v_1, v_i)$ is minimum for all $v_i \in X_\ell$ (*secondary condition*) is said to be *compact* γ-sets. The partitioning described above in respect of a compact γ-set is called a *murtage partition* of $V(G)$.

[7] In the derivative named after U.S.R. Murty, the co-author of [8].

Theorem 2.35 ([24])**.** *Let $|D_{1,i}| = \theta$ for some compact γ-set X_i of G, then:*

$$m(G) = \begin{cases} \theta, & \text{if and only if } v_1 \text{ is not adjacent to any } v_j \in X_i, \\ \theta - 1, & \text{if and only if } v_1 \text{ is adjacent to some } v_j \in X_i. \end{cases}$$

The fact that $m(J_n(x))_{n\in\mathbb{N}} \geq 0$ follows from the definition.

From the definition of a Jaco Graph it follows easily that vertex v_1 dominates $J_1^*(x)$ and $J_2^*(x)$ and vertex v_2 dominates $J_3^*(x)$ hence, $m(J_1^*(x)) = m(J_2^*(x)) = m(J_3^*(x)) = 0$.

For $J_4^*(x)$ and $J_5^*(x)$ it follows that the set $\{v_1, v_3\}$ is a compact γ-set with the d_{om}*-sequences*, $(1,2)$ and $(1,3)$ hence, $m(J_4^*(x)) = m(J_5^*(x)) = 1$.

For the Jaco Graphs $J_6^*(x)$ and $J_7^*(x)$ the sets $\{v_1, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_4\}$, $\{v_2, v_5\}$, $\{v_2, v_6\}$, $\{v_2, v_7\}$ are γ-sets with only $\{v_2, v_4\}$ and $\{v_2, v_5\}$ the compact γ-sets. The corresponding d_{om}*-sequences* are $(2,4)$ and $(2,5)$ hence, $m(J_6^*(x)) = m(J_7^*(x)) = 2$. For $J_8^*(x)$ the sets $\{v_2, v_5\}$, $\{v_2, v_6\}$, $\{v_2, v_7\}$ are γ-sets with $\{v_2, v_5\}$ the unique compact γ-set. The unique corresponding d_{om}*-sequence* is $(2,6)$ so, $m(J_8^*(x)) = 2$.

In respect of $J_9^*(x)$ and $J_{10}^*(x)$ an interesting observation is that exactly two γ-sets, both being compact γ-sets namely, $\{v_2, v_6\}$ and $\{v_2, v_7\}$, exist. The corresponding d_{om}*-sequences* are $(3,6)$ and $(3,7)$ respectively, meaning, $m(J_9^*(x)) = m(J_{10}^*(x)) = 3$.

In the case of $J_{11}^*(x)$ an unique compact γ-set $= \{v_2, v_7\}$ exists with the d_{om}-sequence $(3,8)$. So in this case $m(J_{11}^*(x)) = 3$.

For $J_{12}^*(x)$ and $J_{13}^*(x)$ we note that the sets $\{v_1, v_3, v_8\}$, $\{v_1, v_3, v_9\}$ and $\{v_1, v_3, v_{10}\}$ are the γ-sets with $\{v_1, v_3, v_8\}$ the unique compact γ-set. The corresponding d_{om}-sequences are $(1,3,8)$ and $(1,3,9)$. Hence, $m(J_{12}^*(x)) = m(J_{13}^*(x)) = 1$. Further exploratory analysis led to the next theorem.

Theorem 2.36 ([24])**.** *For any Jaco Graph $J_n^*(x)$, $n, x \in \mathbb{N}$ we have $0 \leq m(J_n^*(x)) \leq 3$. The bounds are obviously sharp as well.*

Corollary 2.37 ([24])**.** *For any finite Jaco Graph $J_n^*(x)$, $n, x \in \mathbb{N}$:*

$$\gamma(J_n(1)) = \gamma(J_\kappa^*(x)) + 1, \quad \kappa = (n - d^-(v_n) - d^-(v_{(n-d^-(v_n))}) - 1).$$

2.9 Brush Numbers of Jaco Graphs

The concept of the brush number $b_r(G)$ of a simple connected graph G was introduced by McKeil [34] and Messinger et al. [36]. The problem is initially set that all edges of a simple connected undirected graph G is *dirty*. A finite number of brushes, $\beta_G(v) \geq 0$ is allocated to each vertex $v \in V(G)$. Sequentially any vertex which has $\beta_G(v) \geq d(v)$ brushes allocated may clean the vertex and send exactly one brush along a dirty edge and in doing so allocate an additional brush to the corresponding adjacent vertex (neighbour). The reduced graph $G' = G - vu$, $\forall\ vu \in A(G)$, $\beta_G(v) \geq d(v)$ is considered for the next iterative cleansing step. Note that a neighbour of vertex v in G say vertex u, now have $\beta_{G'}(u) = \beta_G(u) + 1$.

Clearly for any simple connected undirected graph G the first step of cleaning can begin if and only if at least one vertex v is allocated, $\beta_G(v) = d(v)$ brushes. The minimum number of brushes that is required to allow the first step of cleaning to begin is, $\beta_G(u) = d(u) = \delta(G)$. Note that these conditions do not guarantee that the graph will be cleaned. The conditions merely assure at least the first step of cleaning.

If a simple connected graph G is orientated to become a directed graph, brushes may only clean along an out-arc from a vertex. Cleaning may initiate from a vertex v if and only if $\beta_G(v) \geq d^+(v)$ and $d^-(v) = 0$. The order in which vertices sequentially initiate cleaning is called the *cleaning sequence* in respect of the orientation α_i. The minimum number of brushes to be allocated to clean a graph for a given orientation $\alpha_i(G)$ is denoted $b_r^{\alpha_i}$. If an orientation α_i renders cleaning of the graph undoable we define $b_r^{\alpha_i} = \infty$. An orientation α_i for which $b_r^{\alpha_i}$ is a minimum over all possible orientations is called *optimal*.

Now, since the graph G having $\epsilon(G)$ edges can have $2^{\epsilon(G)}$ orientations, the *optimal orientation* is not necessary unique. Let the set $\mathbb{A} = \{\alpha_i | \alpha_i \textit{ an orientation of } G\}$.

It is important to note that the definition of a Jaco Graph $J_n(x)$, prescribes a well-defined orientation of the underlying Jaco graph. So there is one specific defined orientation of the $2^{\epsilon(J_n(x))}$ possible orientations.

Theorem 2.38 ([18]). *For the finite Jaco Graph $J_n(x)$, $n, x \in \mathbb{N}$, with prime Jaconian vertex v_i it holds that:*

$$b_r(J_n(x)) = \sum_{j=1}^{i}(d^+(v_j) - d^-(v_j)) + \sum_{j=i+1}^{n} \max\{0, (n-j) - d^-(v_j)\}.$$

For illustration the adapted table below follows from the Fisher algorithm for $J_n(x)$, $n \in \mathbb{N}$, $n \leq 16$. Note that v_j^* is the prime Jaconian vertex.

Table 7

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i)$	v_j^*	$b_r(J_i(x))$
1	0	1	v_1	0
2	1	1	v_1	1
3	1	2	v_2	1
4	1	3	v_2	1
5	2	3	v_3	2
6	2	4	v_3	3
7	3	4	v_4	4
8	3	5	v_5	5

$i \in \mathbb{N}$	$d^-(v_i)$	$d^+(v_i)$	v_j^*	$b_r(J_i(x))$
9	3	6	v_5	6
10	4	6	v_6	7
11	4	7	v_7	8
12	4	8	v_7	9
13	5	8	v_8	11
14	5	9	v_8	12
15	6	9	v_9	14
16	6	10	v_{10}	16

From [41] we recall a useful lemma.

Lemma 2.39 ([18]). *For a simple connected directed graph G, we have:*

$$b_r(G) = \min\left\{\sum_{v\in V(G)} \max\{0, d^+(v) - d^-(v)\} : \text{over all } \alpha_i(G) \in \mathbb{A}\right\} = \min\{b_r^{\alpha_i}(G) : \forall\ \alpha_i(G)\}.$$

From Theorem 2.38 and Lemma 2.39 the brush allocations can easily be determined. For example $J_9(x)$ requires the minimum brush allocations, $\beta_{J_9(x)}(v_1)=1$, $\beta_{J_9(x)}(v_2)=0$, $\beta_{J_9(x)}(v_3)=1$, $\beta_{J_9(x)}(v_4)=2$, $\beta_{J_9(x)}(v_5)=1$, $\beta_{J_9(x)}(v_6)=1$, $\beta_{J_9(x)}(v_7)=0$, $\beta_{J_9(x)}(v_8)=0$, $\beta_{J_9(x)}(v_9)=0$. Note that the allocations of $\beta(v_i) > 0$ are located at vertices v_1, v_3, v_4, v_5, v_6. The *end* allocation itself is a minimum allocation associated with an optimal orientation.

2.9.1 Brush Numbers of Mycielski Jaco Graphs

Mycielski [38] introduced an interesting graph transformation in 1955. The transformation can be described as follows:

(1) Consider any simple connected graph G on $n \geq 2$ vertices labelled $v_1, v_2, v_3, \dots, v_n$.

(2) Consider the extended vertex set $V(G) \cup \{x_1, x_2, x_3, \dots, x_n\}$ and add the edges $\{v_i x_j, v_j x_i|$ iff $v_i v_j \in E(G)\}$.

(3) Add one more vertex w together with the edges $\{w x_i |\ \forall\ i\}$.

The transformed graph (*Mycielskian graph of G or Mycielski G*) denoted $\mu(G)$, is the simple connected graph with $V(\mu(G)) = V(G) \cup \{x_1, x_2, x_3, \dots, x_n\} \cup \{w\}$ and $E(\mu(G)) = E(G) \cup \{v_i x_j, v_j x_i :$ iff $v_i v_j \in E(G)\} \cup \{w x_i :\ \forall\ i\}$.

In general, if $\beta_{G'}(v)$ at a particular cleaning step has $\beta_{G'}(v) > d_{G'}(v)$, exactly $\beta_{G'}(v) - d_{G'}(v)$ brushes are left redundant and can clean along new edges linked to vertex v if such are added through transformation of the graph G. The latter observation allows for an adaption of Theorem 2.38 to obtain the *brush number* of $\mu(J_n(x)) n \geq 3$. Note that $\mu(J_1(x)) \simeq K_1 \cup P_2$, hence a disconnected graph. Easy to see that $\mu(J_2(x)) \simeq C_5$ hence $b_r(\mu(J_2(x)) = 2$.

Theorem 2.40 ([32]). *For the Jaco graph $J_n(x)$, $n, x \in \mathbb{N}$, $n \geq 2$ the brush number of the Mycielski Jaco graph is given by:*

$$b_r(\mu(J_n(x)) = 2\sum_{i=1}^{n} d^+_{J_n(x)}(v_i).$$

2.10 Brush Centre of Jaco Graphs

So far cleaning was restricted to a brush transversing a dirty edge only once. If the latter restriction is relaxed to, after the first complete cleaning sequence a brush may transverse an edge for a second time for another complete reversed cleaning sequence, the initial allocation of brushes or a deviation thereof can be obtained. This observation leads to the concept of a *brush centre*. The question is, what is the minimum set of vertices, $\mathbb{B}_r(G) \subseteq V(G)$ (*primary condition*) to allocate the $b_r(G)$ brushes to, to ensure cleaning of graph G and that on return *(second cleaning)* the brushes are clustered as centrally as possible for *maintenance* (*secondary* *condition is the min(max(distance between vertices of the brush centre))*). Finding a *brush centre* of a graph will allow for well located maintenance centres of the brushes prior to a next cycle of cleaning. Because brushes themselves may be technology of kind, the technology in real world application will normally be the subject of maintenance or calibration or virus vetting or alike.

In the defined Jaco graph $J_5(x)$ the brush number is $b_r(J_5(x)) = 2$ ([20, 34]), with the brush allocation $\beta_{J_5(x)}(v_1) = 1$, $\beta_{J_5(x)}(v_2) = 0$, $\beta_{J_5(x)}(v_3) = 1$, $\beta_{J_5(x)}(v_4) = 0$, $\beta_{J_5(x)}(v_5) = 0$. Note that after the first cleaning sequence both brushes are allocated to the vertex v_5. The latter allocation of brushes with an appropriate re-orientation of $J_5(x)$ also clean the Jaco graph. On a second cleaning sequence the brushes can *park* at v_5 for *maintenance*. Clearly the set $\{v_5\}$ with $\beta_{J_5(x)}(v_5) = 2$ is a (*the*) brush centre.

Theorem 2.41 ([32]). *Consider the initial minimal brush allocation of $b_r(J_n(x))$ brushes to the finite Jaco graph, $J_n(x)$, $n,x \in \mathbb{N}$. The location of the brushes at the end of the cleaning sequence represents a brush centre of $J_n(x)$, $n,x \in \mathbb{N}$.*

2.11 Competition Graph of Jaco Graphs

The concept of the competition graph $C(G^{\rightarrow})$ of a simple connected directed graph $G^{\rightarrow}$ on $n \geq 2$ vertices, was introduced by Joel Cohen in 1968 [9]. Much research has followed and recommended reading can be found in ([34, 38, 39] *together with all their references*). The concept of competition graphs found application in amongst others, Coding theory, Channel allocation in communication, Information transmission, Complex systems modelled in energy and economic applications, Decision-making based mainly on opinion influences and Predator-Prey dynamical systems.

For a simple connected directed graph $G^{\rightarrow}$ with vertex set $V(G^{\rightarrow})$ the competition graph $C(G^{\rightarrow})$ is the simple graph *(undirected and possibly disconnected)* having $V(C(G^{\rightarrow})) = V(G^{\rightarrow})$ and the edges $E(C(G^{\rightarrow})) = \{vy$: *if at least one vertex $w \in V(G^{\rightarrow})$ exists such that the arcs $(v,w),(y,w)$ exist*$\}$.

Let $G^{\rightarrow}$ be a simple connected directed graph and let V^* be a non-empty subset of $V(G^{\rightarrow})$ and denote the undirected subgraph induced by V^* by, $\langle V^* \rangle$. In respect of Jaco Graphs the next result was derived.

Theorem 2.42 ([28]). *For the Jaco graph $J_n(x)$, $n,x \in \mathbb{N}$, $n \geq 5$, the competition graph $C(J_n(x))$ is given by:*

$$C(J_n(x)) = \langle V^* \rangle_{V^*=\{v_i:3\leq i\leq n-1\}} - \{v_i v_{m_i} : m_i = i + d^+_{J_n(x)}(v_i), 3 \leq i \leq n-2\} \cup \{v_1, v_2, v_n\}.$$

2.12 Grog Number of Jaco Graphs

For a simple connected graph G on $n \geq 2$ vertices we consider any orientation $G^{\rightarrow}$ thereof. Label the vertices randomly $v_1, v_2, v_3, \ldots, v_n$. The aforesaid vertex labeling is called *subscriptioning* and a specific labeling pattern is called a *subscription* of G. Consider the graph to represent a predator-prey web. A vertex v with $d^+_{G^{\rightarrow}}(v) = 0$ is exclusively *prey.* To the contrary a vertex w

with $d^-_{G^\rightarrow}(w) = 0$ is exclusively *predator*. A vertex z with $d_{G^\rightarrow}(z) = d^+_{G^\rightarrow}(z)_{>0} + d^-_{G^\rightarrow}(z)_{>0}$ is a mix of *predator-prey*.

Let a vertex labeled v_i have an initial *predator*$_{\geq 0}$*-prey*$_{\geq 0}$ *population* of exactly $\rho(v_i) = i$. So generally there is no necessary relationship between the initial *predator*$_{\geq 0}$*-prey*$_{\geq 0}$ *population* $\rho(v_i) = i$ and $d_{G^\rightarrow}(v_i) = d^+_{G^\rightarrow}(v_i)_{\geq 0} + d^-_{G^\rightarrow}(v_i)_{\geq 0}$.

2.12.1 The Grog algorithm [28]

The *predator-prey* dynamics now follow the Grog algorithm.

Grog Algorithm[8]**.**

Step 0: Consider the initial graph $G^\rightarrow$.

Step 1: Choose any vertex v_i and predator along any number $1 \leq \ell \leq d^+_{G^\rightarrow}(v_i) \leq i$ of out-arcs or along any number $1 \leq \ell \leq i < d^+_{G^\rightarrow}(v_i)$ of out-arcs, with only one predator per out-arc provided that the preyed upon vertex v_j has $j \geq 1$.

Step 2: Remove the out-arcs along which were predatored and set $d^+_*(v_i) = d^+_{G^\rightarrow}(v_i) - \ell$, and for all vertices $v_{j\neq i}$ which fell prey, set $d^-_*(v_j) = d^-_{G^\rightarrow}(v_j) - 1$.

Step 3: Set the *predator*$_{\geq 0}$*-prey*$_{\geq 0}$ *populations* $\rho_*(v_i) = i - \ell$ and $\rho_*(v_{j\neq i}) = j - 1$.

Step 4: Consider the next amended graph $G^\rightarrow_*$ and apply Steps 1, 2, 3 and 4 thereto if possible. If not possible, exit.

Observation 2.1. It is observed that since both the *predator*$_{\geq 0}$*-prey*$_{\geq 0}$ *population* of all vertices, and the number of out-arcs embedded in $G^\rightarrow$ as well as those respectively found in the iterative amended graphs $G^\rightarrow_*$ are finite, the Grog algorithm will always terminate. So it can be said informally that the Grog algorithm is well-defined.

Also note that Step 1 allows one to choose any vertex v_i per iteration. Following that, any number of the existing out-arcs from v_i can be chosen to predator along. Collectively, the specific iterative choices will be called the *predator-prey strategy*. Generally, a number of *predator-prey strategies* may exist for a given $G^\rightarrow$ and the set of all possible strategies is denoted, $S(G^\rightarrow)$. Amongst the strategies there will be those which for a chosen vertex v_i, consecutively predator along the maximal number of out-arcs available at v_i. These strategies are called *greedy strategies* and a greedy strategy $s_k \in S(G^\rightarrow)$ is denoted gs_k.

Observation 2.2. It is observed that if a *predator-prey strategy* $s_k \in S(G^\rightarrow)$ is repeated for a specific graph $G^\rightarrow$, the amended graph $G^\rightarrow_*$ found on termination (exit step) is unique. Put differently, it can informally be said the *predator-prey strategy* s_k is well-defined.

Definition 2.10 ([28])**.** For a predator-prey strategy s_k and the final amended graph $G^\rightarrow_*$ (exit step), the cumulative residual, predator$_{\geq 0}$-prey$_{\geq 0}$ population over all vertices is denoted and defined to be $r_{s_k}(G^\rightarrow) = \sum\limits_{\forall v_i} \rho_{G^\rightarrow_*}(v_i)$.

[8] Admittedly, the Grog algorithm has been described informally.

Definition 2.11 ([28])**.** The Grog number[9] of $G^{\rightarrow}$ is defined to be $g(G^{\rightarrow}) = \min(r_{s_k}(G^{\rightarrow}))_{\forall s_k \in S(G^{\rightarrow})}$ or equivalently, $g(G^{\rightarrow}) = \min(r_{gs_k}(G^{\rightarrow}))_{\forall gs_k \in S(G^{\rightarrow})}$.

Definition 2.12 ([28])**.** The grog number of a simple connected graph G is defined to be $g(G) = \min(g(G^{\rightarrow}))$ over all possible orientations of G.

Finding a closed formula for the number of arcs (edges) of a finite Jaco graph will assist in finding closed formula for many recursive results found for Jaco graphs. In the absence of such formula the next lemma is needed to settle Proposition 2.44. In [28] it is noted that in the formulation of the following lemma and proposition, the condition that v_i must be the *prime Jaconian vertex* was erroneously omitted.

Lemma 2.43. *For a Jaco graph, $J_n(x)$, $n, x \in \mathbb{N}$, $n \geq 2$ having prime Jaconian vertex v_i we have $2i - n \geq 0$.*

Proposition 2.44 ([28])**.** *For a Jaco graph, $J_n(x)$, $n, x \in \mathbb{N}$, $n \geq 2$ having prime Jaconian vertex v_i we have $g(J^*_{n+1}(x)) = g(J^*_n(x)) + (2i - n) + 1$.*

Corollary 2.45 ([28])**.** *For a Jaco graph, $J_n(x)$, $n, x \in \mathbb{N}$, $n \geq 2$ we have $g(J^*_{n+1}(x)) > g(J^*_n(x))$.*

2.13 Gutman Index of Jaco Graphs

The concept of the *Gutman index*, denoted $Gut(G)$ was introduced for a connected undirected graph G. In [27] the concept was applied to the underlying graphs of the family of Jaco graphs, (*directed graphs by definition*), and an easy recursive formula for the *Gutman index* $Gut(J^*_{n+1}(x))$ in terms of the *Gutman index* of the underlying Jaco graph, $J^*_n(x)$, $n, x \in \mathbb{N}$ with prime Jaconian vertex v_i was derived. Furthermore, $Gut(J^*_n(x) \rightsquigarrow_{v_1 u_1} J^*_m(x)) = Gut(J^{\rightsquigarrow v_1 u_1}_{n,m}(x))$ was determined in terms of $Gut(J^*_n(x))$ and $Gut(J^*_m(x))$. The aforesaid is the *edge-joint*, $J^*_n(x) \cup J^*_m(x) + v_1 u_1$, $v_1 \in V(J^*_n(x))$ and $u_1 \in V(J^*_m(x))$ (see Section 2.5).

Researching $Gut(J^{\rightsquigarrow v_k u_t}_{n,m}(x))$, $v_k \in V(J^*_n(x))$, $u_t \in V(J^*_m(x))$ with $1 \leq k \leq n$ and $1 \leq t \leq m$ in terms of $Gut(J^*_n(x))$ and $Gut(J^*_m(x))$, will be worthy.

Theorem 2.46 ([27])**.** *For the underlying graph $J^*_n(x)$, $n, x \in \mathbb{N}$, $n \geq 2$ with prime Jaconian vertex v_i we have that recursively:*

$$\begin{aligned} Gut(J^*_{n+1}(x)) = Gut(J^*_n(x)) &+ \sum_{k=1}^{i} \sum_{t=i+1}^{n} d_{J^*_n(x)}(v_k) d_{J^*_n(x)}(v_k, v_t) \\ &+ \sum_{t=i+1}^{n-1} \sum_{q=t+1}^{n} (d_{J^*_n(x)}(v_t) + d_{J^*_n(x)}(v_q)) + (n-i) \\ &\cdot \left(\sum_{k=1}^{i} d_{J^*_n(x)}(v_k) d_{J^*_n(x)}(v_k, v_n) + \sum_{t=i+1}^{n} d_{J^*_n(x)}(v_t) \right) + (n-i-1) + i(n-i). \end{aligned}$$

[9] Named after the author's late father's most detested alcoholic drink i.e. a dash of brandy which is over diluted with H_2O called, a *Grog*.

Theorem 2.47 ([27]). *For the underlying graphs $J^*_n(x)$ and $J^*_m(x)$, $n,m,x \in \mathbb{N}$ and $n \geq m \geq 2$ we have that:*

$$\begin{aligned}Gut(J^{\rightsquigarrow v_1u_1}_{n,m}(x)) &= Gut(J^*_n(x)) + Gut(J^*_m(x)) + \sum_{\ell=2}^{n} d_{J^*_n(x)}(v_\ell) d_{J^*_n(x)}(v_1,v_\ell)\\ &\quad + \sum_{s=2}^{m} d_{J^*_m(x)}(u_s) d_{J^*_m(x)}(u_1,u_s) + \sum_{t=2}^{m}(d_{J^*_n(x)}(v_1)+1) d_{J^*_m(x)}(u_t)(d_{J^*_m(x)}(u_1,u_t)+1)\\ &\quad + \sum_{k=2}^{n}\sum_{t=2}^{m} d_{J^*_n(x)}(v_k) d_{J^*_m(x)}(u_t)(d_{J^*_n(x)}(v_1,v_k) + d_{J^*_m(x)}(u_1,u_t)+1)+4.\end{aligned}$$

2.14 McPherson Number of Jaco Graphs

The recursive concept, called the *McPherson recursion*[10], is a series of *vertex explosions* such that on the first iteration a vertex $v \in V(G)$ explodes to arc (directed edges) to all vertices $u \in V(G)$ for which the edge $vu \notin E(G)$, to obtain the mixed graph G'_1. Now G'_1 is considered on the second iteration and a vertex $w \in V(G'_1) = V(G)$ may explode to arc to all vertices $z \in V(G'_1)$ if edge $wz \notin E(G)$ and arc (w,z) or $(z,w) \notin E(G'_1)$. The *McPherson number* of a simple connected graph G is the minimum number of iterative vertex explosions say ℓ, to obtain the mixed graph G'_ℓ such that the underlying graph of G'_ℓ denoted G^*_ℓ has $G^*_\ell \simeq K_n$. For Jaco graphs the next result was settled.

Theorem 2.48 ([25]). *Consider the Jaco graph $J_n(x)$, $n,x \in \mathbb{N}$, $n \geq 3$. If v_i is the prime Jaconian vertex we have:*

$$\Upsilon(J_n(x)) = \begin{cases} i, & \textit{if the edge } v_iv_n \notin E(J_n(x), \\ i-1, & \textit{otherwise.}\end{cases}$$

The following conjecture is still open.

Conjecture 2.48.1 ([25]). *For a Jaco Graph $J_n(x)$, $n,x \in \mathbb{N}$, $n \geq 3$ we have that $d^+(v_n)$ is unique (*non-repetitive*) if and only if $\Upsilon(J_n(x))$ is unique (*non-repetitive*).*

2.15 Primitive Holes and Pythagorean Primitive Holes

In [21] the concept of a primitive hole was introduced. As a specialised case the concept of a Pythagorean hole was introduced in [22]. Recall from [21] that conventionally a *hole* of a simple connected graph G is loosely defined as a *chordless cycle* C_n, $n \in \mathbb{N}$, $n \geq 4$ in the graph G. The *girth* of a simple connected graph G say k, is the smallest cycle C_k found in G, if any such cycle exists. So, such smallest cycle is necessarily *chordless*. Since many graphs with *girth* 3 exist the concept of a "chordless" C_3 is strongly implied but not conventionally defined. In [21] the convention that; *C_3 is a primitive hole* was proposed. The *girth* of an *acyclic graph G_{ac}* is conventionally defined to be $girth(G_{ac}) = \infty$. In [21] it was proposed that, $girth(G_{ac}) = 0$. This convention will allow the exploration of evolutionary *hole growth* like,

[10] Named after the author's friend, Colonel(ret.) Vic McPherson.

a *hole* C_k may grow over time units $t = 1,2,3,\dots,n$ or ∞ over an integer function $p(t) = j$ to attain j additional cyclic vertices at t. So $girth\big(\lim_{t\to\infty}(C_{k+p(t)}\big) = \infty$. It also implies that for simple connected graphs $G_1,G_2,G_3,\dots,G_n$, that $girth\big(\cup_{i=1}^{n} G_i\big) = \sum_{i=1}^{n} girth(G_i)$, and allows quite naturally that, $girth(\cup_{i=1}^{\infty} G_i) = \sum_{i=1}^{\infty} girth(G_i) = \infty$. These conventions reconcile the inherent conflict between the definitions of *girth* and that of a *hole*. In respect of Jaco graphs the follow results were settled.

Theorem 2.49 ([21]). *For the underlying graph $J_n^*(x)$ $n,x \in \mathbb{N}, n \geq 4$ with Jaconian vertex v_i we recursively have:*

$$h(J_{n+1}^*(x)) = h(J_n^*(x)) + \sum_{j=1}^{(n-i)-1} (n-i)-j.$$

Theorem 2.50 ([21]). *For any primitive hole of the Jaco graph $J_n^*(x)$, $n,x \in \mathbb{N}$ on the vertices v_i, v_j, v_k with $i<j<k$ a primitive hole on the vertices $v_{\ell i}$, $v_{\ell j}$, $v_{\ell k}$ in $J_{n\geq \ell k}^*(1)$, $\ell \in \mathbb{N}$, exists.*

From the famous König's theorem [33] it follows that $J_n(x)$, $1 \leq n \leq 4$ are the only bipartite Jaco graphs since all $J_n(x)$, $n,x \in \mathbb{N}$, $n \geq 5$ contains at least the odd cycle C_3. The smallest Jaco graph having a Pythagorean primitive hole is $J_8^*(x)$. Denote the number of Pythagorean primitive holes of a graph G by $h^p(G)$. Since closed formula for so many invariants of Jaco graphs still escape us the best results are the following.

Corollary 2.51 ([22]). *The Jaco graph $J_n^*(x)$, $n,x \in \mathbb{N}$, $n = 5k + d^+_{J_\infty(x)}(v_{5k})$ has $h^p(J_n(x)) = k$, Pythagorean primitive holes.*

Corollary 2.52 ([22]). *The Jaco graphs $J_n^*(x)$, $n,x \in \mathbb{N}$, $8 \leq n \leq 15$ are the only Jaco graphs with a unique Pythagorean primitive hole.*

Theorem 2.53 ([22]). *The Jaco graph $J_n^*(x)$, $n,x \in \mathbb{N}$, $n \geq 8$ has $h^p(J_n^*(x)) = k$ Pythagorean primitive holes for $5k + d^+_{J_\infty(x)}(v_{5k}) \leq n < 5(k+1) + d^+_{J_\infty(x)}(v_{5(k+1)})$, alternatively $h^p(J_n^*(x)) = \lfloor \frac{n}{8} \rfloor$.*

To construct the adapted Fisher table below an improvement of Theorem 2.46 is required.

Theorem 2.54 ([22]). *For the underlying graph $J_n^*(x)$, $n,x \in \mathbb{N}$, $n \geq 4$ we have recursively:*

$$h(J_{n+1}^*(x)) = h(J_n^*(x)) + \sum_{i=1}^{d^-_{J_\infty(x)}(v_{n+1})-1} i.$$

The adapted Fisher table, $J_\infty(x)$, $35 \geq n \in \mathbb{N}$ depicts the values $h(J_n^*(x))$ and $h^p(J_n^*(x))$ [22].

Table 8

$\phi(v_i) \to i \in \mathbb{N}$	$d^-(v_i) = \nu(\mathbb{H}_{i-1})$	$d^+(v_i) = i - d^-(v_i)$	$h(J_i^*(x))$	$h^p(J_i^*(x))$
$1 = f_2$	0	1	0	0
$2 = f_3$	1	1	0	0
$3 = f_4$	1	2	0	0
4	1	3	0	0
$5 = f_5$	2	3	1	0
6	2	4	2	0
7	3	4	5	0
$8 = f_6$	3	5	8	1
9	3	6	11	1
10	4	6	17	1
11	4	7	23	1
12	4	8	29	1
$13 = f_7$	5	8	39	1
14	5	9	49	1
15	6	9	64	1
16	6	10	79	2
17	6	11	94	2
18	7	11	115	2
19	7	12	136	2
20	8	12	164	2
$21 = f_8$	8	13	192	2
22	8	14	220	2
23	9	14	256	2
24	9	15	292	3
25	9	16	328	3
26	10	16	373	3
27	10	17	418	3
28	11	17	473	3
29	11	18	528	3
30	11	19	583	3
31	12	19	649	3
32	12	20	715	4
33	12	21	781	4
$34 = f_9$	13	21	859	4
35	13	22	937	4

It can be seen that Pythagorean primitive holes are generally scarce within Jaco graphs compared to the number of primitive holes.

3. Finite Jaco Graphs, $\{J_n(mx) : n, m, x \in \mathbb{N}\}$

Initially this family of linear Jaco graphs was denoted $J_n(a)$, [20] but the latter is now substituted by the case $f(x) = mx$, $m, x \in \mathbb{N}$ with $m > 2$, $c = 0$, [29]. Within the new context the notation $J_n(a)$ will represent the case $m = 0$, $c = a$. Of importance is to reflect on the definitions to ensure an understanding of the basic framework of the new family of directed graphs. Note that the underlying graph will be denoted $J_n^*(mx)$ and if the context is clear, both the directed and undirected graph is referred to as a *Jaco graph*. Similarly the difference between *arc* and *edge* and; $d_{J_n(mx)}(v)$ and $d_{J_n^*(mx)}(v)$ will be understood. Note that the definitions below are similar to those found in Section 2, Definitions 2.1 to 2.5.

Definition 3.1 ([29]). *The family of infinite Jaco Graphs denoted by $\{J_\infty(mx) : m, x \in \mathbb{N}\}$ is defined by $V(J_\infty(mx)) = \{v_i : i \in \mathbb{N}\}$, $A(J_\infty(mx)) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(mx))$ if and only if $(m+1)i - d^-(v_i) \geq j$.*

Definition 3.2 ([29]). *The family of finite Jaco Graphs denoted by $\{J_n(mx) : m, n, x \in \mathbb{N}\}$ is defined by $V(J_n(mx)) = \{v_i : i \in \mathbb{N}, i \leq n\}$, $A(J_n(mx)) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j \leq n\}$ and $(v_i, v_j) \in A(J_n(mx))$ if and only if $(m+1)i - d^-(v_i) \geq j$.*

Definition 3.3 ([29]). *The set of vertices attaining degree $\Delta(J_n(mx))$ is called the set of Jaconian vertices or the Jaconian vertices or the Jaconian set of the Jaco Graph $J_n(mx)$ and denoted, $\mathbb{J}(J_n(mx))$ or, $\mathbb{J}_n(mx)$ for brevity.*

Definition 3.4 ([29]). *The lowest numbered (subscripted) Jaconian vertex is called the prime Jaconian vertex of a Jaco Graph.*

Definition 3.5 ([29]). *If v_i is the prime Jaconian vertex, the complete subgraph on vertices $v_{i+1}, v_{i+2}, \cdots, v_n$ is called the Hope subgraph or Hope graph of a Jaco Graph and denoted, $\mathbb{H}(J_n(mx))$ or, $\mathbb{H}_n(mx)$ for brevity.*

The mx-root digraph has four fundamental properties which are:

(i) $V(J_\infty(mx)) = \{v_i : i \in \mathbb{N}\}$, and

(ii) if v_j is the head of an arc then the tail is always a vertex v_i, $i < j$, and

(iii) if v_k for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices v_ℓ, $k < \ell < j$ are tails of arcs to v_j and finally,

(iv) the degree of vertex v_k is $d(v_k) = mk$.

The family of finite Jaco graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and arcs to vertices) v_t, $t > n$. Hence, trivially $d(v_i) \leq mi$ for $i \in \mathbb{N}$. Figure 3 depicts $J_{11}(2x+1)$ (see [29]).

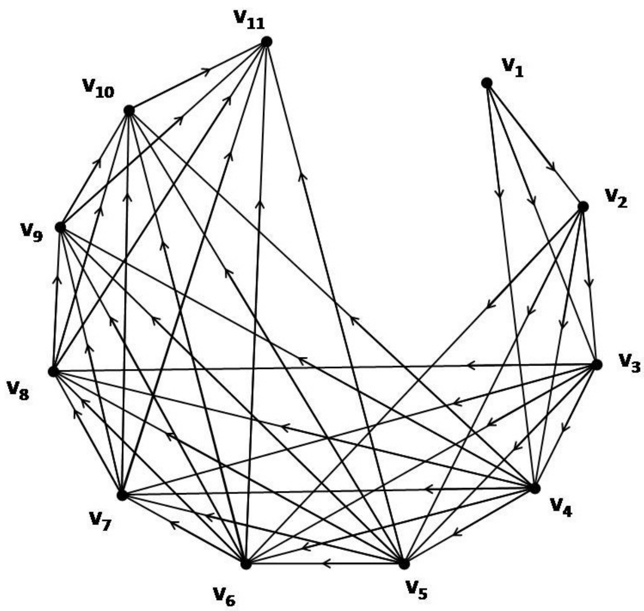

Figure 2. Jaco graph $J_{11}(2x+1)$ [29]

Property 3.1 ([29]). *From the definition of a Jaco Graph $J_n(mx)$, it follows that, if for the prime Jaconian vertex v_i, we have $d(v_i)=ki$ then in the underlying Jaco graph we have $d(v_k)=mk$ for all $k\in\{1,2,3,\dots,i\}$.*

Property 3.2 ([29]). *From the definition of a Jaco Graph $J_n(mx)$, it follows that $\Delta(J_k(mx))\le\Delta(J_n(mx))$ for all $k\le n$.*

Property 3.3 ([29]). *From the definition of a Jaco Graph $J_n(mx)$, it follows that the lowest degree attained by all Jaco Graphs is $0\le\delta(J_n(mx))\le m$.*

Property 3.4 ([29]). *The $d^-(v_k)$ for any vertex v_k of a Jaco Graph $J_n(mx)$, $n\ge k$ is equal to $d(v_k)$ in the underlying Jaco Graph $J_k(mx)$.*

Lemma 3.1 ([29]). *For the Jaco Graphs $J_i(mx)$, $i\in\{1,2,3,\dots,m+1\}$ we have $\Delta(J_i(mx))=i-1$ and $\mathbb{J}(J_i(mx))=\{v_k:1\le k\le i\}=V(J_i(mx))$.*

Lemma 3.2 ([29]). *If in a Jaco Graph $J_n(mx)$, and for smallest i, the arc (v_i,v_n) is defined, then v_i is the prime Jaconian vertex of $J_n(mx)$.*

Lemma 3.3 ([29]). *For all Jaco Graphs $J_n(mx)$, $n,x\in\mathbb{N}$, $n\ge 2$ and, $v_i,v_{i-1}\in V(J_n(mx))$ we have that in the underlying graph $|(d(v_i)-d(v_{i-1})|\le a$.*

Note that $\Delta(J_n(mx))$ might repeat itself as n increases to $n+1$ but on an increase we always obtain $\Delta(J_n(mx))+1$ before $\Delta(J_n(mx))+2$.

Theorem 3.4 ([29]). *The Jaco Graph $J_k(mx)$, $k,m,x\in\mathbb{N}$, $k=m(m+1)+1$ is the smallest Jaco Graph in $\{J_n(mx):n,m,x\in\mathbb{N}\}$ which has $\Delta(J_k(mx))=m(m+1)$ and $\mathbb{J}(J_k(mx))=\{v_{m+1}\}$.*

Observe, perhaps a trivial but important result has not been reported. We shall advance the result to enhance our understanding of Jaco graphs.

Proposition 3.5. *The prime Jaconian vertices of $J_n(x)$, $n,x \in \mathbb{N}$, say v_i, is the prime Jaconian vertex of $J_{i+(mi-d^-_{J_i(mx)}(v_i))}(mx)$.*

Proof. Since v_i is the prime Jaconian vertex of $J_n(x)$ the prime vertex has $d(v_i) = i$. For vertex v_i to be the prime Jaconian vertex of a smallest $J_n(mx)$ it is required that $a \cdot d(v_i) = mi$. Since $d^-(v_i)$ remains constant in $J_{j\geq i}(mx)$ (see Property 3.4) the smallest j is $j = i + (mi - d^-_{J_i(mx)}(v_i))$. □

Definition 3.6 ([29])**.** *For $m \in \mathbb{N}$, we define the series $(c_{m,n})_{n\in\mathbb{N}_0}$ by*

$$c_{m,0} = 0,\ c_{m,1} = 1,\ c_{m,n} = \min\{k < n : mk + c_{m,k} \geq n\} \quad (n \geq 2).$$

The connection between the mx-root digraph $J_\infty(mx)$ and the series $(c_{m,n})$ is explained by the following lemma.

Lemma 3.6 ([20, 29])**.** *Consider the Jaco Graph $J_\infty(mx)$, $m,n,x \in \mathbb{N}$ then the following hold:*

(a) $d^+(v_n) + d^-(v_n) = mn$.

(b) $d^-(v_{n+1}) \in \{d^-(v_n), d^-(v_n)+1\}$.

(c) *If $(v_i, v_k) \in A(J_\infty(mx))$ and $i < j < k$, then $(v_j, v_k) \in A(J_\infty(mx))$.*

(d) $d^+(v_n) = (m-1)n + c_{m,n}$.

Corollary 3.7 ([20, 29])**.** *Note that* (a) *and* (b) *of Lemma* 3.6 *entail that $d^-(v_{n+1}) = (n+1) - c_{m,n+1} \in \{n - c_{m,n}, n - c_{m,n}+1\}$ and that* (d) *then implies that the series $(c_{m,n})$ are well-defined and ascending, more specifically, $c_{m,n+1} \in \{c_{m,n}, c_{m,n}+1\}$ $(n \in \mathbb{N}_0)$.*

Lemma 3.8 ([20, 29])**.** *Let $k \in \mathbb{N}$, and $0 \leq b < m$. Then $c_{m,mk+c_{m,k}-b} = k$.*

3.1 Fibonaccian-Zeckendorf Result related to, $\{J_n(mx) : n,m,x \in \mathbb{N}\}$

Recall that the *generalised Lucas sequence* $U_n(m,-1)$ is defined by

$$U_0 = 0,\ \ U_1 = 1,\ \ U_{n+1} = mU_n + U_{n-1}.$$

It is well known that $U_n = \frac{r^n - s^n}{r-s}$, where $r = \frac{m}{2} + \sqrt{\frac{m^2}{4}+1}$, $s = \frac{m}{2} - \sqrt{\frac{m^2}{4}+1}$. A probably well-known (and not hard to prove) theorem, which in the case $m = 1$ is known as *Zeckendorf's theorem* was required to derive the next important results.

Lemma 3.9 ([29])**.** *Let $n \in \mathbb{N}$ and let $U_0, U_1, \ldots$ be the terms of the Lucas sequence $U(m,-1)$. Then n may be uniquely expressed by a sum $n = \sum\limits_{i\in\mathbb{N}} \alpha_i U_i$, where*

$$0 \leq \alpha_1 < m,\ 0 \leq \alpha_i \leq m\ (i > 2), \text{ and } \alpha_i = m \text{ only if } \alpha_{i-1} = 0\ (i \in \mathbb{N}).$$

Theorem 3.10 ([20, 29])**.** *Let $n \in \mathbb{N}$, $n = \sum\limits_{i\in\mathbb{N}} \alpha_i U_i$ where the requirements of Lemma 3.9 are assumed to be met. Then $b_n = \sum\limits_{i\in\mathbb{N}} \alpha_i U_{i-1} + \tau(n)$.*

3.2 Number of Arcs of Jaco Graphs, $\{J_n(mx) : n, m, x \in \mathbb{N}\}$

It is hoped that as a special case, a closed formula can be found for the number of arcs of a finite Jaco Graph $J_n(mx)$. However, the algorithms discussed in Ahlbach et al. [1] suggest this might not be possible.

Proposition 3.11 ([20, 29])**.** *The number of arcs of a Jaco Graph,* $\epsilon(J_n(mx)) = \frac{1}{2}n(n-1)$ *if* $n \leq m+1$..

Theorem 3.12 ([20, 29])**.** *If for the Jaco Graph* $J_n(mx)$*,* $n, m, x \in \mathbb{N}$ *we have* $\Delta(J_n(mx)) = k$*, then* $\epsilon(J_n(mx)) = \epsilon(\mathbb{H}(J_n(mx))) + \sum\limits_{i=1}^{k} d^+(v_i)$.

Corollary 3.13 ([20, 29])**.** *The number of arcs of a Jaco Graph* $J_n^*(mx)$*,* $n, m, x \in \mathbb{N}$*, having vertex* v_i *as the prime Jaconian vertex, can also be expressed recursively as:*

$$\epsilon(J^*_{n+1}(mx)) = \begin{cases} \epsilon(J_n^*(mx)) - i + n, & \text{if } d(v_i) = mi, \\ \epsilon(J_n^*(mx)) - i + (n+1), & \text{if } d(v_i) < mi. \end{cases}$$

3.3 Number of Shortest Paths in the Jaco Graphs, $\{J_n(mx) : n, m, x \in \mathbb{N}\}$

In the first work the *Liz numbers*[11] were defined (see [20]). Recently Stephan Wagner[12] identified these numbers to be a special case generated from a Horadam Sequence. Reader should note the subtle difference between the $(m,1)$-Fibonacci sequence defined in Kalman et al. [16], and the definition of the Liz numbers. The latter observations with the results from Subsections 2.2 and 3.1 suggest a strong connection between certain properties of Jaco graphs and Number Theory. Furthermore, studies of the generalisation of the concept to polynomial Jaco graphs $J_n(f(x))$ where $f(x) = \sum\limits_{i=1}^{t} a_i x^i + c$, $n, a_i, x \in \mathbb{N}$ and $c \in \mathbb{N}_0$, will justify the generalized Liz numbers as a family of numbers that can be distinguished from the family of Horadam numbers.

Definition 3.7 ([20])**.** Liz numbers are the family of numbers defined by $\mathbb{L} = \{\mathscr{L}_m : \mathscr{L}_0 = 0, \mathscr{L}_1 = 1, \mathscr{L}_2 = 1, \mathscr{L}_i = m\mathscr{L}_{i-1} + \mathscr{L}_{i-2}, m,\ i \in \mathbb{N}, i \geq 3\}$.

Definition 3.8 ([20, 29])**.** The set of distance-root vertices of the Jaco Graph $J_n(mx)$, is the set $\{v_{\mathscr{L}_2}, v_{\mathscr{L}_3}, v_{\mathscr{L}_4}, \ldots, v_{\mathscr{L}_j<n}, \cdots, v_n$: for smallest subscript $\mathscr{L}_j$, arc $(v_{\mathscr{L}_j}, v_n) \in A(J_n(mx))\}$ and is denoted, $\mathbb{D}(J_n(mx))$ or $\mathbb{D}_n(mx)$ for brevity.

Property 3.5 ([20, 29])**.** *From Definitions* 3.7 *and* 3.8 *it follow that besides possibly* v_n*, all other distance-root vertices of the Jaco Graph* $J_n(mx)$*,* $n \in \mathbb{N}$*, have Liz number subscripts.*

Property 3.6 ([20, 29])**.** *The set of Fibonacci numbers* $\mathbb{F} \in \mathbb{L}$*, since* $m = 1$ *for* $\mathbb{F}$.

Lemma 3.14 ([20, 29])**.** *In a Jaco Graph* $J_n(mx)$*,* $n, m, x \in \mathbb{N}$ *we have for smallest* k *such that,* $k + d^+(v_k) \geq i$ *that,* $d_{J_n(mx)}(v_1, v_i) = d_{J_n(mx)}(v_1, v_k) + 1$.

[11] In the derivative named after the author's wife, Elizabeth Kok.

[12] Department of Mathematical Sciences, Stellenbosch University, Republic of South Africa.

Definition 3.9 ([20, 29])**.** Let the number of distinct shortest paths between v_1 and v_n in the Jaco Graph $J_n(x)$, $n,x \in \mathbb{N}$ be denoted by $\psi(v_n)$.

Proposition 3.15 ([20, 29])**.** *Consider vertex v_j, $j \geq 1$ in $J_n(x)$, $n,x \in \mathbb{N}$. The shortest path between v_1 and v_j is unique if and only if, $d^+(v_j) \in \mathbb{F}$.*

Proposition 3.16 ([20, 29])**.** *Consider vertex v_j, $j \geq 1$ in $J_n(x)$, $n,x \in \mathbb{N}$. If and only if $d^+(v_j) \notin \mathbb{F}$, then $\psi(v_j) = \sum\limits_{i=l}^{f_t} \psi(v_i)$, with f_t the largest Fibonacci number less than j and l the smallest integer such that the edge (v_l, v_j) exists.*

Conjecture 3.16.1 ([20, 29])**.** *Let $k \geq 7$ and $f_i < k \leq f_{i+1}$, and $f_i, f_{i+1} \in \mathbb{F}$. If and only if $d^+(v_k)$ is non-repetitive (meaning $d^+(v_{k-1}) \neq d^+(v_k) \neq d^+(v_{k+1})$) then $\psi(v_k)$ is non-repetitive (meaning $\psi(v_{k-1}) \neq \psi(v_k) \neq \psi(v_{k+1})$).*

4. Linear Function Corresponding to a Linear Jaco Graph

It is was reported in [29] that if for a sufficiently large linear Jaco Graph $J_n(f(x))$ there exist two vertices v_i, v_j for which $d(v_i) = f(i)$ and $d(v_j) = f(j)$ then the linear function $f(x)$ can be derived by solving the simultaneous equations:

$$mi + c = d(v_i), \quad mj + c = d(v_j).$$

The smallest linear Jaco graph for which this is possible is for $J_{f(2)+1}(f(x))$ hence, knowing that $d(v_2) = f(2)$ in the given linear Jaco graph.

Proposition 4.1 ([29])**.** *If for a linear Jaco graph, $d(v_i) = f(i)$ and $d(v_{i+1}) = f(i+1)$ then for maximum i', j' for which the arcs $(v_i, v_{i'}), (v_{i+1}, v_{j'})$ exist, we have $j' - i' \in \{m, m+1\}$.*

Each positive integer k can be written as $k+1$ sums of non-negative integers $m+c$, $m \geq 0$, $c \geq 0$. If the lower limit on m is relaxed to allow $m \geq 0$ then for a given k the linear Jaco graphs corresponding to the functions $f_i(x) = m_i x + c_i$, $m_i + c_i = k$ and $1 \leq i \leq k+1$, are f-related Jaco graphs.

For $m = 0$ and $c \geq 0$ two special classes of disconnected linear Jaco graphs exist. For $c = 0$ the Jaco graph $J_n(0)$ is a null graph (*edgeless graph*) on n vertices. For $c = k > 0$, the Jaco graph $J_n(k) = \bigcup\limits_{\lfloor \frac{n}{k+1} \rfloor\text{-copies}} K_{k+1} \bigcup K_{n-(k+1)\cdot\lfloor \frac{n}{k+1} \rfloor}$. Figure 3 depicts the linear Jaco graph $J_{15}(3) = K_4 \bigcup K_4 \bigcup K_4 \bigcup K_3$ (see [29]).

Note that a complete graph K_n is a linear Jaco graph to any linear function $f(x) = mx + c$ if $n \leq m + c + 1$. And, if K_n is a f-related linear Jaco graph, the complete graph corresponds to any of the $m + c + 1$ defined linear functions. It implies that at least a two-component subgraph of complete graphs say $K_s \bigcup K_t$ is required to derive the unique linear function $f(x) = k - 1$, $k = \max\{s, t\}$.

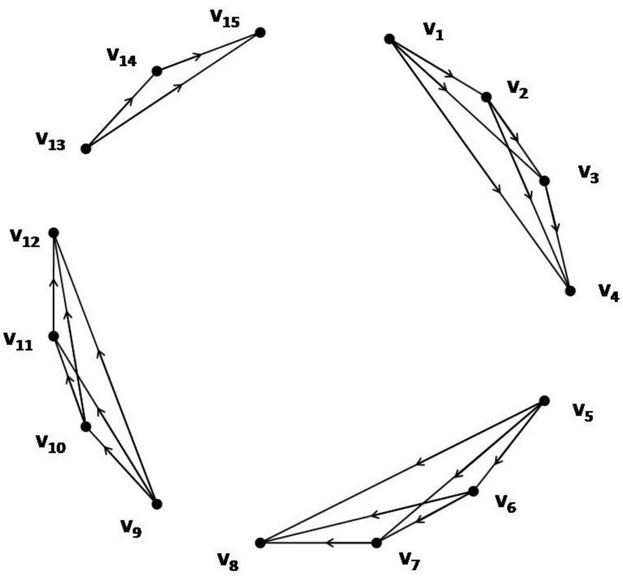

Figure 3. Jaco graph $J_{15}(3) = K_4 \bigcup K_4 \bigcup K_4 \bigcup K_3$ [29]

5. Conclusion

Real world applications of Jaco graphs have not been found as yet. This provides scope for studies into *Applied Jaco Graph Theory*. Much research has been done for the case $f(x) = x$, $x \in \mathbb{N}$ and limited comparative research followed for the case $f(x) = mx$, $m, x \in \mathbb{N}$. Results for the general case $f(x) = mx + c$, $m, c, x \in \mathbb{N}$ are completely open.

Certainly Conjectures 2.4.1, 2.4.2, 2.16.1, 2.48.1 and 3.16.1 call for closure. Also the Grog algorithm requires formalisation.

As stated in [29] the generalisation of the concepts to polynomial Jaco graphs $J_n(f(x))$ where $f(x) = \sum_{i=1}^{t} a_i x^i + c$, $n, a_i, x \in \mathbb{N}$ and $c \in \mathbb{N}_0$, will be worthy to study. Definition 2.1 will have to change to define the orientation of edges resulting from $f(x) < 0$. Also, given a sufficiently large polynomial Jaco graph, deriving the polynomial function can be formalised.

It is the author's considered view that having found a closed formula for the number of arcs (edges of $J_n^*(f(x))$ during the compilation of this critical review paper for the special case $f(x) = x$, $x \in \mathbb{N}$ opened a wide scope for closure of many recursive results. The remaining challenges are to generalised the result for the number of arcs (edges) for $J_n(f(x))$, $f(x) = mx + c$ after which the closure of other general results may follow.

Furthermore, the family of linear Jaco graphs offers a wide scope of formal thesis research at, at least master degree level. The author expresses his willingness to serve as co-supervisor in any such thesis research.

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In memory: The concept of linear Jaco graphs followed from a dream during the night 10/11 January 2013 which was the first dream Kokkie (author) could recall about his daddy, after his daddy passed away in the peaceful morning hours of 24 May 2012, shortly before the awakening of Bob Dylan, celebrating Dylan's 71^{st} birthday. The dream was about, resolve:

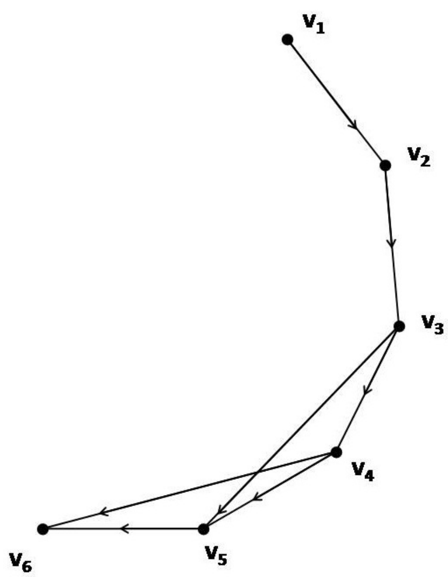

Figure 4. The Dream

It is easy to see that unraveling the secrets of Jaco graphs will take time t, $t \to \infty$. So unfortunately ourselves and the readers will not be around to see it all.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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