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A Study on Quaternionic B_2-Slant Helix in Semi-Euclidean 4-Space

Research Article

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Abstract. In this work, we defined a quaternionic B_2-slant helix in semi-Euclidean space $\mathbb{E}_2^4$. Then we gave Frenet formulae for the quaternionic curve in semi-Euclidean space $\mathbb{E}_2^4$. Also, we investigated some necessary and sufficient conditions for a space curve to be a quaternionic B_2-slant helix according to quaternionic curves in semi-Euclidean space $\mathbb{E}_2^4$.

Keywords. B_2-slant helices; Semi-Euclidean space; Helices; Semi-quaternions

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1. Introduction

The quaternion was first time introduced by Hamilton in 1843 as a successor to complex numbers. In [3], provided a brief introduction of the semi-quaternions. As a set, the quaternions $\mathbb{Q}$ are coincide with $\mathbb{R}^4$-dimensional vector space $\mathbb{R}^4$ over real numbers. According to this feature of quaternions Baharathi and Nagaraj presented the Frenet formulae for a quaternion valued function of a single real variable (quaternionic curves) in $\mathbb{E}^3$ and $\mathbb{E}^4$ [2]. Also, A.C. Çöken and A. Tuna have studied Frenet formulae, harmonic curvatures, inclined curves and some characterizations for a quaternionic curve in the semi-Euclidean space $\mathbb{E}_2^4$ [1]. Specially, in the differential geometry there are some curves satisfying some relationships between their curvatures. One of these curves is a general helix which is defined by the property that the tangent of the curve makes a constant angle with a fixed straight line called the axis of the general helix [5].

In this paper, our main aim is to define quaternionic B_2-slant helix and to obtain some of their necessary and sufficient conditions.

2. Preliminaries

A semi-real quaternion q is an expression form $q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + d\vec{e}_4$ where a, b, c and d are ordinary real numbers and $\vec{e}_i$, $(1 \le i \le 4)$, $e_4 = +1$ are quaternionic units which satisfy the non-commutative multiplication rules

(1) $\vec{e}_i \times \vec{e}_i = \varepsilon_{\vec{e}_i}$; $1 = i \le 3$.

(2) $\vec{e}_i \times \vec{e}_j = \varepsilon_{\vec{e}_i}\varepsilon_{\vec{e}_j}\vec{e}_k$, where $(i\ j\ k)$ is an even permutation of $(1\ 2\ 3)$ in semi-Euclidean space.

Let us denote the algebra of semi-real quaternions by $\mathbb{Q}_v$ and its natural basis is given by $\{e_1, e_2, e_3, e_4\}$. We can write a semi-real quaternion q as a form $q = S_q + V_q$ where $S_q = d$ is scalar part and $V_q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$ is vector part of q. So the conjugate of q is defined by $\gamma q = -a\vec{e}_1 - b\vec{e}_2 - c\vec{e}_3 + d$. Using these basic products we can define the symmetric, non-degenerate, real-valued, bilinear form h as below:

$$(p,q) \to h(p,q) = \frac{1}{2}[-\varepsilon_p\varepsilon_{\gamma q}(p \times \gamma q) - \varepsilon_q\varepsilon_{\gamma p}(q \times \gamma p)]$$

which is called the semi-real quaternion inner product [1]. The norm of semi-real quaternion q is $\|q\|^2 = |h(q,q)| = |\varepsilon_q(q \times \gamma q)| = |-a^2 - b^2 + c^2 + d^2|$. If $\|q\| = 1$, then semi-real quaternion q is called semi-real unit quaternion [4]. Let q and p be two semi-real quaternion in $\mathbb{Q}_v$, then the quaternion product of q and p is given by

$$p \times q = S_pS_q + \langle V_p, V_q\rangle + S_pV_q + S_qV_p + V_pV_q.$$

And then if $q + \gamma q = 0$, then q is called a semi-real spatial quaternion [2]. The four-dimensional semi-Euclidean space $\mathbb{E}_2^4$ is identified with the space of semi-real unit quaternions. Let $I = [0,1]$ be an interval in real line $\mathbb{R}$ and

$$\alpha : I \subset \mathbb{R} \to \mathbb{Q}_v, \quad s \to \alpha(s) = \sum_{i=1}^{4} \alpha_i(s)\vec{e}_i, \quad (1 \le i \le 4),\ e_4 = 1,$$

be a smooth curve in $\mathbb{E}_2^4$ with nonzero curvatures $\{\kappa, \tau, \sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\}$ and $\{T(s), N(s), B_1(s), B_2(s)\}$ denotes the Frenet apparatus of the semi-real quaternionic curve $\alpha(s)$. Let the arc-length parameter s be chosen such that the tangent $T(s) = \alpha'(s)$ has unit magnitude [1]. Then the Frenet equations of the semi-real quaternionic curve $\alpha(s)$ are given by

$$\begin{aligned}
&\frac{d}{ds}T(s) = \varepsilon_N\kappa(s)N(s),\\
&\frac{d}{ds}N(s) = -\varepsilon_t\varepsilon_N\kappa(s)T(s) + \varepsilon_n\tau(s)B_1(s),\\
&\frac{d}{ds}B_1(s) = -\varepsilon_t\tau(s)N(s) + \varepsilon_n[\sigma - \varepsilon_t\varepsilon_T\varepsilon_N](s)B_2(s),\\
&\frac{d}{ds}B_2(s) = -\varepsilon_b[\sigma - \varepsilon_t\varepsilon_T\varepsilon_N\kappa](s)B_1(s), \qquad (1)
\end{aligned}$$

where $\kappa(s) = \varepsilon_N \| \frac{d}{ds} T(s) \|$, $\|N(s)\|^2 = |\varepsilon_N|$, $h(T,T) = \varepsilon_T$, $h(N,N) = \varepsilon_N$, $h(B_1,B_1) = \varepsilon_{B_1}$, $h(B_2,B_2) = \varepsilon_{B_2}$ [1].

3. Quaternionic B_2-Slant Helix in Semi-Euclidean 4-Space

In this section, we give the definition and the necessary and sufficient conditions for quaternionic B_2-slant helices in semi-Euclidean space $\mathbb{E}^4_2$.

Definition 1. *A unit semi-real quaternionic curve $\alpha : I \subset \mathbb{R} \to \mathbb{Q}_v$ is called B_2-slant helix if its second binormal unit vector B_2 makes a constant angle φ with a fixed direction in a unit vector U; that is $h(B_2,U) = \cos\varphi$ is constant along the curve.*

Theorem 1. *A unit semi-real quaternionic curve α in semi-Euclidean space $\mathbb{E}^4_2$ with $\kappa \neq 0$, $\tau \neq 0$ and $\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa \neq 0$ is a quaternionic B_2-slant helix if and only if the condition*

$$\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)^2 + \left(\varepsilon_N \frac{1}{\kappa}\right)^2 \left(\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)\right)^2 = \text{constant} \tag{2}$$

is satisfied.

Proof. Let α be a quaternionic B_2-slant helix with $h(B_2,U) = \cos\varphi$, then differentiating the last equation and by using the Frenet equations given in (1), we get

$$\frac{d}{ds} h(B_2,U) = -\varepsilon_b [\sigma - \varepsilon_t \varepsilon_T \varepsilon_N k] h(B_1,U) = 0$$

that's why U is in the subspace $Sp\{T,N,B_2\}$ and can be written as below

$$U = a_1(s)T(s) + a_2(s)N(s) + a_3(s)B_2(s), \tag{3}$$

where

$$a_1(s) = h(T,U), \ a_2(s) = h(N,U), \ a_3(s) = h(B_2,U) = \cos\varphi = \text{constant}. \tag{4}$$

Taking derivative of (3) with respect to s and by using the Frenet equations given in (1), we obtain

$$\left(\frac{da_1}{ds} - a_2 \varepsilon_t \varepsilon_N \kappa\right) T + \left(\frac{da_2}{ds} + a_1 \varepsilon_N \kappa\right) N + (a_2 \varepsilon_n \tau - a_3 \varepsilon_b [\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa]) B_1 = 0.$$

From this equation we have

$$\frac{da_1}{ds} - a_2 \varepsilon_t \varepsilon_N \kappa = 0, \quad \frac{da_2}{ds} + a_1 \varepsilon_N \kappa = 0, \quad a_2 \varepsilon_n \tau - a_3 \varepsilon_b [\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa] = 0,$$

that is,

$$a_2 = \varepsilon_n \varepsilon_b \frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau} a_3 = \varepsilon_t \varepsilon_N \frac{1}{\kappa} \frac{da_1}{ds}, \tag{5}$$

$$\frac{da_2}{ds} = -\varepsilon_N \kappa a_1. \tag{6}$$

Differentiating (5) with respect to s and by using (6), we get the second order linear differential equation for a_1 as below

$$\frac{d^2a_1}{ds^2}-\frac{\kappa'}{\kappa}\frac{da_1}{ds}+\varepsilon_t\varepsilon_N\kappa^2a_1=0. \tag{7}$$

By changing the variables in (7) as $t=\int_0^s\varepsilon_t\varepsilon_N\kappa(s)ds$ we obtain

$$\frac{d^2a_1}{dt^2}+a_1=0.$$

The general solution of the above differential equation is

$$a_1=A\cos t+B\sin t, \tag{8}$$

where A and B are constants. With the help of (5), (6) and (8) we have

$$a_2=\varepsilon_n\varepsilon_b\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}a_3=-A\sin t+B\cos t, \tag{9}$$

$$a_1=-\varepsilon_n\varepsilon_b\varepsilon_N\frac{1}{\kappa}\left[\frac{d}{ds}\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)\right]a_3=A\cos t+B\sin t. \tag{10}$$

From (9) and (10) it follows that the constants A and B are

$$A=-\varepsilon_n\varepsilon_b a_3\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{t}\sin t+\varepsilon_N\frac{1}{\kappa}\left[\frac{d}{ds}\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)\right]\cos t\right), \tag{11}$$

$$B=\varepsilon_n\varepsilon_b a_3\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\cos t-\varepsilon_N\frac{1}{\kappa}\left[\frac{d}{ds}\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)\right]\sin t\right). \tag{12}$$

Using the equations (4), (11) and (12) we get

$$A^2+B^2=\varepsilon_n\varepsilon_b\left[\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)^2+\left(\varepsilon_N\frac{1}{\kappa}\right)^2\left[\frac{d}{ds}\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)\right]^2\right]\cos^2\varphi=\sin^2\varphi.$$

Thus, we have

$$\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)^2+\left(\varepsilon_N\frac{1}{\kappa}\right)^2\left[\frac{d}{ds}\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)\right]^2=\varepsilon_n\varepsilon_b\tan^2\varphi=\text{constant}. \tag{13}$$

Conversely, for a unit semi-real quaternionic curve the condition (2) is satisfied we can always find a constant unit vector U which makes a constant angel with the second binormal vector of the semi-real quaternionic curve. By considering the unit vector U and using the equations (4), (9) and (10) we get

$$U=\left[-\varepsilon_n\varepsilon_b\varepsilon_N\frac{1}{\kappa}\left(\frac{d}{ds}\left(\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}\right)\right)T+\varepsilon_n\varepsilon_b\frac{\sigma-\varepsilon_t\varepsilon_T\varepsilon_N\kappa}{\tau}N+B_2\right]\cos\varphi.$$

Taking derivative of the above equation with the help of (13), gives that $\frac{du}{ds}=0$, this means that the unit vector U is constant vector. Consequently, the unit semi-real quaternionic curve α is a quaternionic B_2-slant helix in semi-Euclidean space. □

Theorem 2. *A unit semi-real quaternionic curve α in the semi-Euclidean space $\mathbb{E}_2^4$ is a quaternionic B_2-slant helix if and only if there exists a C^2-function f such that*

$$\varepsilon_N \kappa f(s) = \frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right), \quad \frac{d}{ds} f(s) = -\varepsilon_N \kappa \frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}. \tag{14}$$

Proof. We assume that α is a quaternionic B_2-slant helix. Differentiation of (13) with respect to s gives

$$\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau} \frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right) + \varepsilon_N \frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right) \frac{d}{ds}\left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)\right] = 0. \tag{15}$$

Therefore, we have

$$\varepsilon_N \frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right) = -\frac{\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)}{\frac{d}{ds}\left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)\right]}.$$

If we take

$$f(s) = -\frac{\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)}{\frac{d}{ds}\left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_N \kappa}{\tau}\right)\right]},$$

then the above equation becomes

$$\varepsilon_N \kappa f(s) = \frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right). \tag{16}$$

Therefore, (15) is rewritten as

$$\frac{d}{ds}\left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)\right] = -\varepsilon_N \kappa \frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}. \tag{17}$$

By differentiating (16) with respect to s, we get

$$\frac{d}{ds} f(s) = \varepsilon_N \frac{d}{ds}\left[\frac{1}{\kappa}\frac{d}{ds}\left(\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}\right)\right]. \tag{18}$$

From (17) and (18), we obtain

$$\frac{d}{ds} f(s) = -\varepsilon_N \kappa \frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}. \tag{19}$$

Conversely, if the condition (14) holds, then from the equations (4), (9), (10) and (16) we can write the unit constant vector U as

$$U = \left[-\varepsilon_n \varepsilon_b f(s) T + \varepsilon_n \varepsilon_b \frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau} N + B_2\right] \cos\varphi.$$

From this equation the second binormal vector B_2 of α makes a constant angle φ with a fixed direction U; that is $h(B_2, U) = \cos\varphi = \text{constant}$. Thus, α is a quaternionic B_2-slant helix. □

Theorem 3. *Let α be a unit semi-real quaternionic curve in the semi-Euclidean space $\mathbb{E}_2^4$. Then is α a quaternionic B_2-slant helix if and only if the following condition is satisfied;*

$$\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau} = C_1 \cos\omega + C_2 \sin\omega, \tag{20}$$

where C_1 and C_2 are constants.

Proof. Let α be a unit semi-real quaternionic B_2-slant helix. Then the condition (14) is holds. By using this condition let us define C^2-function $\omega(s)$ by

$$\omega(s) = \int_0^s \varepsilon_N \kappa(s) ds, \tag{21}$$

and C^1-functions $g(s)$ and $r(s)$ by

$$\begin{aligned} g(s) &= \frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau} \cos\omega - f(s) \sin\omega, \\ r(s) &= \frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau} \sin\omega + f(s) \cos\omega. \end{aligned} \tag{22}$$

By differentiating equations (22) with respect to s and using equations (16), (19) and (21) we get that $\frac{d}{ds} g(s) = 0$ and $\frac{d}{ds} r(s) = 0$ are both identically zero. Therefore, $g(s) = C_1$ and $r(s) = C_2$, where C_1 and C_2 are constants. By replacing these in (22) and solving the result that getting from (22) for $\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau}$ we have

$$\frac{\sigma - \varepsilon_t \varepsilon_T \varepsilon_N \kappa}{\tau} = C_1 \cos\omega + C_2 \sin\omega.$$

Conversely, suppose that condition (20) holds. Then by solving the equations in (22) we have

$$f(s) = -C_1 \sin\omega + C_2 \cos\omega,$$

this function satisfies the condition (14). Therefore, α is a quaternionic B_2-slant helix. □

4. Conclusion

For a space curve to be a quatermionic B_2-slant helix, we obtain necessary and sufficient conditions according to quaternionic curves in semi-Euclidena space $\mathbb{E}_2^4$.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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