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Research Article

Sensitivity Analysis in Linear Bilevel Programs

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Abstract. Sensitivity analysis is the study of changing in the input parameters of the original problem and description of their impacts on the optimal solution. Sensitivity analysis is important in practice, where parameter values of the practical problems may be estimates. This paper studies the link between sensitivity analysis of linear programming program where there are extensive literature on it, and the sensitivity analysis of the linear bilevel programming problem. Numerical examples are provided to illustrate the approach.

Keywords. Linear bilevel programming problem; Sensitivity analysis; Multiple objective linear programming; Simplex algorithm

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1. Introduction

Over the past few decades *bilevel programming problems* (BLP) have received a great deal of attention and have been successfully applied to a variety of fields such as network design, transportation systems, economics and so on. It has a hierarchical structure, where the set of all variables is partitioned between two vectors x (upper-level or leader decision variables) and y (lower level or follower decision variables), and y is to be chosen as an optimal solution of a second mathematical programming problem parameterized in x.

$$z = \min_{x,y} d_1^T x + d_2^T y \tag{1.1}$$

$$\text{s.t. } A_1 x \le b_1 \tag{1.2}$$

$$x \ge 0 \tag{1.3}$$

$$\begin{aligned} &\min_{y} c^T y \\ &\text{s.t. } A_2x + A_3y \le b_2 \\ &\qquad\qquad\quad y \ge 0 \end{aligned} \tag{1.4}$$

Here x and d_1 are vectors in R^{n_x}, y and d_2 are vectors in R^{n_y}, b_1 is a vector in R^{m_1}, b_2 is a vector in R^{m_2}, A_1 is $m_1 \times n_x$, A_2 is $m_2 \times n_x$ and A_3 is $m_2 \times n_y$. We refer to the entire problem as linear BLP. The objective (1.1) is the upper level or leader's objective function, the constraints (1.2) and (1.3) are the upper level constraints, the problem (1.4) is the lower or follower's problem and one of the upper level constraints is that y solves the lower level problem (1.4). In this formulation, the leader controls the x variables and the follower selects the y variables based on the leader's selection of x.

Early work on BLP problem dates back to the 1970, But it was not until the early 1980 that usefulness of these mathematical programs in modeling hierarchical decision processes and engineering design problems prompted researchers to pay close attention to BLP problems. Since that time there have been broad interests to this problem both from the practical and the theoretical points of view and many interesting papers have introduced theoretical properties, selected applications, and solution algorithms of BLP problems.

Generally, in formulating the practical applications as optimization models, e.g. BLP problem, input parameters are not known exactly and are often estimates. It is therefore important to study of the behavior of the optimal solution with respect to changes in the input parameters of the original optimization problem.

In other words, after solving the original problem, we may discover that some of the entries have to be changed or some factors may be overlooked at early stages of problem formulation. Solving the modified problem from scratch will be wasteful. It is important to update the current solution in a way that takes care of these factors. The main idea of this paper is to study the sensitivity analysis of the linear BLP problem. Here, we apply the proposed algorithm by Glackin et al. (2009).

The rest of this paper is organized as follows. In Section 2, we study sensitivity analysis of some variations in the linear BLP problem. In Section 3, we use a numerical example to explore the proposed approach. Finally, Section 4 concludes the paper.

2. Sensitivity Analysis of A Linear BLP Problem

Definition 2.1. Extended region of the linear BLP problem is defined as follows:

$$ER = \{(x,y) \mid A_1x \le b_1, A_2x + A_3y \le b_2, x, y \ge 0\}.$$

Theorem 2.2 (Fulop (1993)). *A point (x,y) is a feasible solution for linear BLP problem if and only if (x,y) is an efficient solution of following multiple objective linear programming (MOLP) problem*

$$\min\left\{\begin{bmatrix} -I & 0 \\ e^T & 0 \\ 0 & e^T \end{bmatrix} (x,y)^T \mid (x,y) \in ER \right\}.$$

The technical report by Fulop (1993) [1] about relationship of linear BLP problem and *multiple objective linear programming* (MOLP) is widely used for introducing solution algorithms for BLP problem. Based on this relationship, Glackin et al. (2009) [2] proposed an algorithm that uses simplex pivots for minimization of leader objective function on efficient set of the above MOLP problem. In other words, their algorithm evaluate leader objective function on extreme points of extended region ER and using simplex pivots for moving on the extreme efficient points of the MOLP problem, along with the consideration of improvement on the objective function.

Algorithm (Glackin et al. (2009))**.**

Step 1. Minimize the upper level objective function over the feasible region, obtaining the optimal point $z_0 = (x_0, y_0)$ with leader's objective value $F(z_0)$.

Step 2. Use the $e^T s$ test to determine that the point z_0 is efficient or find an initial efficient extreme point z'.

- If the point z_0 is efficient, then z_0 solves BLP; stop.
- If the point z_0 is not efficient but $F(z') = F(z_0)$, then z' solves BLP; stop.
- If the point z_0 is not efficient and $F(z_0) < F(z')$, continue with Step 3.

Step 3. Pivot along improving efficient edges (identified by the $c^T e$ test) successively, without revisiting vertices, until no further improving efficient edges are found. Update z' to the best efficient point found so far.

Step 4. Add or update the previously added cut to enforce the constraint $F(z) \le F(z')$.

Step 5. Examine each efficient vertex of the cutting hyperplane and use the $c^T e$ test to find an improving efficient edge.

- If an improving efficient edge is found, pivot along the improving edge and go to Step 3.
- If no improving efficient edges can be found from any of the efficient vertices of the cutting hyperplane (including all of their degenerate representations) then z' solves BLP; stop.

Now, suppose we take a practical problem, formulate it as a linear BLP problem and by using Glackin et al's algorithm we find an optimal efficient basis B. Note that, according to Glackin et al's algorithm, there is no improving efficient edges for optimal efficient basis B. We shall describe how to make use of this optimality conditions (feasibility and no improvement in leader's function, together) in order to find the new optimal solution if some of the problem data change without resolving the problem from scratch. In particular, the effect of the following variations in the problem will be considered

- Change in the cost vector of leader's function d.
- Change in the cost vector of follower's function c.

Changing in the cost vector can occur in the leader or follower's function which will be investigated separately due to difference in losing optimality condition. We consider one change at a time (e.g. how to get the new optimal feasible solution if the value of only one c_j has to be changed) and for several changes we make them one at a time to take care of several simultaneous changes.

We can change the coefficients of A_1, A_2, A_3, b_1 or b_2, after solution of BLP problem and want to know the effect of these variations on the optimal solution. Note that in these cases, the extended region of BLP problem and thereby the efficient set of the corresponding MOLP will be changed. Then, we can run the Glackin et al's algorithm from Step 2 of it (by considering optimal solution z^* as z_0) and using the $e^T s$ test on z^* to determine that the point z^* is efficient or find an initial efficient point z' which is near the z^* and dominated it.

2.1 Change in the cost vector of leader's function

Given an optimal efficient solution for linear BLP problem and suppose that the cost coefficient of one of the variables in leader function is changed from d_k to d'_k. As mentioned before, it is necessary to investigate two cases as BLP's optimality conditions: Feasibility and no improvement in leader's function, simultaneously. Note that, due to the Theorem **??** for transformation of the feasible region of the linear BLP problem into the efficient set of a multiple objective linear program, changing the coefficients of leader's function do not affect feasibility region of linear BLP problem. Therefore, the effect of this change on the final tableau will occur in the leader's function cost row and no improvement in leader's function may be lost. Consider the following cases:

Case I: x_k **(or** y_k**) is nonbasic.**

In this case d_B is not affected and hence $z_j = d_B B^{-1} a_j$ is not changed for any j. Therefore, we should calculate $z_k - d_k^{\text{new}}$ while the others remain unchanged. Then, run the Glackin et al's algorithm from Step 3 of it. Note that, if $z_k - d_k^{\text{new}} > 0$, then x_k will be eligible to enter the basis and B^*, X^*, Z^* may be changed (in this case, the point corresponding to subsequent tableau is efficient), otherwise B^*, X^*, Z^* will remain unchanged .

Case II: x_k **(or** y_k**) is basic.**

Here d_B is replaced by d_B^{new}. Let the new value of z_j be z_j^{new}, and calculate $z_j^{\text{new}} - d_j$ for all nonbasic variables in cost row. Then, run the Glackin et al.'s algorithm from Step 3 of it. Note that, if there exist some j with $z_j^{\text{new}} - d_j > 0$, then B^*, X^* and z^* may be changed. If no improving efficient edges can be found then both B^* and X^* will remain fixed but z^* will be modified according to the associated d_B.

2.2 Change in the cost vector of follower's function

Suppose that the cost coefficient c_k will be changed while the others remain fixed at their present values. According to the coefficients of follower's function in determining the efficient region or the feasible region of BLP problem, any change in c may change the efficient region and make the optimal solution corresponding to the previous efficient region infeasible. So we should check the efficiency of the optimal solution, which can be done by $e^T s$ test. If the optimal point is efficient, the BLP was easy and (x^*, y^*) solves BLP, but if (x^*, y^*) is not efficient, then we find (x', y') that is an efficient extreme point and continue the algorithm from this point.

3. Numerical Examples

Consider the following example from Glackin et al. (2009).

$$\begin{aligned}
&\min_{x,y} \ -2x_1 + 4x_2 + 3y \\
&\text{s.t} \quad x_1 - x_2 \le -1 \\
&\qquad\ \ x_1, x_2 \ge 0 \\
&\min_{y} \ -y \\
&\text{s.t} \quad x_1 + x_2 + y \le 4 \\
&\qquad\ \ 2x_1 + 2x_2 + y \le 6 \\
&\qquad\ \ y \ge 0
\end{aligned} \tag{3.1}$$

The optimal tableau is given by Table 1.

Table 1. Optimal tableau

	x_1	x_2	y	S_5	S_6	S_7	S_8	RHS
Z	0	0	$-\frac{5}{2}$	-3	0	$\frac{1}{2}$	0	6
x_2	0	1	$\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	0	2
S_6	0	0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	1
x_1	1	0	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$	0	1
S_8	0	0	$\frac{5}{2}$	3	0	$-\frac{1}{2}$	1	0

(a) Suppose that $d_y = 3$ (coefficient of y) replaced by $\frac{1}{3}$. Since y is nonbasic and this change is in the leader's function, then $z_y - d_y^{\text{new}} = \frac{1}{6}$ and all other $z_j - d_j$ are unaffected. The new $z_y - d_y$ is now positive so y can enter the basis, from this point we find other improving efficient edges neither at Step 3 (the next point $(0, 1, \frac{2}{3})$ is not efficient) nor at Step 4 with added cutting hyperplane $-2x_1 + 4x_2 + \frac{1}{3}y \le 6$. Therefore, previous point is still optimum.

(b) Let d_2 represents the cost coefficient of x_2 whose present value is 4. Suppose d_2 has to be changed to 6. After changing the cost coefficient of x_2 from the present value 4 to 6 the modified relative cost coefficient will be $(0, 0, -2, -4, 0, 1)$. So s_7 can be brought in to the basic vector, but during the running of step 3, no improving efficient edges are found. Hence, at step 4 we add a cut $-2x_1 + 6x_2 + 3y \le 10$. In step 5, we examine each vertex of cutting hyperplane and find that the previous optimal value remains fixed.

(c) In linear BLP (3.1), suppose we change cost coefficient of variable y from its present value of -1 to -2. Then, based on the running of Step 2 we have the model (3.2) and due to $e^T s$ test we find that the previous optimal point is efficient and remains optimum for linear BLP problem (3.1).

$$
\begin{aligned}
\min_{x,S} \quad & -S_1 - S_2 - S_3 - S_4 \\
\text{s.t.} \quad & -x_1 + S_1 = -1 \\
& -x_2 + S_2 = -2 \\
& x_1 + x_2 + S_3 = 3 \\
& -2y + S_4 = 0 \\
& x_1 - x_2 \le -1 \\
& x_1 + x_2 + y \le 4 \\
& 2x_1 + 2x_2 + y \le 6 \\
& x_1, x_2, y \ge 0 \\
& S_1, S_2, S_3, S_4 \ge 0
\end{aligned}
\tag{3.2}
$$

4. Conclusion

This paper is deal with some variations in linear bilevel programming problems and their effects on the final optimal solution. The performance of the sensitivity analysis approach might be improved by using a more sophisticated method to find a feasible solution of linear BLP or to optimize the linear function over the efficient set.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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