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Weighted G^0- and G^1-Degree Reduction of Disk Bézier Curves

Research Article

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Abstract. A Bézier curve in the plane whose control points are disks is called a disk Bézier curve. In this paper we introduce a novel approach to find weighted degree reduction of disk Bézier curve with G^0- and G^1- continuity at the boundary. Numerical examples are provided to demonstrate the efficiency and simplicity of the proposed method. Moreover some figures are provided to illustrate the comparisons with other methods.

Keywords. Disk Bézier curves; Degree reduction; G^0- and G^1-continuity.

MSC. 41A10; 68U07

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1. Preliminaries

In this section Bernstein polynomials and Bézier curves are defined and some of their properties are stated. The Bernstein polynomials of degree n are defined by

$$B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i, \quad i = 0,1,\dots,n.$$

The product of two Bernstein polynomials with chebyshev weight function of second kind is given by

$$B_i^m(t)B_j^n(t)2t(1-t) = \frac{2\binom{m}{i}\binom{n}{j}}{\binom{m+n+2}{i+j+1}}B_{i+j+1}^{m+n+2}(t).$$

The integral of these product with weight function is an $(m+1)\times(n+1)$-matrix $G_{m,n}$, whose elements are

$$g_{ij}=\int_0^1 B_i^m(t)B_j^n(t)2t(1-t)dt=\frac{2\binom{m}{i}\binom{n}{j}}{(m+n+3)\binom{m+n+2}{i+j+1}},\quad i=0,\dots,m,\ j=0,1,\dots,n. \tag{1.1}$$

The matrix $G_{m,m}$ is real, symmetric, and positive definite [8].

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}^+$ be the set of non-negative real numbers.

A disk centred at $p=(x_0,y_0)\in\mathbb{R}^2$ with radius $r_0\in\mathbb{R}^+$ is given by

$$(p):=(x_0,y_0)_{r_0}:=\{(x,y)\in\mathbb{R}^2:(x-x_0)^2+(y-y_0)^2\le {r_0}^2\}. \tag{1.2}$$

where $(x_0,y_0)\in\mathbb{R}^2$ is the center and $r_0\in\mathbb{R}^+$ is the radius. For any two disks $(p)=(x_0,y_0)_{r_0}$ and $(q)=(x_1,y_1)_{r_1}$, addition and scalar multiplication are defined as follows:

$$(p)+(q)=(x_0+x_1,y_0+y_1)_{(r_0+r_1)} \tag{1.3}$$

$$s(p)=(sx_0,sy_0)_{|s|r_0},\quad \text{for } s\in\mathbb{R}, \tag{1.4}$$

where $|s|$ is the absolute value of s.

For constants s_i and disks $(x_i,y_i)_{r_i}$, $i=0,1,\dots,n$, the last definition can be generalized as

$$\sum_{i=0}^n s_i(x_i,y_i)_{r_i}=\left(\sum_{i=0}^n s_ix_i,\sum_{i=0}^n s_iy_i\right)_{\sum_{i=0}^n|s_i|r_i}. \tag{1.5}$$

Now, we are ready to define the disk Bézier curves in the following definition.

Definition 1.1 (Disk Bézier curves)**.** A disk Bézier curve of degree n corresponding to $n+1$ disks $(p_i)=(x_i,y_i)_{r_i}$, $i=0,1,\dots,n$, is defined as follows:

$$(P_n)(t):=\sum_{i=0}^n(p_i)B_i^n(t),\quad 0\le t\le 1, \tag{1.6}$$

where $B_i^n(t)$ are the Bernstein polynomials of degree n, and (p_i), $i=0,1,\dots,n$, are the control disks.

An example of quadratic disk Bézier curve is shown in Figure 1.

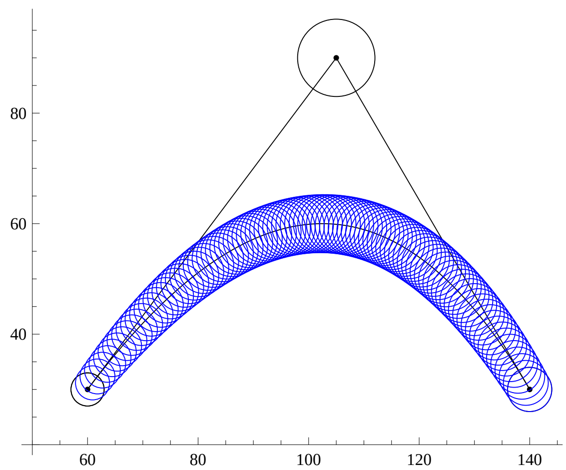


Figure 1. Quadratic disk Bézier curve.

Disk Bézier curve $(P_n)(t)$ can be expressed explicitly in terms of center curve and radius curve as

$$(P_n)(t) := (p_n(t))_{r(t)}, \tag{1.7}$$

where

$$p_n(t) := \sum_{i=0}^{n} p_i B_i^n(t) = \sum_{i=0}^{n} (x_i, y_i) B_i^n(t)$$

and

$$r(t) = \sum_{i=0}^{n} r_i B_i^n(t)$$

are the center curve and the radius curve of $(P_n)(t)$ with control points $p_i = (x_i, y_i)$, $i = 0, 1, \dots, n$ and r_i, $i = 0, 1, \dots, n$, respectively.

The Delta operator Δ is defined on disks as analogous generalization of Δ on Bézier points (see [2]).

Definition 1.2 (Delta operator)**.** Define the operator Δ on the disk (p_i) as follows:

$$\Delta^0(p_i) = (p_i),\ \Delta^k(p_i) = \Delta^{k-1}(p_{i+1}) - \Delta^{k-1}(p_i), \quad k \ge 1,\ i = 0, 1, \dots, n-k.$$

The k-th derivatives of disk Bézier curves can be given in formulas similar to the formulas of the derivatives of Bézier curves (see [2]).

The k-th derivatives of the disk Bézier curve in (1.6) at $t = 0, 1$ are given in terms of the delta operator as follows:

$$\frac{d^k}{dt^k}(P_n)(0) = \frac{n!}{(n-k)!}\Delta^k(p_0), \tag{1.8}$$

$$\frac{d^k}{dt^k}(P_n)(1) = \frac{n!}{(n-k)!}\Delta^k(p_{n-k}). \tag{1.9}$$

In the following section we review geometric continuity of disk Bézier curves and discuss the problem of degree reduction of disk Bézier curves.

2. G^k-Degree Reduction of Disk Bézier Curves

Geometric Continuity is denoted by G^k. In [12], weighted G^1 multi degree reduction of Bézier curves is considered. Disk Bézier curves $(P_n)(t)$ and $(Q_m)(t)$ are said to be G^k-continuous at $t = 0, 1$ if there exists a strictly increasing parametrization $s(t) : [0, 1] \to [0, 1]$ with $s(0) = 0$, $s(1) = 1$, and

$$(Q_m)^{(i)}(t) = (P_n)^{(i)}(s(t)), \quad t = 0, 1, \quad i = 0, 1, \dots, k. \tag{2.1}$$

The problem of degree reduction of disk Bézier curve can be stated as follow: for a given disk Bézier curve $(P_n)(t)$ of degree n, find a disk Bézier curve $(Q_m)(t)$ of degree m, where $m < n$, such that $(Q_m)(t)$ bounds $(P_n)(t)$ as tight as possible. In this paper, we included Chebyshev weight function of second kind and consider geometric continuity conditions between the adjacent disk Bézier curves. This means $(Q_m)(t)$ has to satisfy the following conditions:

(1) $(P_n)(t)$ and $(Q_m)(t)$ are G^k-continuous,

(2) the L_2-error between $(P_n)(t)$ and $(Q_m)(t)$ is minimum, and

(3) $(P_n)(t) \subseteq (Q_m)(t)$, $0 \le t \le 1$.

The curves $(P_n)(t)$ and $(Q_m)(t)$ can be written in matrix form as

$$(P_n)(t) = \sum_{i=0}^{n} (p_i) B_i^n(t) =: B_n(P_n), \qquad 0 \le t \le 1, \tag{2.2}$$

$$(Q_m)(t) = \sum_{i=0}^{m} (q_i) B_i^m(t) =: B_m(Q_m), \quad 0 \le t \le 1, \tag{2.3}$$

where $B_n = (B_0^n(t), B_1^n(t), \dots, B_n^n(t))$ and $(P_n) = ((p_0), \dots, (p_n))^t$ are row vectors formed by Bernstein polynomials and column vectors formed by the Bézier disks respectively. Similarly, B_m and (Q_m) are defined alike.

We use L_2-norm to measure distances between the center Bézier curves p and q, and the radius Bézier curves r and $\tilde{r}$. Our strategy in this paper is to minimize

$$\varepsilon = \int_0^1 \| B_n(P_n) - B_m(Q_m) \|^2 2t(1-t)dt. \tag{2.4}$$

Under the satisfaction of one of the conditions:

(1) G^0-continuity at the boundaries, and

(2) G^1-continuity at the boundaries.

In the following sections, we investigate, in particular, the cases of G^0-, and G^1-continuity with degree reduction of disk Bézier curves.

3. G^0-Degree Reduction

G^0-continuity of $(Q_m)(t)$ and $(P_n)(t)$ at the disks corresponding to $t = 0, 1$, requires the satisfaction of the following two conditions:

$$(Q_m)(t) = (P_n)(s(t)), \quad t = 0, 1. \tag{3.1}$$

This means the two curves have common end disks:

$$(q_0) = (p_0), \quad (q_m) = (p_n). \tag{3.2}$$

The disks (q_0) and (q_m) are determined by G^0-continuity conditions at the boundaries. The elements of (Q_m) are decomposed into two parts. The part of constraints control disks $(Q_m)^c = [(q_0), (q_m)]^t$ and the part of free control disks $(Q_m)^f = (Q_m) \setminus (Q_m)^c = [(q_1), \dots, (q_{m-1})]^t$. Similarly, B_m is decomposed in the same way. The distance between $(Q_m)(t)$ and $(P_n)(t)$ is measured using L_2-norm; therefore, the error term becomes

$$\begin{aligned} \varepsilon &= \int_0^1 \| B_n(P_n) - B_m(Q_m) \|^2 t(1-t)dt \\ &= \int_0^1 \| B_n(P_n) - B_m^c(Q_m)^c - B_m^f(Q_m)^f \|^2 t(1-t)dt. \end{aligned} \tag{3.3}$$

Differentiating with respect to the unknown control disks $(Q_m)^f$ we get

$$\frac{\partial \varepsilon}{\partial (Q_m)^f} = 2\int_0^1 B_m^f \ (B_n(P_n) - B_m^c(Q_m)^c - B_m^f(Q_m)^f)\ t(1-t)dt.$$

Evaluating the integral and equating to zero gives

$$\frac{\partial \varepsilon}{\partial (Q_m)^f} = G_{m,n}^p(P_n) - G_{m,m}^c(Q_m)^c - G_{m,m}^f(Q_m)^f = 0, \tag{3.4}$$

where

$$\begin{aligned} G_{m,n}^p &:= G_{m,n}(1,\dots,m-1;0,1,\dots,n),\\ G_{m,m}^c &:= G_{m,m}(1,\dots,m-1;0,m),\\ G_{m,m}^f &:= G_{m,m}(1,\dots,m-1;1,\dots,m-1),\end{aligned}$$

and $G_{m,n}(\dots;\dots)$ is the sub-matrix of $G_{m,n}$ formed by the indicated rows and columns.

Now the case of G^0-degree reduction is illustrated. The center curve of disk Bézier curve is expanded into x and y components together with their radius curve. Therefore, our system of equations has $\tilde{x}_k$, $\tilde{y}_k$, $\tilde{r}_k$ variables for $k = 1,\dots,m-1$.

The following vectors are defined to express the linear system in explicit form:

$$\begin{aligned} P_n &= [x_0,\dots,x_n,y_0,\dots,y_n,r_0,\dots,r_n]^t,\\ Q_m^F &= [\tilde{x}_1,\dots,\tilde{x}_{m-1},\tilde{y}_1,\dots,\tilde{y}_{m-1},\tilde{r}_1,\dots,\tilde{r}_{m-1}]^t,\\ Q_m^C &= [\tilde{x}_0,\ \tilde{x}_m,\ \tilde{y}_0,\ \tilde{y}_m,\ \tilde{r}_0,\ \tilde{r}_m]^t.\end{aligned}$$

Let $\oplus$ be the direct sum. Define the matrices

$$\begin{aligned} G_{m,n}^P &= G_{m,n}^p \oplus G_{m,n}^p \oplus G_{m,n}^p,\\ G_{m,m}^C &= G_{m,m}^c \oplus G_{m,m}^c \oplus G_{m,m}^c,\\ G_{m,m}^F &= G_{m,m}^f \oplus G_{m,m}^f \oplus G_{m,m}^f.\end{aligned} \tag{3.5}$$

The matrix $G_{m,m}^F$ inherits the properties of the Gram matrix $G_{m,m}^f$.

The coordinate form of the expansion of (3.4) becomes

$$G_{m,m}^F Q_m^F = G_{m,n}^P P_n - G_{m,m}^C Q_m^C. \tag{3.6}$$

From (3.6) we can find the unknowns as

$$Q_m^F = (G_{m,m}^F)^{-1}\Big(G_{m,n}^p P_n - G_{m,m}^C Q_m^C\Big). \tag{3.7}$$

Note that the matrix $G_{m,m}^F$ is not singular. Moreover, it is real, symmetric, and positive definite; therefore, the solution of the system always exist.

4. G^1-Degree Reduction

G^1-continuity of $(Q_m)(t)$ and $(P_n)(t)$ at the disks corresponding to $t = 0,1$, requires the two curves $(P_n)(t)$ and $(Q_m)(t)$ to be G^0-continuous and satisfy the following conditions

$$(Q_m)'(t) = s'(t)(P_n)'(s(t)), \quad s'(t) > 0,\ t = 0,1. \tag{4.1}$$

This means that the direction of the tangents at the two end disks of (Q_m) and (P_n) should coincide, but they need not to be of equal length. As in [10, 11] $s'(i) = \delta_i$, $i = 0,1$, are used to get

$$(Q_m)'(t) = \delta_i(P_n)'(i), \quad i = 0,1. \tag{4.2}$$

The two control disks at either end of the curve are obtained by solving (3.1) and (4.2):

$$\begin{aligned}
&(q_0) = (p_0), && (q_m) = (p_n),\\
&(q_1) = (p_0) + \frac{n}{m}\Delta(p_0)\delta_0, && (q_{m-1}) = (p_n) - \frac{n}{m}\Delta(p_{n-1})\delta_1.
\end{aligned}$$

The disks $(q_0),(q_1),(q_{m-1})$ and (q_m) are determined by G^1-continuity conditions at the boundaries; accordingly, the elements of (Q_m) are decomposed into two parts. The part of constraints control disks $(Q_m)^c = [(q_0),(q_1),(q_{m-1}),(q_m)]^t$ and the part of free control disks $(Q_m)^f = (Q_m)\backslash(Q_m)^c = [(q_2),\ldots,(q_{m-2})]^t$. Similarly, B_m is decomposed in the same way. The distance between $(Q_m)(t)$ and $(P_n)(t)$ is measured using L_2-norm; therefore, the error term becomes

$$\begin{aligned}
\varepsilon &= \int_0^1 \|B_n(P_n) - B_m(Q_m)\|^2 t(1-t)dt\\
&= \int_0^1 \|B_n(P_n) - B_m^c(Q_m)^c - B_m^f(Q_m)^f\|^2 t(1-t)dt.
\end{aligned} \tag{4.3}$$

The error $\varepsilon := \varepsilon((Q_m)^f, \delta_0, \delta_1)$ is a function of $(Q_m)^f, \delta_0$, and δ_1. Differentiating with respect to the unknown control disks $(Q_m)^f$ we get

$$\frac{\partial\varepsilon}{\partial(Q_m)^f} = 2\int_0^1 B_m^f.(B_n(P_n) - B_m^c(Q_m)^c - B_m^f(Q_m)^f).t(1-t)dt.$$

Evaluating the integral and equating to zero gives

$$\frac{\partial\varepsilon}{\partial(Q_m)^f} = G_{m,n}^p(P_n) - G_{m,m}^c(Q_m)^c - G_{m,m}^f(Q_m)^f = 0, \tag{4.4}$$

where

$$\begin{aligned}
G_{m,n}^p &:= G_{m,n}(2,\ldots,m-2;0,1,\ldots,n),\\
G_{m,m}^c &:= G_{m,m}(2,\ldots,m-2;0,1,m-1,m),\\
G_{m,m}^f &:= G_{m,m}(2,\ldots,m-2;2,\ldots,m-2),
\end{aligned}$$

and $G_{m,n}(\ldots;\ldots)$ is the sub-matrix of $G_{m,n}$ formed by the indicated rows and columns.

Differentiating (4.3) with respect to δ_i and equating to zero gives

$$\frac{\partial \varepsilon}{\partial \delta_0} = \left(G^1_{m,n}(P_n) - G^{1;c}_{m,m}(Q_m)^c - G^{1;f}_{m,m}(Q_m)^f\right) \cdot \Delta(p_0) = 0, \tag{4.5}$$

$$\frac{\partial \varepsilon}{\partial \delta_1} = \left(G^{m-1}_{m,n}(P_n) - G^{m-1;c}_{m,m}(Q_m)^c - G^{m-1;f}_{m,m}(Q_m)^f\right) \cdot \Delta(p_{n-1}) = 0, \tag{4.6}$$

where for $j = 1,\ m-1$:

$$\begin{aligned} G^j_{m,n} &:= G_{m,n}(j;0,1,\dots,n), \\ G^{j;c}_{m,m} &:= G_{m,m}(j;0,1,m-1,m), \\ G^{j;f}_{m,m} &:= G_{m,m}(j;2,\dots,m-2). \end{aligned} \tag{4.7}$$

The center curve of disk Bézier curve is expanded into x and y components together with their radius curve. Therefore, the variables of our system of equations are $\tilde{x}_k$, $\tilde{y}_k$, $\tilde{r}_k$, $k = 2,\dots,m-2$, δ_0 and δ_1. To express the system in a clear form, we have to decompose each of q_1 and q_{m-1} into a constant part and a part involving δ_0 and δ_1, respectively. Let v_1 and v_{m-1} be the constant part of q_1 and q_{m-1} respectively. Similarly $\tilde{r}_1$ and $\tilde{r}_{m-1}$ are decomposed alike. Let s_1 and s_{m-1} be the constant parts of $\tilde{r}_1$ and $\tilde{r}_{m-1}$ respectively. Hence

$$v_1 = p_0, \quad v_{m-1} = p_n,$$

$$s_1 = r_0, \quad s_{m-1} = r_n.$$

The following vectors are defined to express the linear system in explicit form:

$$\begin{aligned} P_n &= [x_0,\dots,x_n,y_0,\dots,y_n,r_0,\dots,r_n]^t, \\ Q^F_m &= [\tilde{x}_2,\dots,\tilde{x}_{m-2},\tilde{y}_2,\dots,\tilde{y}_{m-2},\tilde{r}_2,\dots,\tilde{r}_{m-2},\delta^c_0,\delta^c_1,\delta^r_0,\delta^r_1]^t, \\ Q^C_m &= [\tilde{x}_0,\ v^x_1,\ v^x_{m-1},\ \tilde{x}_m,\ \tilde{y}_0,\ v^y_1,\ v^y_{m-1},\ \tilde{y}_m,\tilde{r}_0,\ s_1,\ s_{m-1},\ \tilde{r}_m]^t. \end{aligned}$$

Define the matrices A, B, $L^c_{m,n}$, $L^{cc}_{m,m}$, $L^{fc}_{m,m}$, $L^r_{m,n}$, $L^{cr}_{m,m}$, $L^{fr}_{m,m}$ as follows

$$A = \begin{bmatrix} \Delta p_0 & 0 \\ 0 & \Delta p_{n-1} \end{bmatrix} \begin{bmatrix} G_{m,m}(1,1) & G_{m,m}(1,m-1) \\ G_{m,m}(m-1,1) & G_{m,m}(m-1,m-1) \end{bmatrix} \begin{bmatrix} \Delta p_0 & 0 \\ 0 & \Delta p_{n-1} \end{bmatrix},$$

$$B = \begin{bmatrix} \Delta r_0 & 0 \\ 0 & \Delta r_{n-1} \end{bmatrix} \begin{bmatrix} G_{m,m}(1,1) & G_{m,m}(1,m-1) \\ G_{m,m}(m-1,1) & G_{m,m}(m-1,m-1) \end{bmatrix} \begin{bmatrix} \Delta r_0 & 0 \\ 0 & \Delta r_{n-1} \end{bmatrix},$$

$$L^c_{m,n} = \begin{bmatrix} G^1_{m,n}\Delta x_0 & G^1_{m,n}\Delta y_0 \\ G^{m-1}_{m,n}\Delta x_{n-1} & G^{m-1}_{m,n}\Delta y_{n-1} \end{bmatrix}, \qquad L^r_{m,n} = \begin{bmatrix} G^1_{m,n}\Delta r_0 \\ G^{m-1}_{m,n}\Delta r_{n-1} \end{bmatrix},$$

$$L^{cc}_{m,m} = \begin{bmatrix} G^{1;c}_{m,m}\Delta x_0 & G^{1;c}_{m,m}\Delta y_0 \\ G^{m-1;c}_{m,m}\Delta x_{n-1} & G^{m-1;c}_{m,m}\Delta y_{n-1} \end{bmatrix}, \qquad L^{cr}_{m,m} = \begin{bmatrix} G^{1;c}_{m,m}\Delta r_0 \\ G^{m-1;c}_{m,m}\Delta r_{n-1} \end{bmatrix},$$

$$L^{fc}_{m,m} = \begin{bmatrix} G^{1;f}_{m,m}\Delta x_0 & G^{1;f}_{m,m}\Delta y_0 \\ G^{m-1;f}_{m,m}\Delta x_{n-1} & G^{m-1;f}_{m,m}\Delta y_{n-1} \end{bmatrix}, \qquad L^{fr}_{m,m} = \begin{bmatrix} G^{1;f}_{m,m}\Delta r_0 \\ G^{m-1;f}_{m,m}\Delta r_{n-1} \end{bmatrix}.$$

Let $\oplus$ be the direct sum. Define the matrices

$$
\begin{aligned}
G^{p++}_{m,n} &= G^{p}_{m,n} \oplus G^{p}_{m,n} \oplus G^{p}_{m,n}\,,\\
G^{c++}_{m,m} &= G^{c}_{m,m} \oplus G^{c}_{m,m} \oplus G^{c}_{m,m}\,,\\
G^{f++}_{m,m} &= G^{f}_{m,m} \oplus G^{f}_{m,m} \oplus G^{f}_{m,m}\,.
\end{aligned} \tag{4.8}
$$

Further define $L^{+}_{m,n}, L^{c+}_{m,m}, L^{f+}_{m,m}$ as

$$
\begin{aligned}
L^{+}_{m,n} &= L^{c}_{m,n} \oplus L^{r}_{m,n}\,,\\
L^{c+}_{m,m} &= L^{cc}_{m,m} \oplus L^{cr}_{m,m}\,,\\
L^{f+}_{m,m} &= L^{fc}_{m,m} \oplus L^{fr}_{m,m}\,.
\end{aligned}
$$

After some mathematical manipulations the coordinate form of the expansion of (4.4) together with (4.5) and (4.6) becomes

$$
G^{F}_{m,m} Q^{F}_{m} = G^{P}_{m,n} P_n - G^{C}_{m,m} Q^{C}_{m}, \tag{4.9}
$$

where

$$
G^{P}_{m,n} = \begin{bmatrix} G^{p++}_{m,n} \\ L^{+}_{m,n} \end{bmatrix}, \quad G^{C}_{m,m} = \begin{bmatrix} G^{c++}_{m,m} \\ L^{c+}_{m,m} \end{bmatrix}, \quad G^{F}_{m,m} = \begin{bmatrix} G^{f++}_{m,m} & \frac{n}{m}(L^{f+}_{m,m})^t \\ L^{f+}_{m,m} & \frac{n}{m}(A \oplus B) \end{bmatrix}.
$$

The square matrix $G^{F}_{m,m}$ is a block matrix formed by $G^{f++}_{m,m}$, $(L^{f+}_{m,m})^t$, $L^{f+}_{m,m}$, and $A \oplus B$. The matrix $G^{f++}_{m,m}$ is a positive definite, and the matrix $A \oplus B$ excluding Δc_0 and Δc_{n-1} parts, is also positive definite. Therefore, the matrix $G^{F}_{m,m}$ is non-singular [11].

From (4.9) we can find the unknowns as

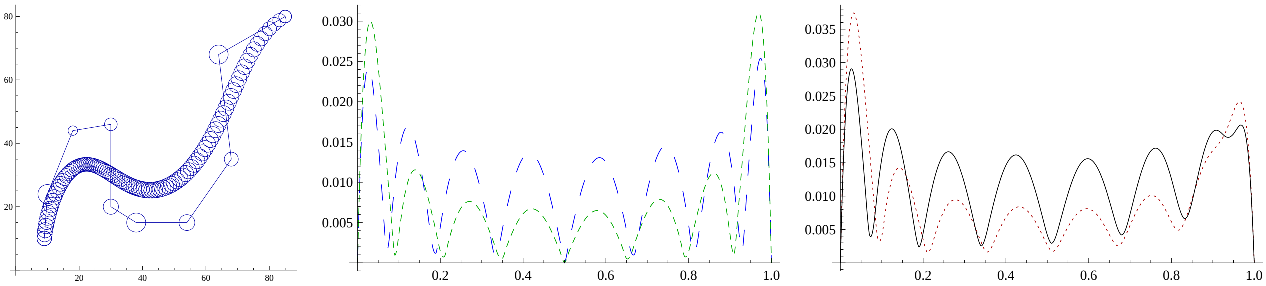
$$
Q^{F}_{m} = (G^{F}_{m,m})^{-1}\left(G^{P}_{m,n} P_n - G^{C}_{m,m} Q^{C}_{m}\right). \tag{4.10}
$$

5. Examples and Comparisons

In this section, we illustrate four examples to demonstrate the effectiveness of the proposed method and compares the error functions produced by weighted G^0-, weighted G^1-, G^0-, and G^1-degree reduction. For the purpose of comparison different kind of lines are used as follows:

- long-dashed: Weighted G^0 (WG^0),
- short-dashed: G^0,
- dotted: Weighted G^1 (WG^1),
- solid: G^1.

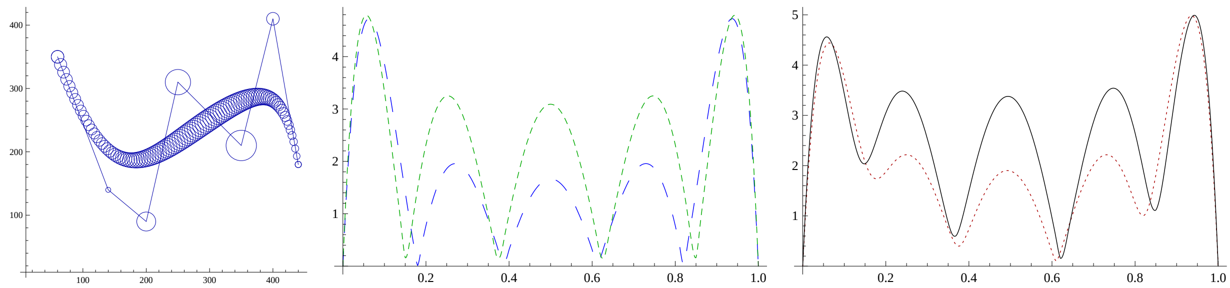
Example 1 (see Example 1 in [11], see also [1])**.** A disk Bézier curve $(P_n)(t)$ of degree nine is reduced to disk Bézier curve $(Q_m)(t)$ of degree eight using WG^0- and WG^1-degree reduction methods. Figure 2 depicts the curve and comparisons of the error functions.



(a) (b) (c)

Figure 2. Example 1, (a) original curve, (b) comparison of WG^0 and G^0 (c) comparison of WG^1 and G^1.

Example 2 (see Example 2 in [11], see also [1]). A disk Bézier curve $(P_n)(t)$ of degree six is reduced to disk Bézier curve $(Q_m)(t)$ of degree five using WG^0- and WG^1-degree reduction methods. Figure 3 depicts the curve and comparisons of the error functions.



(a) (b) (c)

Figure 3. Example 2, (a) original curve, (b) comparison of WG^0 and G^0 (c) comparison of WG^1 and G^1.

Example 3 (see Example 3 in [11], see also [3]). A disk Bézier curve $(P_n)(t)$ of degree eight is reduced to disk Bézier curve $(Q_m)(t)$ of degree five using WG^0- and WG^1-degree reduction methods. Figure 4 depicts the curve and comparisons of the error functions.

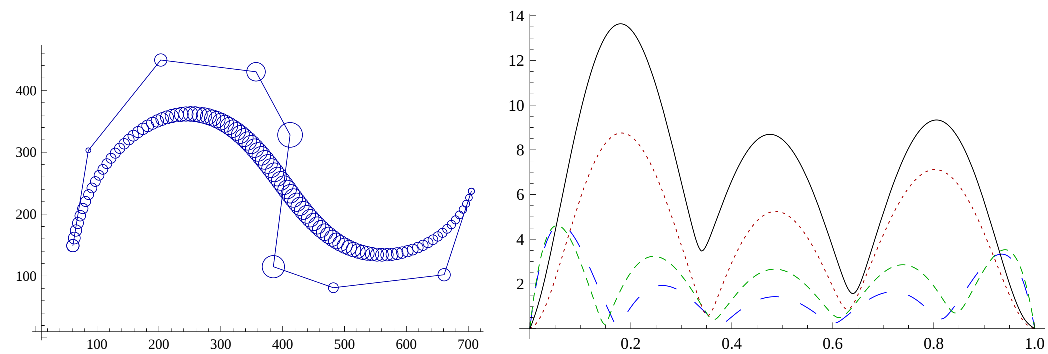


Figure 4. Example 3, original curve (left), comparisons of the error functions (right).

Example 4 (see Example 4 in [11], see also [4]). A disk Bézier curve $(P_n)(t)$ of degree seven is reduced to disk Bézier curve $(Q_m)(t)$ of degree six using WG^0- and WG^1-degree reduction methods. Figure 5 depicts the curve and comparisons of the error functions.

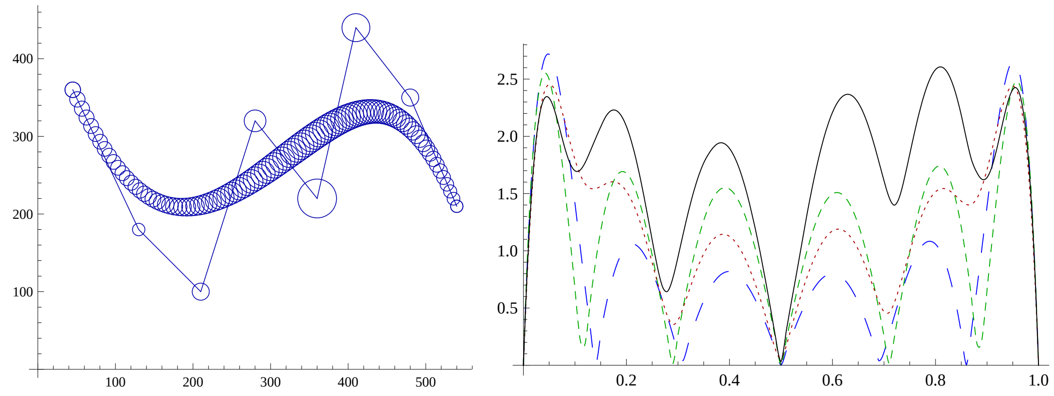


Figure 5. Example 4, original curve (left), comparisons of the error functions (right).

6. Conclusions

In this paper we introduced a weighted degree reduction of disk Bézier curve with G^0- and G^1-continuity at the end disks. Due to the effect of the weight function, our proposed WG^0 and WG^1 has a smaller approximation error at the center than the methods in [11].The examples and figures show the efficiency, simplicity, and applicability of the method.

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Competing Interests

The authors declare that he has no competing interests.

Author's Contributions

All the authors read and approved the final manuscript.

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