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Research Article

Eigenvalues variation of the *p*-Laplacian under the Yamabe Flow on *SM*

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Abstract. Let (M,F) be a closed Finsler manifold. Studying geometric flows and the eigenvalues of geometric operators are powerful tools when dealing with geometric problems. In this article we will consider the eigenvalue problem for the p-laplace operator for Sasakian metric acting on the space of functions on SM. We find the first variation formula for the eigenvalues of p-Laplacian on SM evolving by the Yamabe flow on M and give some examples.

Keywords. Yamabe flow; Finsler manifold; p-Laplace operator

MSC. 53C44; 58C40

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1. Introduction

For a closed Finsler manifold (M,F), the eigenvalues of geometric operators is important in geometric analysis. In studying of the p-Laplace equation several parts of mathematics for instance: Calculus of Variation, Partial Differential Equation, Potential Theory and Analytic Function have a momentous impress. Recently, there are many research about properties of the eigenvalues of p-Laplacian on Finsler manifolds and Riemannian manifolds to estimate the spectrum in terms of the other geometric structures of the manifold ([6, 8, 14, 16]).

Also, geometric flows have been a topic of active research interest in mathematics and other sciences ([10–13]). Yamabe flow ([7, 9, 17]) which is extension of Hamilton's Ricci flow ([4, 10]) is the best known example of a geometric evolution equation. The Yamabe flow is related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold. One of the aims of such flows is to obtain metrics with special properties. Special cases arise when the metric is invariant under a group of transformations and this property is preserved by the flow.

Let M be a manifold with a Finsler metric g_0 (or F_0), the family $g(t)$ (or F_t) of Finsler metrics on M is called an un-normalized Yamabe flow when it satisfies the equations

$$\partial_t(g_{ij}) = -H_g g_{ij}, \quad g(0) = g_0, \tag{1.1}$$

it implies that

$$\partial_t \log F = -\frac{1}{2}H_g, \quad F(t=0) = F_o, \tag{1.2}$$

where $H_g = g^{ij}Ric_{ij}$, which introduced by Akbarzadeh in [1] and Ric is the Ricci tensor of $g(t)$, $Ric_{ij} = (\frac{1}{2}F^2\mathcal{R}ic)_{y^iy^j}$. In fact the Yamabe flow is a system of partial differential equations of parabolic type which was introduced by Hamilton on Riemannian manifolds for the first time in 1982 and author with A. Razavi (see [3]) studied Yambe flow equation in Berwald manifold. The Yamabe flow has been proved to be a very useful tool to improve metrics in Finsler geometry, when M is compact. One often considers the normalized Yamabe flow

$$\partial_t g = (-H_g + Avg(H_g))g, \quad g(0) = g_0, \tag{1.3}$$

it implies that

$$\partial_t \log F = \frac{1}{2}(-H_g + Avg(H_g)), \quad F(0) = F_0, \tag{1.4}$$

where $Avg(H_g) = \frac{1}{Vol_{SM}}\int_{SM} H_g dV_{SM}$ and under this normalized Yamabe flow, the volume of the solution metrics remains constant in time. Short time exitance and uniqueness for solution to the Yamabe flow on $[0,T)$ have been shown by T. Aubin in [2] and by A. Bahri in [5] for Riemannian manifolds and by the author with A. Razavi in [3] for Berwald manifolds.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. For a point $x \in M$, denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. Any element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$.

Definition 2.1. A Finsler metric on a manifold M is a function $F : TM_0 \to [0,\infty)$ which has the following properties:

(i) $F(x,\alpha y) = \alpha F(x,y)$, $\forall\ \alpha > 0$;

(ii) $F(x,y)$ is C^∞ on TM_0 ;

(iii) For any non-zero tangent vector $y \in T_xM$, the associated quadratic form $g_y : T_xM \times T_xM \to \mathbf{R}$ on TM is an inner product, where

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial r}\left[F^2(x,y+su+rv)\right]\Big|_{s=r=0}.$$

The pair (M,F) is called a Finsler manifold.

Let us denote by S_xM the set consisting of all rays $[y] := \{\lambda y | \lambda > 0\}$, where $y \in T_xM_0$. The sphere bundle of M, i.e. SM, is the union of S_xM's:

$$SM = \cup_x S_xM$$

SM has a natural $(2n-1)$-dimensional manifold structure. We denote the elements of SM by $(x,[y])$ where $y \in T_xM_0$. If there is not any confusion we write (x,y) for $(x,[y])$. In a local coordinate system (x^i, y^i) we have $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$ and $(g^{ij}) := (g_{ij})^{-1}$.

The geodesics of F are characterized locally by $\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$ where

$$G^i = \frac{1}{4} g^{il}\left\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right\} y^j y^k. \tag{2.1}$$

Definition 2.2. The coefficients of the Riemann curvature $R_y = R^i_k dx^i \otimes \frac{\partial}{\partial x^i}$ are given by

$$R^i_{\ k} := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.2}$$

The Ricci scalar function of F is given by $\mathcal{R}ic := \frac{1}{F^2} R^i_{\ i}$. A companion of the Ricci scalar is the Ricci tensor

$$Ric_{ij} := \left(\frac{1}{2} F^2 \mathcal{R}ic\right)_{y^i y^j}. \tag{2.3}$$

Definition 2.3. A Finsler metric is said to be an Einstain metric if the Ricci scalar function is a function of x alone, equivalently $Ric_{ij} = \mathcal{R}(x) g_{ij}$ (see [15]).

Definition 2.4. Let (M,F) be a Finsler manifold, the Sasakian metric $\tilde{g}$ of g on TM_0 is defined as

$$\tilde{g} = g_{ij} dx^i \otimes dx^j + g_{ij}\frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F} \tag{2.4}$$

then $\tilde{g}$ is a Riemannian metric on TM_0 and $\left\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^i}\right\}$ is a coordinate bases on TM_0, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j}$ and $\left\{dx^i, \frac{\delta y^i}{F}\right\}$ is the dual of $\left\{\frac{\delta}{\delta x^i}, F\frac{\partial}{\partial y^i}\right\}$ where $\delta y^i = dy^i + G^i_j dx^j$.

Remark 2.5. The Levi-Civita connection $\tilde{\nabla}$ on TM_0 with respect to the Sasakian metric $\tilde{g}$ is locally expressed as follows:

$$\begin{aligned}
\tilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} &= -\left(C^k_{\ ij} + \frac{1}{2} R^k_{\ ij}\right)\frac{\partial}{\partial y^k} + F^k_{\ ij}\frac{\delta}{\delta x^k},\\
\tilde{\nabla}_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} &= C^k_{\ ij}\frac{\partial}{\partial y^k} - g_{ih}(F^h_{\ jk} - G^h_{\ jk}) g^{hk}\frac{\delta}{\delta x^k},\\
\tilde{\nabla}_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} &= F^k_{\ ij}\frac{\partial}{\partial y^k} + \left(C^k_{\ ij} + \frac{1}{2} g_{ih} R^h_{\ lj} g^{lk}\right)\frac{\delta}{\delta x^k} = \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} + G^k_{\ ij}\frac{\partial}{\partial y^k},
\end{aligned} \tag{2.5}$$

where

$$C^k_{\ ij} = \frac{1}{2} g^{kh}\frac{\partial g_{ij}}{\partial y^h}, \quad F^k_{\ ij} = \frac{1}{2} g^{kh}\left(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\right), \quad G^k_{\ ij} = \frac{\partial G^k_j}{\partial y^i},$$

and

$$R^k_{ij} = \frac{\delta G^k_i}{\delta x^j} - \frac{\delta G^k_j}{\delta x^i}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k_{ij}\frac{\partial}{\partial y^k}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = G^k_{ij}\frac{\partial}{\partial y^k}.$$

Lemma 2.6 ([15]). *For a Sasakian metric $\tilde{g}$ and any $f : TM \to \mathbf{R}$, there exists a unique vector field $Y \in \mathcal{X}(TM)$ such that*

$$\tilde{g}(Y, \tilde{X}) = df(\tilde{X}), \quad \forall\ \tilde{X} \in \mathcal{X}(TM), \tag{2.6}$$

where

$$\tilde{X} = X^i_1 \frac{\delta}{\delta x^i} + X^i_2 F \frac{\partial}{\partial y^i},$$

and X^i_1, X^i_2 are C^∞ function on TM. Here we take $Y = 0$ if $df = 0$.

Denote the vector field Y in (2.6) by $\tilde{\nabla} f$. We call $\tilde{\nabla} f$ the gradient of f and define the divergence $\operatorname{div}\tilde{X}$ as follows:

$$\operatorname{div}\tilde{X} = \operatorname{tr}\tilde{\nabla}\tilde{X} = \operatorname{tr}(\tilde{\nabla}\tilde{X})(\cdot,\cdot).$$

Definition 2.7. According to the above definition,the gradient of a function f is

$$\tilde{\nabla} f = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial}{\partial y^j}, \tag{2.7}$$

therefore, the norm of $\tilde{\nabla} f$ with respect to the Riemannian metric $\tilde{g}$ is given by

$$|\tilde{\nabla} f|^2 = \tilde{g}(\tilde{\nabla} f, \tilde{\nabla} f) = g^{ij}\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}. \tag{2.8}$$

Definition 2.8. Let M be a compact Finsler manifold. The Laplace operator of f on TM is defined as follows:

$$\begin{aligned}
\Delta f &= \operatorname{div}(\tilde{\nabla} f)\\
&= \tilde{g}^{ij}\partial_i\partial_j f - \tilde{g}^{ij}\tilde{\Gamma}^k_{ij}\partial_k f\\
&= g^{ij}\frac{\delta^2 f}{\delta x^i \delta x^j} + g^{ij}F^2\frac{\partial^2 f}{\partial y^i \partial y^j} - g^{ij}({}^1\tilde{\Gamma}^k_{ij})\frac{\delta f}{\delta x^k} - F g^{ij}({}^2\tilde{\Gamma}^k_{ij})\frac{\partial f}{\partial y^k}
\end{aligned}$$

where $\tilde{\Gamma}^k_{ij}$ is Christoffel symbols of $\tilde{\nabla}$ and

$${}^1\tilde{\Gamma}^k_{ij} = \left(\tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\right)dx^k \quad \text{or} \quad {}^1\tilde{\Gamma}^k_{ij} = \left(\tilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j}\right)\frac{\delta y^k}{F},$$

and

$${}^2\tilde{\Gamma}^k_{ij} = \left(\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}\right)dx^k \quad \text{or} \quad {}^2\tilde{\Gamma}^k_{ij} = \left(\tilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j}\right)\frac{\delta y^k}{F},$$

therefore using (2.7), we have:

$$\begin{aligned}
\Delta f = g^{ij}&\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial G^r_j}{\partial x^i}\frac{\partial f}{\partial y^r} - G^r_j\frac{\partial^2 f}{\partial x^i \partial y^r} - G^s_i\frac{\partial^2 f}{\partial y^s \partial x^j} + G^r_j G^s_i \frac{\partial^2 f}{\partial y^s \partial y^r}\right) + g^{ij}F^2\frac{\partial^2 f}{\partial y^i \partial y^j}\\
&+ g^{ij}\left(C^k_{ij} + \frac{1}{2}R^k_{ij}\right)\frac{\delta f}{\delta x^k} - F g^{ij}F^k_{ij}\frac{\delta f}{\delta x^k} - F g^{ij}C^k_{ij}\frac{\partial f}{\partial y^k} - F^2 g^{ij}g_{ih}(G^h_{jl} - F^h_{jl})g^{lk}\frac{\delta f}{\delta y^k}.
\end{aligned}$$

Definition 2.9. Let M be a compact Finsler manifold. The p-Laplace operator of $f : TM \to \mathbf{R}$, $f \in W^{1,p}(TM)$ for $1 < p < \infty$ is defined as follows:

$$\triangle_p f = \operatorname{div}(|\widetilde{\nabla} f|^{p-2}\widetilde{\nabla} f) = |\widetilde{\nabla} f|^{p-2}\Delta f + (p-2)|\widetilde{\nabla} f|^{p-4}(Hess\, f)(\widetilde{\nabla} f, \widetilde{\nabla} f), \tag{2.9}$$

where

$$(Hess\, f)(X,Y) = \widetilde{\nabla}(\widetilde{\nabla} f)(X,Y) = Y \cdot (X \cdot f) - (\widetilde{\nabla}_Y X) \cdot f, \quad X, Y \in \mathcal{X}(SM),$$

and in local coordinate, we have:

$$(Hess\, f)(\partial_i, \partial_j) = \partial_i \partial_j f - \widetilde{\Gamma}^k_{ij} \partial_k f.$$

Note 2.10. If f is a function of x alone, or suppose that is the lifting of $f : M \to \mathbf{R}$ then

$$\Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - g^{ij}({}^1\widetilde{\Gamma}^k_{ij})\frac{\partial f}{\partial x^k}\right). \tag{2.10}$$

2.1 Eigenvalues of the p-Laplacian

Definition 2.11. Let (M^n, F) be a compact Finsler manifold and $f : SM \to \mathbf{R}$. We say that λ is an eigenvalue of the p-Laplace operator whenever

$${}^{\tilde{g}}\Delta_p f + \lambda |f|^{p-2} f = 0 \tag{2.11}$$

then f is said to be the eigenfunction associated to λ, or equivalently they satisfy in

$$\lambda = \frac{\int_{SM} |\widetilde{\nabla} f|^p dv}{\int_{SM} |f|^p dv}. \tag{2.12}$$

Normalized eigenfunctions are defined as follows:

$$\int_{SM} f|f|^{p-2} dv = 0, \quad \int_{SM} |f|^p dv = 1. \tag{2.13}$$

Suppose that $(M^n, g(t))$ is a solution of the yamabe flow on the smooth manifold (M^n, g_0) in the interval $[0,T)$ and

$$\lambda(t) = \int_{SM} |\widetilde{\nabla} f(x,y)|^p dv_t \tag{2.14}$$

defines the evolution of an eigenvalue of p-Laplacian under the variation of $g(t)$ whose eigenfunction associated to $\lambda(t)$ is normalized. Suppose that for any metric $g(t)$ on M^n

$$\underset{p}{Spec}(\tilde{g}) = \{0 = \lambda_0(g) \le \lambda_1(g) \le \lambda_2(g) \le \cdots \le \lambda_k(g) \le \cdots\}$$

is the spectrum of $\Delta_p = {}^{\tilde{g}}\Delta_p$.

In what follows we assume the existence and C^1-differentiability of the elements $\lambda(t)$ and $f(t)$, under a Yamabe flow deformation $g(t)$ of a given initial metric. We prove some propositions about the problem of the spectrum variation under a deformation of the metric given by a Yamabe flow equation.

3. Variation of $\lambda(t)$

In this part, we will give some useful evolution formulas for $\lambda(t)$ under the Yamabe flow. Let $(M^n, g(t))$, $t \in [0,T)$, be a deformation of Finsler metric g_0. Assume that $\lambda(t)$ is the eigenvalue of

Δ_p, $f = f(x,y,t)$ satisfies

$$\Delta_p f + \lambda |f|^{p-2} f = 0$$

and $\int_{SM} |f|^p dv = 1$, using (2.8), we have:

$$\frac{d}{dt}|\tilde{\nabla} f|^2 = \frac{\partial}{\partial t}(g^{ij})\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + g^{ij}\frac{\partial}{\partial t}\left(\frac{\delta f}{\delta x^i}\right)\frac{\delta f}{\delta x^j} + g^{ij}\frac{\delta f}{\delta x^i}\frac{\partial}{\partial t}\left(\frac{\delta f}{\delta x^j}\right) + \frac{\partial (F^2)}{\partial t} g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}$$
$$+ F^2\frac{\partial}{\partial t}(g^{ij})\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j} + 2F^2 g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}, \tag{3.1}$$

where

$$\frac{\partial}{\partial t}(g^{ij}) = -g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk}) \tag{3.2}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\delta f}{\delta x^i}\right) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x^i} - G_i^r\frac{\partial f}{\partial y^r}\right)$$
$$= \frac{\partial f'}{\partial x^i} - G_i^r\frac{\partial f'}{\partial y^r} - \frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r}$$
$$= \frac{\delta f'}{\delta x^i} - \frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r} \tag{3.3}$$

therefore, a substitution of (3.2) and (3.3) in (3.1), implies that:

Proposition 3.1. *Let $(M^n, g(t))$ be a deformation of Finsler manifold (M^n, g_0), then*

$$\frac{d}{dt}|\tilde{\nabla} f|^p = \frac{p}{2}|\tilde{\nabla} f|^{p-2}\left\{-g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk})\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + 2g^{ij}\frac{\delta f'}{\delta x^i}\frac{\delta f}{\delta x^j} - 2g^{ij}\frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}\right\}$$
$$+\frac{p}{2}|\tilde{\nabla} f|^{p-2}\left\{2F\frac{\partial F}{\partial t}g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j} - F^2 g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk})\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j} + 2F^2 g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}\right\}. \tag{3.4}$$

On the other hand we have

$$\frac{d}{dt}(dv) = \left\{g^{ij}\frac{\partial}{\partial t}(g_{ij}) - n\frac{\partial}{\partial t}(\log F)\right\}dv. \tag{3.5}$$

Now, we get the following two integrability conditions:

$$0 = \frac{d}{dt}\int_{SM}|f|^p dv = p\int_{SM} f f' |f|^{p-2} dv + \int_{SM}|f|^p\frac{d}{dt}dv$$

which implies

$$p\int_{SM} f f' |f|^{p-2} dv = -\int_{SM}|f|^p\left\{g^{ij}\frac{\partial}{\partial t}(g_{ij}) - n\frac{\partial}{\partial t}(\log F)\right\}dv. \tag{3.6}$$

Now, if we suppose that $g(t)$ is a solution of the un-normalized Yamabe flow (1.2) and (1.1), then we have:

$$\frac{d\lambda}{dt} = \int_{SM}\left(\frac{d}{dt}|\tilde{\nabla} f|^p\right)dv + \int_{SM}|\tilde{\nabla} f|^p\frac{d}{dt}(dv)$$
$$= \frac{p}{2}\int_{SM}\left\{-g^{il}g^{jk}(-H_g g_{lk})\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + 2g^{ij}\frac{\delta f'}{\delta x^i}\frac{\delta f}{\delta x^j} - 2g^{ij}\frac{\partial}{\partial t}(G_i^r)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}\right\}|\tilde{\nabla} f|^{p-2}dv$$

$$+\frac{p}{2}\int_{SM}\left\{2F^2\left(-\frac{1}{2}H_g\right)g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}-F^2g^{il}g^{jk}(-H_g g_{lk})\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}+2F^2g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}\right\}|\widetilde{\nabla}f|^{p-2}dv$$

$$+\int_{SM}|\widetilde{\nabla}f|^p\left\{g^{ij}(-H_g g_{ij})-n\left(-\frac{1}{2}H_g\right)\right\}dv$$

$$=\frac{p-n}{2}\int_{SM}H_g|\widetilde{\nabla}f|^p dv+p\int_{SM}\widetilde{g}(\widetilde{\nabla}f',\widetilde{\nabla}f)|\widetilde{\nabla}f|^{p-2}dv$$

$$-p\int_{SM}g^{ij}\frac{d}{dt}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla}f|^{p-2}dv-\frac{p}{2}\int_{SM}F^2H_g g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla}f|^{p-2}dv$$

where $\frac{\partial}{\partial t}(G^r_i)$ is obtained as follows:

$$G^r=\frac{1}{4}g^{rl}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k,\quad G^r_i=\frac{\partial G^r}{\partial y^i}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}(G^r_i)&=\frac{\partial}{\partial y^i}(G^r)'\\&=\frac{\partial}{\partial y^i}\left\{\frac{1}{4}\left(\frac{\partial}{\partial t}g^{rl}\right)\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k\right\}\\&\quad+\frac{\partial}{\partial y^i}\left\{\frac{1}{4}g^{rl}\left\{2\frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial t}g_{jl}\right)-\frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}y^jy^k\right\}\\&=\frac{\partial}{\partial y^i}\left\{-\frac{1}{2}\frac{\partial H_g}{\partial x^k}y^ry^k+\frac{1}{4}g_{jk}g^{rl}\frac{\partial H_g}{\partial x^l}y^jy^k\right\}\end{aligned}\tag{3.7}$$

Using (3.6) we obtain

$$p\int_{SM}\widetilde{g}(\widetilde{\nabla}f',\widetilde{\nabla}f)|\widetilde{\nabla}f|^{p-2}dv=p\lambda\int_{SM}f'f|f|^{p-2}dv=-\frac{n\lambda}{2}\int_{SM}|f|^pH_g dv.$$

We have thus proved the following proposition:

Proposition 3.2. *Let $(M^n,g(t))$ be a solution of the un-normalized Yamabe flow on the smooth Finsler manifold (M^n,g_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then*

$$\begin{aligned}\frac{d\lambda}{dt}&=\frac{p-n}{2}\int_{SM}H_g|\widetilde{\nabla}f|^p dv+\frac{n\lambda}{2}\int_{SM}H_g|f|^p dv-p\int_{SM}g^{ij}\frac{\partial}{\partial t}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\widetilde{\nabla}f|^{p-2}dv\\&\quad-\frac{p}{2}\int_{SM}F^2H_g g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla}f|^{p-2}dv,\end{aligned}\tag{3.8}$$

where f is the associated normalized evolving eigenfunction.

Note 3.3. Let $f:SM\to\mathbf{R}$ be a lifting of $f:M\to\mathbf{R}$. We have:

$$\frac{d\lambda}{dt}=\frac{p-n}{2}\int_{SM}H_g|\widetilde{\nabla}f|^p dv+\frac{n\lambda}{2}\int_{SM}H_g|f|^p dv,$$

and in this case, if H_g is a constant, then

$$\frac{d\lambda}{dt}=\frac{p\lambda}{2}H_g.$$

Corollary 3.4. *Let $(M^n, g(t))$ be a solution of the un-normalized Yamabe flow on the smooth Riemannian manifold (M^n, g_0), i.e. F_t, F_0 are Riemannian metric. If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then $H_g = R_g$ hence*

$$\frac{d\lambda}{dt} = \frac{p-n}{2}\int_{SM} R_g |\tilde{\nabla} f|^p dv + \frac{n\lambda}{2}\int_{SM} R_g |f|^p dv - p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\tilde{\nabla} f|^{p-2}dv$$
$$-\frac{p}{2}\int_{SM} F^2 R_g g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\tilde{\nabla} f|^{p-2}dv, \tag{3.9}$$

where R_g is the scalar curvature of M.

Corollary 3.5. *Let $(M^n, g(t))$ be a solution of the un-normalized Yamabe flow on the smooth homogenous Riemannian manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then:*

$$\frac{d\lambda}{dt} = \frac{pR\lambda}{2} - p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\tilde{\nabla} f|^{p-2}dv - \frac{pR}{2}\int_{SM} F^2 g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\tilde{\nabla} f|^{p-2}dv,$$

where R is the scalar curvature of M.

Proof. Since the evolving metric remains homogenous and a Riemannian homogenous manifold has constant scalar curvature, so the corollary is obtained by (3.8). □

Now, we give a variation of $\lambda(t)$ under the normalized Yamabe flow which is similar to the pervious proposition.

Proposition 3.6. *Let $(M^n, g(t))$ be a solution of the normalized Yamabe flow on the smooth Finsler manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then:*

$$\frac{d\lambda}{dt} = -\frac{p\lambda}{2}\mathrm{Avg}(H_g) + \frac{p-n}{2}\int_{SM} H_g |\tilde{\nabla} f|^p dv + \frac{n\lambda}{2}\int_{SM} H_g |f|^p dv$$
$$-p\int_{SM} g^{ij}\frac{\partial}{\partial t}(G^s_i)\frac{\partial f}{\partial y^s}\frac{\delta f}{\delta x^j}|\tilde{\nabla} f|^{p-2}dv - p\int_{SM} F^2(H_g - \mathrm{Avg}(H_g))g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\tilde{\nabla} f|^{p-2}dv, \tag{3.10}$$

where f is the associated normalized evolving eigenfunction, $\mathrm{Avg}(H_g) = \frac{\int_{SM} H_g dv}{\mathrm{Vol}(SM)}$.

Proof. In the normalized case, the integrability conditions read as follows

$$p\int_{SM} f' f |f|^{p-2} dv = -\frac{n}{2}\int_{SM}(-H_g + Avg(H_g))|f|^p dv, \tag{3.11}$$

using (3.4), (3.7) and the above equation, we can then write

$$\frac{d\lambda}{dt} = \int_{SM}\left(\frac{d}{dt}|\tilde{\nabla} f|^p\right)dv + \int_{SM}|\tilde{\nabla} f|^p \frac{d}{dt}(dv_t)$$
$$= \frac{p}{2}\int_{SM}\left\{-g^{il}g^{jk}(-H_g + Avg(H_g))g_{lk}\frac{\delta f}{\delta x^i}\frac{\delta f}{\delta x^j} + 2g^{ij}\frac{\delta f'}{\delta x^i}\frac{\delta f}{\delta x^j}\right\}|\tilde{\nabla} f|^{p-2}dv$$
$$+\frac{p}{2}\int_{SM}\left\{2F^2(-H_g + Avg(H_g))g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j} - F^2 g^{il}g^{jk}(-H_g + Avg(H_g))g_{lk}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}\right\}|\tilde{\nabla} f|^{p-2}dv$$

$$+\frac{p}{2}\int_{SM}\left\{-2g^{ij}\frac{\partial}{\partial t}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}+2F^2g^{ij}\frac{\partial f'}{\partial y^i}\frac{\partial f}{\partial y^j}\right\}|\tilde{\nabla}f|^{p-2}dv+\frac{n}{2}\int_{SM}(-H_g+Avg(H_g))|\tilde{\nabla}f|^p dv$$

$$=-\frac{p-n}{2}\int_{SM}(-H_g+Avg(H_g))|\tilde{\nabla}f|^p dv+p\int_{SM}\tilde{g}(\tilde{\nabla}f',\tilde{\nabla}f)|\tilde{\nabla}f|^{p-2}dv$$

$$-p\int_{SM}g^{ij}\frac{d}{dt}(G^r_i)\frac{\partial f}{\partial y^r}\frac{\delta f}{\delta x^j}|\tilde{\nabla}f|^{p-2}dv-\frac{p}{2}\int_{SM}F^2(-H_g+Avg(H_g))g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\tilde{\nabla}f|^{p-2}dv \tag{3.12}$$

but

$$p\int_{SM}\tilde{g}(\tilde{\nabla}f',\tilde{\nabla}f)|\tilde{\nabla}f|^{p-2}dv=p\lambda\int_{SM}f'f|f|^{p-2}dv$$

$$=-\frac{n\lambda}{2}\int_{SM}|f|^p(-H_g+Avg(H_g))dv \tag{3.13}$$

and $\frac{\partial}{\partial t}(G^s_i)$ is obtained by replacing F' and g'_{ij} from (1.4) and (1.3), respectively, in (3.7). Thus the proposition is obtained by replacing (3.13) in (3.12).

Corollary 3.7. *Let $(M^n,g(t))$ be a solution of the normalized Yamabe flow on the smooth Riemannian manifold (M^n,g_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then:*

$$\frac{d\lambda}{dt}=-\frac{p\lambda}{2}\text{Avg}(R_g)+\frac{p-n}{2}\int_{SM}R_g|\tilde{\nabla}f|^p dv+\frac{n\lambda}{2}\int_{SM}R_g|f|^p dv$$

$$-p\int_{SM}g^{ij}\frac{\partial}{\partial t}(G^s_i)\frac{\partial f}{\partial y^s}\frac{\delta f}{\delta x^j}|\tilde{\nabla}f|^{p-2}dv-p\int_{SM}F^2(R_g-\text{Avg}(R_g))g^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\tilde{\nabla}f|^{p-2}dv, \tag{3.14}$$

where R is the scalar curvature of M.

Corollary 3.8. *Let $(M^n,g(t))$ be a solution of the normalized Yamabe flow on the smooth homogenous Riemannian manifold (M^n,g_0). If $\lambda(t)$ denotes the evolution of an eigenvalue under the Yamabe flow, then:*

$$\frac{d\lambda}{dt}=-p\int_{SM}g^{ij}\frac{\partial}{\partial t}(G^s_i)\frac{\partial f}{\partial y^s}\frac{\delta f}{\delta x^j}|\tilde{\nabla}f|^{p-2}dv$$

where R is the scalar curvature of M.

Proof. Since the evolving metric remains homogenous and a Riemannian homogenous manifold has constant scalar curvature, so the corollary is obtained by (3.10). □

4. Examples

In this section, we will find the variational formula for some of Finsler manifolds.

Example 4.1. Let (M^n,g_0) be an Einstein manifold with constant Ricci i.e. there exists a constant a such that $Ric(F_0)=aF_0^2$. Therefore $Ric_{ij}(g_0)=ag_{ij}(0)$. Assume we have a solution to the Yamabe flow which is of the form

$$g(t)=u(t)g_0,\quad u(0)=1$$

where $u(t)$ is a positive function. We compute

$$\frac{\partial g}{\partial t} = u'(t)g_0, \quad H_g = \frac{an}{u(t)},$$

for this to be a solution of the un-normalized Yamabe flow, we require

$$u'(t)g_0 = -H_g g = -\frac{an}{u(t)}u(t)g_0 = -ang_0,$$

this shows that

$$u'(t) = -na,$$

therefore

$$u(t) = -nat + 1,$$

so that we have

$$g(t) = (1 - nat)g_0$$

which says that $g(t)$ is an Einstein metric. Therefore

$$H_g = \frac{an}{1 - nat}.$$

Also

$$\begin{aligned} G^i(t) &= \frac{1}{4}g^{il}\left\{2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right\}y^j y^k \\ &= \frac{1}{4}(g_0)^{il}\left\{2\frac{\partial (g_0)_{jl}}{\partial x^k} - \frac{\partial (g_0)_{jk}}{\partial x^l}\right\}y^j y^k = G^i(0), \end{aligned}$$

therefore

$$\frac{\partial}{\partial t}(G^i_r) = 0.$$

Using the un-normalized Yamabe flow equation (1.1) and (3.8) ,we obtain the following relation:

$$\begin{aligned} \frac{d\lambda}{dt} &= \frac{p-n}{2}\int_M \frac{an}{1-nat}|\widetilde{\nabla} f|^p dv + \frac{n\lambda}{2}\int_M |f|^p \frac{an}{1-nat}dv - \frac{p}{2}\int_{SM} F_0^2 \frac{an}{1-nat} g_0^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv \\ &= \frac{pan\lambda}{2(1-nat)} - \frac{pan}{2(1-nat)}\int_{SM} F_0^2 g_0^{ij}\frac{\partial f}{\partial y^i}\frac{\partial f}{\partial y^j}|\widetilde{\nabla} f|^{p-2}dv. \end{aligned}$$

Remark 4.2. Let (M^n, F_0) be a Finsler manifold of dimension $n \geq 3$. Suppose that the flag curvature $k = k(x)$ is isotropic and a function of $x \in M$ alone then $k =$ constant and therefore (M^n, F_0) is Einstein and the variation of its eigenvalues is similar to Example 4.1.

Example 4.3. If we suppose that $F_t = u(t)F_0$, $u(0) = 1$ is a solution of the Yamabe flow, then:

$$H_g = (u(t))^{-2}H_{g_0}.$$

Now the Yamabe flow (1.2) implies that

$$\frac{\partial (u(t))^{-2}}{\partial t} = -H_{g_0}.$$

By integration we have:

$$u^{-2}(t) = -H_{g_0}t + c,$$

with condition $u(0) = 1$ we have:

$$u^{-2}(t) = 1 - H_{g_0} t,$$

therefore

$$F_t^2 = \frac{1}{1 - H_{g_0} t} F_0^2. \tag{4.1}$$

By replacing above identities in (3.8) we obtain the variation of an eigenvalue.

In next example we determine the behavior of the evolving spectrum on the Yamabe solitons.

Definition 4.4. Let $(M, g(t))$ is a solution of the Yamabe flow. We says $g(t)$ is Yamabe soliton, when satisfies in $g_t^2 = u(t)\varphi_t^* g_0^2$ where φ_t is a family of diffeomorphisms.

Example 4.5. Let (M, F) and $(\overline{M}, \overline{F})$ be two closed Finsler manifolds and

$$\varphi : (M, g) \to (\overline{M}, \overline{F})$$

an isometry, then for $p = 2$ we have

$$^{\widetilde{g}}\Delta \circ \varphi^* = \varphi^* \circ {}^{\widetilde{\widetilde{g}}}\Delta.$$

Therefore given a diffeomorphism $\varphi : M \to M$ we have that

$$\varphi : (SM, \varphi^* \widetilde{g}) \to (SM, \widetilde{g})$$

is an isometry, hence we conclude that $(SM, \varphi^* \widetilde{g})$,and $(SM, \widetilde{g})$ have the same spectrum

$$Spec(\widetilde{g}) = Spec(\varphi^* \widetilde{g})$$

with eigenfunction f_k and $\varphi^* f_k$ respectively. If $g(t)$ is a Yamabe soliton on (M^n, g_0) then

$$Spec(\widetilde{g}(t)) = \frac{1}{u(t)} Spec(\widetilde{g}_0),$$

so that $\lambda(t)$ satisfies

$$\lambda(t) = \frac{1}{u(t)}, \quad \frac{d\lambda}{dt} = -\frac{u'(t)}{(u(t))^2}.$$

5. Conclusion

In this paper we obtain the evolution formulas for the eigenvalue of p-Laplacian on SM under the Yamabe flow. Furthermore, we find the variational formula for Einstein Finsler manifold and determine the behavior of evolving eigenvalue on the Yamabe soliton.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

Author write, read and approved the final manuscript.

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