Journal of Informatics and Mathematical Sciences
Vol. 8, No. 1, pp. 1–16, 2016
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

On A Variance Gamma Model (VGM) in Option Pricing: A Difference of Two Gamma Processes

Research Article

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Abstract. The Variance-Gamma (VG) process is a three parameter stochastic process with respect to a Brownian motion. Here, we consider in our presentation, a detailed study of the VG process expressed as a difference of two gamma processes. As a result, we obtain the basic moments of the process using the characteristic function of the VG process with regard to the parameters of a differenced gamma processes. Also, the Levy-Khintchine formula for the process is derived via the Frullani's integral. Finally, a modified European call option VG model incorporating a difference of two gamma processes is proposed.

Keywords. Option pricing; Variance gamma model; Levy processes; Levy-Khintchine formula

MSC. 91G80; 97M30; 33E50

Received: December 19, 2015 **Accepted:** December 28, 2015

1. Introduction

The bedrock of any economy viewed from the aspect of finance, actuarial or management sciences [11] is the financial market; and embedded in the financial market is a derivative security or a contingent claim, which is a financial contract whose value at an expiration date T, which is written into the contract is determined by the price of some assets called the underlying. Also, in the financial Market, there are four main kinds of derivative securities namely: *options*, *forwards*, *futures* and *swaps*.

An option is a derivative security which gives the holder the right but not the obligation to buy or sell an asset in the future for a price that is agreed on today. Types of options include: a call option which gives a financial agent the right to sell, and he is said to hold a short position. There are several kinds of options which include the European options, American options, Asian options, Bermudian options, look back options etc.

In financial modeling, the *Geometric Brownian Motion* (GBM) model has been a very useful tool in the modeling of stock prices, this was suggested by Samuelson (1965) in [22], following the imperfection which includes negative stock of the Bachelier's model in [2]. The most celebrated stock price model is the Black Scholes' Model in [4, 12], in which the demonstration of how to price European options based on the geometric Brownian model was given. In [1], a stochastic analysis of stock market price models is considered using a proposed log-normal distribution.

In recent times, it has been made clear after some investigations by some researchers that the Black Scholes Model presents unrealistic assumptions such as the log returns of stocks are not normally distributed as in the B-S model [20, 23, 27]. They are skewed and have an actual kurtosis higher than that of the normal distribution. More so, it has been observed that the parameters of uncertainty estimated (volatility) change stochastically over time and are clustered. The above listed points suggest that the Black Scholes model does not give a perfect description of financial asset prices adequately.

In order to overcome the above challenges in financial modeling, more flexible stochastic processes with independent and stationary increments, whose distributions are infinitely divisible, were proposed for modeling financial data [23]. These distributions are able to capture aspect of skewness and excess kurtosis, while the associated processes are called the Levy processes. Since the inception of the Levy models, many models have been proposed, they are in two groups: the first group takes care of local volatility modeling, and the second considers inputting jump component in the price dynamics. An exponential Levy model describing the effect of the duality principle in option pricing was considered in [28]. To be useful in financial modeling, the infinitely divisible distributions need to be able to represent skewness and excess kurtosis. Examples of such distributions, which can take into account skewness and excess kurtosis, are the *Variance Gamma* (VG), the *Normal Inverse Guassian* (NIG), the CGMY, the Generalized Hyperbolic model and the Meixner distributions. Madan and Seneta [18] have proposed a Levy process with VG distributed increments. The Hyperbolic Model was proposed by Eberlein and Keller in [10]. Barndorff-Neilson proposed the NIG Levy process [3]. Recently, the CGMY Model was introduced by Carr *et al.* in [7], and Meixner model was used in [23]. As one of the Levy models, the VG model was introduced in 1987 by Madan and Seneta using methods of Chebyshev polynomial approximations and characteristic function estimation, which was used to model stock returns. Madan and Milne in [19] priced the European Call Option when the uncertainty driving stock prices follows the VG stochastic process introduced by Madan and Seneta in [18] and they considered the symmetric case. Using the log return mean, variance and kurtosis, Madan and Milne in [19] approximated and identified the incomplete markets equilibrium change of measure. A comparative analysis with the B-S model shows that the VG option values are higher, particularly for out of the money options with long maturity in stocks with high means, low variances and high kurtosis.

As a three parameter stochastic process (VG process) which in practical terms, can be likened to volatility, kurtosis and skewness with respect to the Brownian motion was developed as a model for the dynamics of log stock prices as described in [17]. They obtained this process by evaluating Brownian motion with drift at a random time by a gamma process. The drift of the Brownian motion and the volatility of the time change were the two derived additional parameters. Their results show that the two additional parameters control the skewness and the kurtosis of the return distribution. Assuming the characteristic function of the risk-neutral density is known, analytically, the authors in [5, 6, 26] had numerically solved for the delta and for the risk neutral of "finishing in-the-money". Two Fourier transforms in terms of the characteristics function of the log of the terminal stock price which follows a VG process were analytical developed in [8]. They illustrated their methods for the VG option pricing model and their result shows that the use of the FFT is considerably faster than most available methods, and that the methods described in [5, 6, 26] can be both slow and inaccurate [13]. In this work, our interest is not only in the study of the VG process in terms of three parameters mentioned but also in the VG process when it is expressed as a difference of two gamma processes.

The structure of the remaining part of the paper is as follows: Section 2 deals with basic definitions and concepts that provide fundamentals for constructing financial models. The VG process in terms of the parameters of the differenced gamma processes is given in Section 3, an estimation of the moments of the VG process in terms of the parameters of the gamma processes is derived, more so, the Levy-Khintchine formula for a variance gamma process as a difference of two gamma processes was derived via the Frullani's integral. Given the characteristic function of the VG model with respect to the parameters of the differenced gamma processes, a formula for pricing European Call option is derived for the model in Section 4, while Section 5 is on discussion of results and a concluding remark.

2. Basic Definitions, Concept and Theorems

In this section a brief summary of several concepts, definitions, lemmas, notations and theorems is given [14–16, 24, 25]. All these provide the fundamentals for constructing the financial models used in this work.

Definition 2.1 (Characteristics Function)**.** Let X be a random variable whose distribution function is given $F(x) = P(X \le x)$. The characteristic function $\phi_X(u)$ of the distribution or the random variable is the Fourier-Stieltjes transform of the distribution. Thus we define $\phi_X(u)$ as:

$$\phi_X(u) = E(\exp(iuX)) = \int (iux)dF(x), \quad i^2 = -1. \tag{2.1}$$

Note 2.1. For two independent random variables, X and $Y : \phi_{X+Y}(u) = \phi_X(u)\phi_Y(u)$.

Definition 2.2 (Characteristic exponent)**.** The characteristics exponent sometimes called the cumulant characteristic function is given as

$$\psi(u) = \log E(\exp(iuX)) = \log \phi_X(u) \quad \Rightarrow \quad \psi(u) = \exp(\varphi(u)).$$

From the above,

$$k(u) = \log E[\exp(-uX) = \log \phi_X(u)$$

and

$$v(u) = E[\exp(uX)] = \phi(-iu)$$

are the cumulant function and the moment-generating function respectively.

The characteristic function of a gamma distribution with *probability density function* (pdf) is defined and denoted as:

$$f(x) = \begin{cases} \dfrac{\beta e^{-\beta x}(\beta x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0,\ \beta > 0,\ \alpha > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where X is said to be a continuous gamma random variable and α, β are positive real numbers. Thus, upon simplification, the characteristic function is:

$$\phi_{Gamma}(u) = \left(\frac{\beta - iu}{\beta}\right)^{-\alpha} = \left(1 - \frac{iu}{\beta}\right)^{-\alpha}.$$

It is useful to note that the characteristic exponent of a Gamma distribution is

$$\psi_{X^{Gamma}}(u) = \alpha \log\left(1 - \frac{iu}{\beta}\right), \quad \alpha > 0,\ \beta > 0.$$

2.1 The Levy Process

The geometric Brownian motion does not take into account stochastic volatility, skewness and excess kurtosis. This is due to the fact that, log-returns are normally distributed, hence a more flexible distribution would be needed.

In the sequel, we give the description of the following notations to enable us define the Levy process

Definition 2.3 (Independent increments)**.** The increments of a process are called independent if the increments $X_s - X_t$ and $X_u - X_v$ are independent random variables, whenever the two time intervals do not overlap.

Definition 2.4 (Stationary increments)**.** The increments are called stationary, if the probability distribution of any increment $X_s - X_t$ depends only on the length $s - t$ of the time interval, that is, increments with equally long time intervals are identically distributed.

Definition 2.5 (Cadlag)**.** A function $f : \mathbb{R}^+ \to \mathbb{R}^d$ is a cadlag if it is right continuous with left limits, i.e. $f(t_-) = \lim_{s\to t} f(s)$, $s < t$ and $f(t_+) = \lim_{s\to t} f(s)$, $s > t$.

Definition 2.6 (Levy process)**.** A stochastic process $X = \{X_t,\ 0 \leq t \leq T\}$ defined on (Ω, β, μ) with $X_0 = 0$ is called a Levy process if it possesses the following properties:

(a) *Independent increments*: for any time t, such that: $0 < t_0 < t_1 < \cdots < t_n$ the random variables

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}} \text{ are independent.}$$

(b) *Stationary increments*: the law of $X_{t_{t+h}} - X_t$ does not depend on t but only on the length h of the interval $[t, t+h]$.

(c) *Stochastic continuity*: for $\varepsilon > 0$, $\lim_{h\to 0} p(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

(d) *Cadlag*: sample path of X have left limit and are right continuous.

2.2 Properties of Levy Processes

We discuss here, some properties of Levy process and important results of Levy process.

Definition 2.7 (Infinitely divisible distributions)**.** Given the characteristic function $\phi_X(u)$ of a random variable X, we define the corresponding distribution as an infinitely divisible distribution if for every positive integer n, $\phi_X(u)$ is also the power of the characteristic function.

In terms of X, this means that, one could write for any n:

$$X = Y_1^{(n)} + Y_2^{(n)} + \ldots + Y_n^{(n)},$$

where Y_i^n, $i = 1, 2, \ldots, n$ are *independent and identically distributed* (i.i.d) random variables, with characteristic function $\phi(u)$.

The Normal distribution $N(\mu, \sigma^2)$ is infinitely divisible.

In fact,

$$\phi_N(u; \sigma^2, \mu) = \big(\exp(iu\mu)\big) \exp\left(-\frac{1}{2n}\sigma^2 u^2\right) = (\phi_n(u))^n$$

where $\phi_n(u)$is the characteristic function of a Normal Distribution $ND = N\left(\frac{\mu}{n}, \frac{\sigma^2}{n}\right)$.

If Y_i^n, $i = 1, 2, \ldots, n$ are i.i.d normal random variables with ND, then:

$$X = Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)} \sim Normal(\mu, \sigma^2).$$

Remark 2.1. There exist a close relationship between the distribution of a Levy process at time t and the concept of infinitely divisible distribution. De Finetti introduced the notion of infinitely divisible distribution and showed that they have a close relationship with Levy processes. The following theorems show the relationship between infinitely divisible distributions and Levy processes.

Theorem 2.1 ([24])**.** Let $X = \{X_t, t \geq 0\}$ be a Levy process and $F(dx)$ be the distribution of X_1. Then the distribution is infinitely divisible. Conversely, given an infinitely divisible distribution $F(dx)$, then there exist a Levy process such that the distribution of X_1 is $F(dx)$.

Theorem 2.2 (Levy-Khintchine formula [13]). *Suppose $\alpha \in R$, $\sigma \geq 0$ and $v(dx)$ is a measure concentrated on $\{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)v(dx) < \infty$, where we have the Levy triplet $(\alpha, \sigma^2, v(dx))$ defined for each $\alpha \in \mathbb{R}$. Then, there exists a probability space (Ω, β, μ) on which a Levy process is defined having characteristic exponent $\psi(u)$.*

Where

$$\psi(u) = i\alpha u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (1 - e^{iux} + iux1_{(|x|<1)})v(dx) \tag{2.2}$$

for as given in Definition 2.5 and $1_{(|x|<1)}$ is an indicator function with $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge x^2)v(dx) < \infty$.

Equation (2.2) is called the Levy-Khinchine formula which gives an expression for characteristic exponent $\psi(u)$.

Remark 2.2. From the above, we can deduce that a Levy process can be decomposed into three independent components

(i) a deterministic drift with rate α,
(ii) a continuous path diffusion volatility σ^2,
(iii) a jump process with Levy measure $v(dx)$.

Thus, $(\alpha, \sigma^2, v(dx))$ is referred to as Levy triplet.

Theorem 2.3 (Levy-Ito decomposition [16, 24]). *Consider a triplet $(\alpha, \sigma^2, v(dx))$, where $\alpha \in \mathbb{R}$, $\sigma > 0$ and $v(dx)$ is a measure satisfying 0 and $\int_{\mathbb{R}}(1 \wedge x^2)v(dx) < \infty$. Then, there exists a probability space (Ω, β, μ) on which four independent Levy processes exist, $X^{(1)}$, $X^{(2)}$, $X^{(3)}$ and $X^{(4)}$ where $X^{(1)}$ a constants drift, $X^{(2)}$ is a Brownian motion, $X^{(3)}$ is a compound Poisson process and $X^{(4)}$ is a square integrable (pure jump) martingale with almost surely countable number of jumps of magnitude less than one (1) on each finite time interval.*

Taking $X = X^{(1)} + X^{(2)} + X^{(3)} + X^{(4)}$, we have a Levy process $X = \{X_t, t \geq 0\}$ with characteristics exponent:

$$\psi(u) = i\alpha u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 + iux1_{(|x|<1)}\right)v(dx), \quad \text{for all } u \in \mathbb{R}. \tag{2.3}$$

Proof. We shall first split the Levy exponent into four parts:

$$\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \psi^{(4)}$$

where $\psi^{(1)}(u) = iu\alpha$, $\psi^{(2)}(u) = \frac{u^2\sigma^2}{2}$, $\psi^{(3)}(u) = \int_{|x|\geq 1}(e^{iux} - 1)v(dx)$,
$\psi^{(4)}(u) = \int_{|x|<1}(e^{iux} - 1 - iux)v(dx)$.

The first part corresponds to a deterministic linear process (drift) with parameter α, the second one to a Brownian motion with coefficient $\sqrt{\sigma_*^2} = \sigma$ and the third part corresponds to a compound Poisson process with arrival rate $\lambda := v(R \setminus (-1, 1))$ and jump magnitude

$$F(dx) := \frac{v(dx)}{v(R \setminus (-1, 1))} 1_{|x|\geq 1}.$$

For the last part, let $\Delta X^{(4)}$ denote the jumps of the Levy process, such that:

$$\Delta X^{(4)} = X_t^{(4)} - X_{t-}^{(4)}$$

and let $\mu^{X^{(4)}}$ denote the random measure counting the jumps of $X^{(4)}$.

We now construct a compensated compound Poisson process

$$\begin{aligned} X_t^{(4,\xi)} &= \sum_{0\le s\le t} \Delta X_s^{(4)} 1_{\{1>|\Delta X_s^{(4)}|>\xi\}} - t\left(\int_{1>|x|>\xi} xv(dx)\right) \\ &= \int_0^t \int_{0>|x|>\xi} x\mu^{X^{(4)}}(dx,ds) + t\left(\int_{1>|x|>\xi} xv(dx)\right). \end{aligned} \tag{2.4}$$

Next is to show that the jumps of $X^{(4)}$ form a Poisson process, given the following stated conditions:

Consider a set $A \in \beta(\mathbb{R}\setminus\{0\})$ with $0 \notin A$ and a function $f:\mathbb{R}\to\mathbb{R}$, Borel measurable and finite on A.

$$E\left[\exp\left(iu\int_0^t \int_A f(x)\mu^L(ds,dx)\right)\right] = \exp\left[t\int_A (e^{iuf(x)}-1)v(dx)\right].$$

If $f \in X'(A)$, then,

$$E\left[\int_0^t \int_A f(x)\mu^A(ds,dx)\right] = t\int_A f(x)v(dx).$$

Combining the above conditions with the definition of characteristic exponent, and applying it to (2.4) we have:

$$\begin{aligned} \psi_t^{(4,\xi)} &= E\left(e^{iuX_t^{(4,\xi)}}\right) \\ &= E(\exp(iu)\int_0^t \int_{0<|x|<\xi} x\mu^{X^{(4)}}(dx,ds)) + E\left(\exp iut\left(\int_{1>|x|>\xi} xv(dx)\right)\right) \end{aligned} \tag{2.5}$$

$$\Rightarrow \qquad \psi^{(4,\xi)}(u) = \int_{\xi<|x|<1} (e^{iux}-1-iux)v(dx).$$

Then, there exists a Levy process $X^{(4)}$ which is a square integrable martingale, such that $X^{(4,\xi)} \to X^{(4)}$ uniformly on $[0,T]$ as $\xi \to 0^+$. Clearly, the Levy exponent of the latter Levy process is ψ^4.

Therefore, we can decompose any Levy process into four independent Levy processes:

$$X_t = X^{(1)} + X^{(2)} + X^{(3)} + X^{(4)} \quad \text{as follows:}$$

$$X_t = \alpha t + \sigma W_t + \int_0^t \int_{|x|\ge 1} x\mu^X(ds,dx) + \int_0^t \int_{|x|<1} x(\mu^X - v^X)(ds,dx)$$

where $v^X(ds,dx) = v(dx)(ds)$. □

This result is the Levy-Ito decomposition of a Levy process.

Lemma 2.1 ([16]). *A Levy process with Levy-Khintchine exponent corresponding to the triplet* $(\alpha, \sigma^2, v(dx))$ *has paths of bounded variation if and only if:*

$$\sigma = 0 \quad and \quad \int_{\mathbb{R}} (1 \wedge |x|) v(dx) < \infty. \tag{2.6}$$

Due to the finiteness of the integral in (2.6), $\psi(u)$*for such bounded variation process can be written as:*

$$\psi(u) = idu + \int_{\mathbb{R}} (1 - e^{iux}) v(dx), \quad d \in \mathbb{R} \tag{2.7}$$

where d*, the drift term can be expressed in terms of*α*and*v*as*

$$d = -\left(\alpha + \int_{|x|<1} x v(dx)\right). \tag{2.8}$$

For instance, suppose that $X = \{X_t; t \geq 0\}$ *is a gamma process as defined above, for the Levy-Khintchine decomposition, we have* $\sigma = 0$, $v(dx) = \beta x^{-1} e^{-\alpha x} dx$ *concentrated on* $(0, \infty)$ *and* $\alpha = -\int_0^1 x v(dx)$*, thus*

$$X_t = dt + \int_{[0,t]} \int_{\mathbb{R}} x N(ds, xdx), \quad t \geq 0$$

Thus, substituting into (2.7) *for* $\sigma, v(dx)$ *and* α *gives:*

$$\psi(u) = \int_{\mathbb{R}} (1 - e^{iux}) \beta x^{-1} e^{-\alpha x} dx. \tag{2.9}$$

Definition 2.8 (Subordinators). Levy processes whose paths are non-decreasing are called subordinators. Examples are gamma processes, inverse Gaussian process etc.

Remark 2.3. (i) Any Levy process of bounded variation can be written as the difference of two independent subordinators.

(ii) In the next section, we shall establish (2.9) via the Frullani's integral.

2.3 The Levy-Khintchine Formula for A Gamma Process via the Frullani's Integral

In this subsection, we state the Frullani's identity which is a useful tool to determine the Levy-Khintchine formula for a gamma distribution.

Lemma 2.2 (Frullani's integral [16]). *For all* $\alpha, \beta > 0$ *and* $z \in \mathbb{R}$ *such that* $R_z < 0$*, we have:*

$$\left(1 - \frac{z}{\alpha}\right)^{-\beta} = \exp\left(\int_0^{\infty} (1 - e^{zx}) \beta x^{-1} e^{-\alpha x} dx\right), \quad x > 0. \tag{2.10}$$

3. The Variance Gamma Process [25]

The VG distribution is characterized by a triplet of positive parameters (C, G, M), which is defined on the entire real line and the pdf is given as

$$f(z; C, G, M) = \frac{(GM)^C}{\sqrt{\prod}\, \Gamma(C)} \exp\left(\frac{(G-M)z}{2}\right) \left(\frac{|z|}{G+M}\right)^{C-0.5} K_{C-0.5}(0.5(G+M)|z|), \tag{3.1}$$

where $K_v(x)$ is the modified Bessel function of the third kind with index v.

3.1 The VG Process as A Difference of Two Subordinators

In this section, we refer to the work of Madan *et al.* in [17], wherein the VG process was expressed as the difference of two independent increasing gamma processes (subordinators). A study of the explicit relation between the parameters of the gamma processes differenced and the parameters of the VG process is looked into.

The VG process defined as $X^{VG}(t;\sigma,v,\theta)$ has three parameters viz: σ the volatility of the Brownian motion, v the variance of the gamma time change, and θ the drift in the Brownian motion with drift.

In Madan *et al.*, a variance gamma random variable can be constructed as the difference of two gamma random variables

Thus,

$$X(t;\sigma,v,\theta)=\Gamma_1(t;\alpha_1\beta_1)-\Gamma_2(t;\alpha_2\beta_2). \tag{3.2}$$

The gamma process $\Gamma(t;\alpha,\beta)$ with mean rate α and variance rate β is the process of independent gamma increments over non-overlapping intervals of time $(t,t+h)$.

Given that:

$$m=y(t+h;\alpha,\beta)-y(t;\alpha,\beta). \tag{3.3}$$

Note that the differenced gamma processes have been used to model queues by Gourieroux *et al.* in [15].

The density function of (3.3) is given as

$$f_h(m)=\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha^2h}{\beta}}\left(\Gamma\left(\frac{\alpha^2h}{\beta}\right)\right)^{-1}m^{\frac{\alpha^2h}{\beta}-1}\exp\left(-\frac{\alpha}{\beta}m\right),\quad m>0. \tag{3.4}$$

By definition, (3.4) has a characteristic function $\phi_{y(t)}(u)=E[\exp(iuy(t;\alpha,\beta)]$ given by

$$\phi_{y(t)}(u)=\left(1-iu\frac{\beta}{\alpha}\right)^{-\frac{\alpha^2}{\beta}t}.$$

Thus, for $y_1(t;\alpha_1,\beta_1)$ and $y_2(t;\alpha_2,\beta_2)$

$$\phi_{y_1(t)}(u)=\left(1-iu\frac{\beta_1}{\alpha_1}\right)^{-\frac{\alpha_1^2}{\beta_1}t}$$

and

$$\phi_{y_2(t)}(u)=\left(1-iu\frac{\beta_2}{\alpha_2}\right)^{-\frac{\alpha_2^2}{\beta_2}t}$$

$$\Rightarrow \qquad \phi_{X^{VG}}(u)=\phi_{y_1-y_2}(u)=\phi_{y_1}(u)\phi_{-y_2}(u).$$

By employing the property of characteristic function of the VG process

$$\phi_{-y_2(t)}(u) = \phi_{y_2(t)}(-u) = \left(1 + \frac{iu\beta_2}{\alpha_2}\right)^{-\frac{\alpha_2^2}{\beta_2}t}$$

Thus, the characteristic function of the VG process is:

$$\phi_{X^{VG}}(u;\sigma,v,\theta) = \left(1 - iu\frac{\beta_1}{\alpha_1}\right)^{\frac{-\alpha_1^2}{\beta_1}t}\left(1 + iu\frac{\beta_2}{\alpha_2}\right)^{\frac{-\alpha_2^2}{\beta_2}t}.$$

For $\frac{\alpha_1^2}{\beta_1} = \frac{\alpha_2^2}{\beta_2} = \frac{1}{v}$, we have:

$$\phi_{X^{VG}}(u;\alpha_1,\alpha_2,\beta_1,\beta_2) = \{(\alpha_1\alpha_2)(\alpha_1\alpha_2 - i\alpha_2\beta_1 u + i\alpha_1\beta_2 u - i^2\beta_1\beta_2 u)^{-1}\}^{\frac{t}{v}}$$

$$\Rightarrow \qquad \phi_{X^{VG}}(u) = \left(1 - i\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)u + \left(\frac{\beta_1\beta_2}{\alpha_1\alpha_2}\right)u^2\right)^{-\frac{t}{v}}. \tag{3.5}$$

Equation (3.5) is the characteristic function of a VG process with respect to the parameters of the differenced gamma processes $(\alpha_1,\alpha_2,\beta_1,\beta_2)$.

Now, we determine the relation between the parameters of the VG process and the gamma processes as follows:

We have that:

$$\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2} = \theta v,\ 2\beta_1\beta_2 = \sigma^2 v\alpha_1\alpha_2 \ \text{ and } \ \frac{\alpha_1^2}{\beta_1} = \frac{\alpha_2^2}{\beta_2} = \frac{1}{v}. \tag{3.6}$$

Letting $\zeta = \frac{\beta_1}{\alpha_1}$, $\mu = \frac{\beta_2}{\alpha_2}$, $a = \theta v$, $b = \frac{\sigma^2 v}{2}$ into (3.6) gives:

$$\zeta - \mu = a \quad \text{and} \quad \zeta\mu = b. \tag{3.7}$$

So considering ζ and μ from (3.7) in terms of a and b, we get:

$$\alpha_1 = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} + \frac{\theta}{2}, \tag{3.8}$$

$$\alpha_2 = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} - \frac{\theta}{2}, \tag{3.9}$$

$$\beta_1 = \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} + \frac{\theta}{v}\right)^2 v, \tag{3.10}$$

$$\beta_2 = \left(\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} - \frac{\theta}{2}\right)^2 v. \tag{3.11}$$

Remark 3.1. *The equations in* (3.8)-(3.11) *show the relation between the parameters of the VG process and parameters of the Gamma process* (*see Appendix in* [17]).

3.2 Moments of the VG Process in Terms of the Parameters of the Gamma Processes

By method of deriving moments, given a characteristic function, [21]

$$M_X(n) = E(X^n) = \frac{1}{i^n}\frac{d^n \phi_X(u)}{du}, \quad \text{at } u = 0. \tag{3.12}$$

Using (3.12) and (3.5), we derive the moments of the VG process. The result is summarized in Table 1 below showing the properties of the VG process in terms of the parameters of the differenced Gamma processes.

Table 1. Moments of the VG process in terms of the parameters $VG(\alpha_1, \alpha_2, \beta_1, \beta_2)$

	$VG(\alpha_1,\alpha_2,\beta_1,\beta_2)$	$VG(\alpha_1 \equiv \beta_1; \alpha_2 \equiv \beta_2)$, Symmetric case
Mean	$\alpha_1 - \frac{\alpha_1^2\beta_2}{\alpha_2\beta_1} = \frac{\alpha_2^2\beta_1}{\alpha_1\beta_2} - \alpha_2$	0
Variance	$\frac{2\beta_2\alpha_1}{\alpha_2} + \frac{\beta_1}{\alpha_1^2}\left(\alpha_1 - \frac{\alpha_1^2\beta_2}{\alpha_2\beta_1}\right)^2$	$\frac{2\beta_2\alpha_1}{\alpha_2} \equiv 2\alpha_1$ for $\alpha_2 = \beta_2$
Skewness	$\dfrac{\left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)\left[\frac{6\beta_2\alpha_1}{\alpha_2} + \frac{2\beta_1}{\alpha_1^2}\left(\alpha_1 - \frac{\alpha_1^2\beta_2}{\alpha_2\beta_1}\right)^2\right]}{\left[\frac{2\beta_2\alpha_1}{\alpha_2} + \frac{\beta_2}{\alpha_1^2}(\alpha_1 - \frac{\alpha_1^2\beta_2}{\alpha_2\beta_1})^2\right]^{\frac{2}{3}}}$	0
Kurtosis	$3 + \frac{6\beta_1}{\alpha_1^2} - \frac{12\beta_2^3}{\alpha_2^2}\left[\frac{2\beta_2\alpha_1}{\alpha_2} + \frac{\beta_1}{\alpha_1^2}(\alpha_1 - \frac{\alpha_1^2\beta_2}{\alpha_2\beta_1})^2\right]^{-2}$	$3\left(1 + \frac{\beta_1}{\alpha_1^2}\right) \equiv 3\left(1 + \frac{1}{\alpha_1}\right)$ for $\alpha_1 = \beta_1$

Remark 3.2. We note here that the VG distribution is symmetric if the parameters $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Under the conditions of $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, the results lead to a negative skewness. Also the parameter $\alpha_1$controls the kurtosis.

3.3 The Levy-Khintchine Formula for A Variance Gamma Process as A Difference of Two Gamma Processes

In the previous sections, we had discussed extensively on the derivation of the Levy-Khintchine formula for a gamma process via the Frullani's integral. It has also been established that the VG process can be expressed as a difference of two gamma processes. Thus, we now apply the concept of finding the Levy-Khintchine formula for the VG process in terms of the parameters of the gamma processes.

We can deduce from (2.10) that:

$$\left(1 - iu\frac{\beta_1}{\alpha_1}\right)^{\frac{-\alpha_1^2}{\beta_1}t} = \exp\left(-\int_0^1 (1 - e^{iux})\frac{\alpha_1^2 t}{\beta_1} e^{-\frac{\alpha_1}{\beta_1}x} dx\right), \quad x > 0 \tag{3.13}$$

and

$$\left(1 + iu\frac{\beta_2}{\alpha_2}\right)^{\frac{-\alpha_2^2}{\beta_2}t} = \exp\left(-\int_0^1 (1 - e^{iux})\frac{\alpha_2^2}{\beta_2} t\, e^{\frac{\alpha_2}{\beta_2}x} dx\right), \quad x > 0. \tag{3.14}$$

Thus, it implies from (2.10), (3.5), (3.13) and (3.14) that:

$$\left(1-i\left(\frac{\beta_1}{\alpha_1}-\frac{\beta_2}{\alpha_2}\right)u+\left(\frac{\beta_1\beta_2}{\alpha_1\alpha_2}\right)u^2\right)^{-\frac{t}{v}}$$
$$=\exp\left(-\int_0^\infty(1-e^{iux})\frac{\alpha_1^2}{\beta_1}te^{\frac{-\alpha_1}{\beta_1}x}dx-\int_0^\infty(1-e^{iux})\frac{\alpha_2^2}{\beta_2}te^{\frac{\alpha_2}{\beta_2}x}dx\right). \tag{3.15}$$

4. Option Pricing for the VG Model using the Characteristic Function [29]

Let X_t be an exponential VG process, which is also an exponential Levy process where $X_0 = 0$ at $t = 0$.

Assume that the price of a risky asset in a risk-neutral world is given by

$$S_t = S_0 e^{X_t} \tag{4.1}$$

where S_0 is the price of the asset at time $t = 0$

$$E(e^{X_t}) = 1. \qquad \text{(Exponential martingale condition)}$$

Given that the pdf of X_t is $f_t(x)$, then the price of any European option with expiration date T, and payoff $h(S_T)$ is given as:

$$P = V_T\int_{-\infty}^{\infty} S_T f_T(x)dx \quad \Rightarrow \quad P = V_T\int_{-\infty}^{\infty} h(S_0e^{X_T})f_T(x)dx \tag{4.2}$$

where V_T is the discounted price (factor) $V_T = e^{-rT}$.

In [9], the price of a European call option with strike at K is given by:

$$C(K) = F_TV_T\int_K^\infty (e^x-e^K)f_T(x)dx, \quad K = \ln\left(\frac{K}{F_T}\right). \tag{4.3}$$

Recall that:

$$\phi_{X_t}(u) = E(e^{iux}) = \int_{-\infty}^{\infty} e^{iux}f_t(x)dx$$

which is also called the Fourier transform of the pdf $f_t(x)$.

We remark as follows that:

(i) The VG model is an example of models that do not offer a closed form for the terminal density but they do have closed forms for the characteristic functions [29].

(ii) We can write the price of the European options as the integral of the product of the Fourier transform of the payoff and the characteristic function.

Thus, given the Fourier transform (FT) of the payoff and the characteristic function as:

$$\int_{-\infty}^{\infty} e^{iux} h(S_T)dx = D(u) \quad \text{and} \quad \phi_T(u) = \int_{-\infty}^{\infty} e^{iux} f_T(x)dx.$$

Respectively, we can therefore denote the product of $D(u)$ and $\phi_T(u)$ by $P(T)$ such that:

$$P(T) = V_T \int_{-\infty}^{\infty} \left((G(u))^c \phi_T(u)\right) du, \quad c \in \mathbb{C} \text{ is a complex conjugation.}$$

The European call option (28) with strike at K can be rewritten as:

$$C(K) = F_T V_T \int_{-\infty}^{\infty} \left[e^{-(1+\theta)x}(e^x - e^K)^+\right]\left[e^{(1+\theta)x} f_T(x)dx\right]. \tag{4.4}$$

We take the Fourier transform (FT) of the terms in the square brackets to get the following.

For the first square bracket, we get:

$$\int_{-\infty}^{\infty} e^{iux} e^{-(1+\theta)x}(e^x - e^K)dx.$$

Thus, with a strike at K, we have:

$$\begin{aligned}\int_K^{\infty} e^{iux} e^{-(1+\theta)x} e^x dx - \int_K^{\infty} e^{iux} e^{-(1+\theta)x} e^K dx &= \frac{e^{K(\theta-iu)}}{(\theta - iu)} - \frac{e^{K(\theta-iu)}}{(1+\theta) - iu} \\ &= \frac{e^{K\theta} e^{-iuK}}{\theta^2 + \theta - u^2 - i(2\theta+1)u}.\end{aligned} \tag{4.5}$$

Similarly, the FT of the terms in the second square bracket gives:

$$\int_{-\infty}^{\infty} e^{iux} e^{(1+\theta)x} f_T(x)dx = \phi(u - (1+\theta)i). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) gives:

$$C(K) = F_T V_T \int_{-\infty}^{\infty} \frac{e^{K\theta} e^{iuk} \phi(u - (1+\theta)i)}{\theta^2 + \theta - u^2 + i(2\theta+1)u} du.$$

The modified European call option is thus given as:

$$\Rightarrow \qquad C(K) = F_T V_T \frac{e^{K\theta}}{2\prod} \int_{-\infty}^{\infty} \frac{e^{iuk} \phi(u - (1+\theta)i)}{\theta^2 + \theta - u^2 + i(2\theta+1)u} du. \tag{4.7}$$

Equation (4.7) is true since by the inner product form of Parseval's theorem [28],

$$\int_{-\infty}^{\infty} g(x)(h(x))^c dx = \frac{1}{2\prod} \int_{-\infty}^{\infty} G(u)(H(u))^c du$$

where $G(\cdot)$ and $H(\cdot)$ are the FT of $g(\cdot)$ and $h(\cdot)$, respectively.

For a detailed derivation of (4.7), we refer the reader to Carr *et al.* [9].

We now derive the price of the European call option for a VG model when it is expressed as a difference of two gamma processes as follows:

Given the characteristic function of the VG process in (3.5), we determine $\phi(u-(1+\theta)i)$, for $u = u-(1+\theta)i$, we get

$$\begin{aligned}
&\phi_{X_T^{VG}}(u-(1+\theta)i,\alpha_1,\alpha_2,\beta_1,\beta_2) \\
&= \frac{1}{1-i\frac{\beta_1}{\alpha_1}u+\frac{\beta_2}{\alpha_1}(1+\theta)i+i\frac{\beta_2}{\alpha_2}u+\frac{\beta_2}{\alpha_2}(1+\theta)+\frac{\beta_1\beta_2}{\alpha_1\alpha_2}u-\frac{\beta_1\beta_2}{\alpha_1\alpha_2}(1+\theta)i} \\
&= \frac{\alpha_1\alpha_2}{\alpha_1\alpha_2-i(\beta_1\alpha_2-\alpha_1\beta_2)u-(\beta_1\alpha_2-\alpha_1\beta_2)(1+\theta)+\beta_1\beta_2(u-(1+\theta)i)} \\
&= \alpha_1\alpha_2 H^{-1}(u),
\end{aligned}$$

where $H(u) = \alpha_1\alpha_2 - [iu+(1+\theta)][\beta_1\alpha_2-\alpha_1\beta_2]+\beta_1\beta_2[u-(1+\theta)i]$.

Hence, the modified European call option with strike at K for a VG model expressed as a difference of two gamma processes $y_1(t;\alpha_1\beta_1)$ and $y_2(t;\alpha_2,\beta_2)$ is given as:

$$\widehat{C}(K) = F_T V_T \frac{e^{K\theta}}{2\prod}\alpha_1\alpha_2\int_K^\infty\left(\frac{e^{-iuK}H^{-1}(u)}{\theta^2+\theta-u^2-i(2\theta+1)u}\right)du\,. \tag{4.8}$$

5. Discussion of Results and Concluding Remarks

In this paper, we considered a detailed study of a three parameter stochastic process known as *Variance-Gamma* (VG) process with respect to a Brownian motion. A particular case of the VG process expressed as a difference of two gamma processes was highlighted. We obtained the moments of the VG process in terms of the parameters of the gamma processes using characteristic function of the VG process with regard to the parameters of a differenced gamma process. It is noted that the VG distribution is symmetric if the parameters $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Also, under the conditions: $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, we have a case of negative skewness; while the parameter α_1 in the VG distribution controls the kurtosis. In addition, the Levy-Khintchine formula for the process was derived via the Frullani's integral. Finally, we proposed a modified European call option model incorporating a difference of two gamma processes via the application of the characteristic function corresponding to the VG.

Acknowledgements

The authors are grateful to the anonymous reviewer(s) for valuable and constructive comments aimed at improving the paper. Authors[2,3] wish to thank Covenant University for financial support.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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