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Solving Fractional Order Differential Problems using Fuzzy Transform

Research Article

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Abstract. In this paper we give some background on the main concepts of the theory of fractional calculus and Grünwald formula for the fractional derivative, we also introduce Fuzzy Transform as a new technique for solving fractional differential equations. Fuzzy transform already proved itself in solving many problems in different branches, which encourage us to use it as a technique to approximate the solution of Fractional Differential Equations. The Fuzzy transform will be applied on a fractional order differential equations. The numerical algorithm will be implemented as a user-subroutine to the mathematical code MATLAB. We have introduced a numerical example of fractional order differential equation. Results are obtained for different fractional values and different partitions with triangular and Sinusoidal shaped basic functions and compared with the analytical solution.

Keywords. Fractional order differential equation; Fuzzy transform; Basic functions; Numerical algorithm

MSC. 65N22

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1. Introduction

Our understanding of nature depends basically on calculus, which in turn depends on the intuitive concept of the derivative. It's descriptive power comes from the fact that it analyses the behavior at scales small enough that its properties change linearly to avoid complexities which arise at large ones. Fractional calculus deals with the generalization of differentiation and integration of real orders [17], allowing calculations such as deriving a function to $\alpha = \frac{1}{3}$ order. The term fractional is used to denote this kind of derivatives. Despite it seems not to have significant applications in fundamental physics, various mathematicians have built up a large body of mathematical knowledge on fractional integrals and derivatives [20]. Fractional calculus seems to be valuable, and has played a significant role in engineering, finance and applied sciences. There are many interesting applications of fractional calculus, for example in physics it is used to model anomalous diffusion and in Hamiltonian chaos fractional partial differential equations can be used [8]. Other applications to physics involve fractional mechanics and fractional oscillators [19].

Several different first-order accurate numerical methods to solve fractional order differential equations have been presented before [23]. Many finite difference approximations for the fractional difference equations are based on some form of the Grünwald estimates, and these estimates are only first-order accurate. Meerschaert in [21] introduced a practical numerical algorithm for solving multidimensional fractional partial differential equations with variable coefficients, using modified Euler method. He used the shifted Grünwald finite difference approximation formula.

Also Kai Diethelm [6] has a pioneer work in presenting algorithms for solving linear and non-linear differential equations with fractional order used in modeling plasticity and visco-plasticity. In [21] Tadjeran and Meerschaert used finite difference to develop a stable numerical algorithm for solving fractional order differential equations with non-constant coefficients.

The Fuzzy transform is considered a new technique, it is published for the first time in 2001 by Irina Perfiliev [14], it depends basically on Fuzzy Set Theory which was published in 1965 by Lotfi Zadeh [23], Fuzzy Set Theory is one of the most important theories, it has many applications in a wide range of scientific area [1, 3, 23].

Fragments of the Fuzzy set theory has been used to establish a new field called fuzzy approximation, focusing on approximation properties of fuzzy models. Fuzzy transform is belonging to these models.

Instead of wavelet transform, Dankova and Valasek [4] considered fuzzy transform as an alternative approach to the solution of image fusion problem. The F-transform has been used successfully by many authors to introduce numerical algorithms for solving ordinary and partial differential equations with integer order derivatives [1, 15, 19, 20].

F-transform introduces an approximate representation of continuous functions defined on closed interval $[a,b] \subset R$. This technique actually like other transforms, consists of two transforms "direct F-transform and inverse F-transform".

2. The Fuzzy Transform

Although Fuzzy transform is a new mathematical technique for solving differential equations, it proved itself in solving many problems in different branches, [1, 12, 13, 14, 15] which encourage us to use it as a technique to approximate the solution of Ordinary Fractional Differential Equations. Now we will include some basic definitions to demonstrate the F-transform. Consider $[a,b]$ as a common domain for all functions in our study, this domain is partitioned into subintervals "Fuzzy Partition" to define membership functions to introduce fuzzy sets. We recall here fundamental facts about the technique of fuzzy transform

Definition 2.1 ([14])**.** Let $x_0 = x_1 < x_2 < \cdots < x_n = x_{n+1}$ be fixed nodes within $D = [a,b]$ such that $x_1 = a$, $x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1,\ldots,A_n$ are basic functions and form a fuzzy partition of D if the following conditions hold true for each $i = 1,\ldots,n$:

- $A_i : [a,b] \to [0,1]$, $A_i(x_i) = 1$,
- $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$,
- A_i is continuous on D,
- A_i strictly increases on $[x_{i-1}, x_i]$ and strictly decreases on $[x_i, x_{i+1}]$,
- $\sum\limits_{i=1}^{n} A_i(x) = 1$, for all $x \in D$.
  - If the nodes $x_1,\ldots,x_n$ are equidistant, then $h = \frac{b-a}{n-1}$, and $x_i = a + h(i-1)$, for all $i = 1,\ldots,n$
  - so the following properties are hold:
- $A_i(x_i - c) = A_i(x_i + c)$, for all $c \in [0,h]$, $i = 2,\ldots,n-1$, $n > 2$,
- $A_{i+1}(x) = A_i(x-h)$, for all $x \in [a+h,b]$, $i = 2,\ldots,n-1$, $n > 2$,
- $A_i(x_i + c) = A_{i+1}(x_{i+1} - c)$, for all $c \in [0,h]$, $i = 1,\ldots,n-1$, $n \geq 2$.

The most popular basic functions are triangular shaped and Sinusoidal shaped [14].

Definition 2.2 ([12])**.** Let $A_1,\ldots,A_n$ be basic functions which form a fuzzy partition on $[a,b]$ and $f \in C[a,b]$. We say that the n-tuple of real numbers $[F_1,\ldots,F_n]$ given by

$$F_i = \frac{\displaystyle\int_a^b f(x)A_i(x)dx}{\displaystyle\int_a^b A_i(x)dx}, \quad i = 1,\ldots,n \tag{2.1}$$

is the direct integral Fuzzy transform of f with respect to $A_1,\ldots,A_n$, and to transform the integral F-transform back we use the inverse F-transform formula

$$f_{F,n}(x) = \sum_{k=0}^{n} F_k A_k(x). \tag{2.2}$$

If the original function $f(x)$ is known only at some nodes $p_1,\ldots,p_l \in [a,b]$, then we can not use the direct F-transform in the previous definition, so we will need to use the discrete F-transform which defined as follows.

Definition 2.3. Let a function $f(x)$ be given at nodes $p_1,\ldots,p_l \in [a,b]$ and $A_1(x),\ldots,A_n(x)$, $n \ll l$, be basic functions which form a fuzzy partition of $[a,b]$. We say that the n-tuple of real numbers $[F_1,\ldots,F_n]$ is the direct (discrete) F-transform of f with respect to $A_1,\ldots,A_n$ if

$$F_k = \frac{\sum\limits_{j=1}^{l} f(p_j)A_k(p_j)}{\sum\limits_{j=1}^{l} A_k(p_j)}, \quad k = 1,\ldots,n. \tag{2.3}$$

And to transform the discrete F-transform back we use the same inverse F-transform formula, but we consider that function only at points where the original function is given as:

$$f_{F,n}(p_i) = \sum_{k=0}^{n} F_k A_k(p_i), \quad i = 1, \dots, l. \tag{2.4}$$

In [11] Perfilieva and Novak proved that the sequence of the inverse F-transform gives a continuous function and uniformly converges to the original function f.

3. The Fuzzy Transform Method for the Fractional Differential Equation

This section is mainly devoted to develop the basic concepts of solving fractional order differential equations by F-transform on the finite domain $D = [a, b]$.

The normal defintiion of the first derivative of the function in differential calculus is

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{3.1}$$

Iterating this approach yields an expression for the nth derivative of a function.

As can be easily seen — and proved by induction — for any natural number n

$$\frac{d^n f}{dx^n} = \lim_{\Delta x \to 0} (\Delta x)^{-n} \sum_{m=0}^{n} (-1)^m \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} f(x + (x - m\Delta x)) \tag{3.2}$$

also obtained by letting in the above formula, and $n = 1$ correspond to the classical first derivative.

Consider the following general form of fractional order differential equation

$$y^{\alpha}(x) + \beta y(x) = g(x), \quad 0 < \alpha < 1 \tag{3.3}$$

where

$$a \le x \le b, \quad \beta \in R, \ y(a) = 0. \tag{3.4}$$

Define a uniform partition $a = x_1 < x_2 < \cdots < x_n = b$ with equidistant step $h = \frac{b-a}{n}$ on $[a, b]$, and define a basic function $\{A_1, \dots, A_n\}$ of $[a, b]$, so we construct a fuzzy partition.

By applying F-transform as a linear operator [13] on (3.3) we get the equation

$$F^1[y^{\alpha}] + \beta F^1[y] = F^1[g], \tag{3.5}$$

where

$F^1[y^{\alpha}] = [Y_1^{\alpha}, Y_2^{\alpha}, \dots, Y_n^{\alpha}]^T$ is the F-transform components of y^{α},

$F^1[y] = [Y_1, Y_2, \dots, Y_n]^T$ is the F-transform components of y, and

$F^1[G] = [G_1, G_2, \dots, G_n]^T$ is the F-transform components of g.

So for $i = 1,\dots,n$ we have

$$Y_i^\alpha + \beta Y_i = G_i. \tag{3.6}$$

Discrete approximation to the fractional derivative term in equation (3.6) will be derived from the following Grünwald formula [16]

$$y_n^\alpha \approx \frac{1}{h^\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)} y_{n-k} \tag{3.7}$$

where $\omega_k^{(\alpha)}$ is the Grünwald weights and computed with the recurrence relationships

$$\omega_0^{(\alpha)} = 1, \omega_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right)\omega_{k-1}^{(\alpha)}, \quad k = 1,2,\dots \tag{3.8}$$

applying the F-transform to the Grünwald formula

$$Y_n^\alpha = \frac{\int_a^b y' A_i(x)dx}{\int_a^b A_i(x)dx} \approx \frac{\int_a^b \left(\frac{1}{h^\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)} y_{n-k}\right) A_i(x)dx}{\int_a^b A_i(x)dx},$$

$$Y_n^\alpha = \frac{1}{h^\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)} \frac{\int_a^b y_{n-k} A_i(x)dx}{\int_a^b A_i(x)dx} = \frac{1}{h^\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)} Y_{n-k}.$$

Therefore

$$Y_n^\alpha = \frac{1}{h^\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)} Y_{n-k}. \tag{3.9}$$

With the initial condition

$$Y_1 = \frac{\int_a^b y(a) A_i(x)dx}{\int_a^b A_i(x)dx} = 0. \tag{3.10}$$

By substituting (3.9) into (3.6) we get the recursive equation

$$Y_i = \frac{1}{1+\beta h^\alpha}\left(h^\alpha g_i - \sum_{k=2}^{i}\omega_{k-1}^{(\alpha)} Y_{n-k+1}\right) \tag{3.11}$$

where $i = 2,3,\dots,n$.

4. A Numerical Example

Consider the following fractional differential equation

$$y^{\alpha}(x)+y(x)=\frac{2}{\Gamma(3-\alpha)}x^{2-\alpha}-\frac{1}{\Gamma(2-\alpha)}x^{1-\alpha}+x^2-x,\quad 0<\alpha<1,\ 0\le x\le 2,\ y(0)=0 \tag{4.1}$$

where the exact solution to the above problem is given by $y(x)=x^2-x$.

Which can be verified by direct differentiation of the given solution and substituting in the fractional differential equation (the initial conditions are clearly satisfied).

Figures 1–6 show the numerical solution obtained by applying the F-transform method discussed above to the differential equation for fractional order $\alpha=0.5$, 0.75 and number of partitions $n=10$, 100 with triangular and Sinusoidal shaped basic functions. The numerical results compare well with the exact analytical solution to the fractional differential equation. Results show clearly that computed solutions get closer to the exact analytical solution as the number of partitions n increasing.

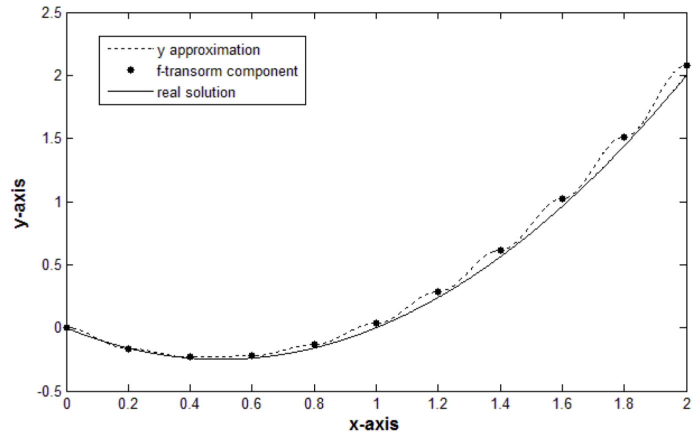

Figure 1. The approximated solution at $\alpha=0.5$ with Sinusoidal shaped basic functions for $n=10$.

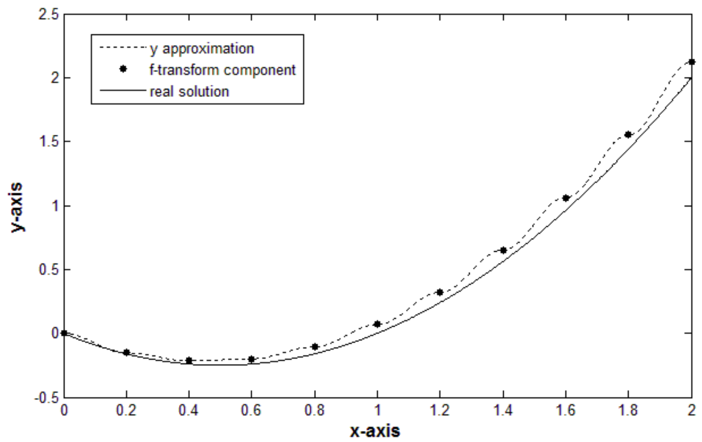

Figure 2. The approximated solution at $\alpha=0.75$ with Sinusoidal shaped basic functions for $n=10$.

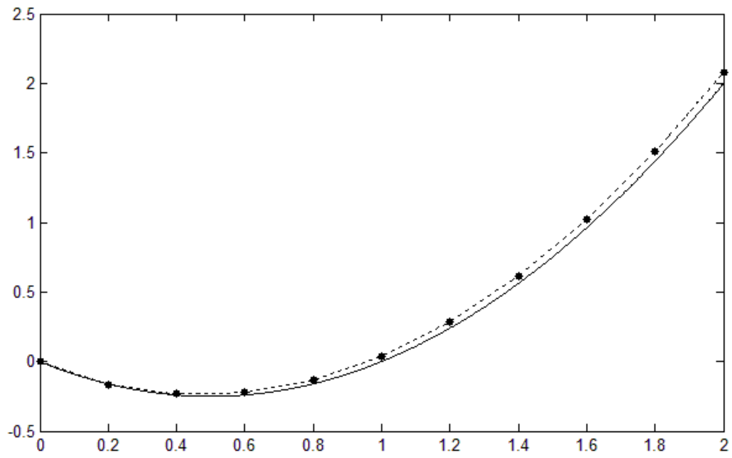

Figure 3. The approximated solution at $\alpha = 0.5$ with triangular shaped basic functions for $n = 10$.

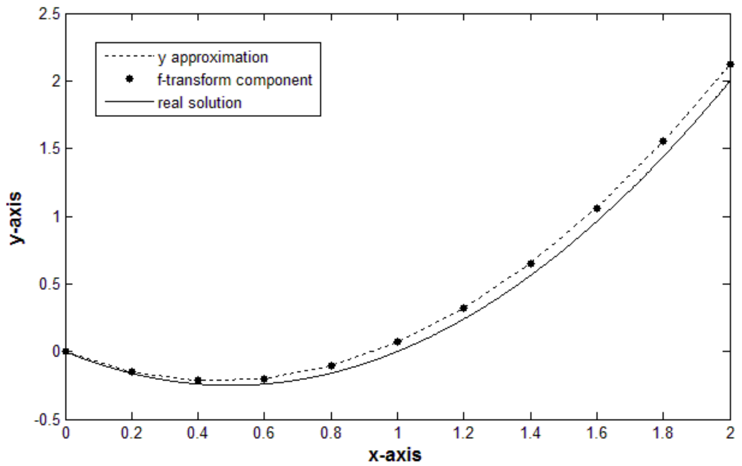

Figure 4. The approximated solution at $\alpha = 0.75$ with triangular shaped basic functions for $n = 10$.

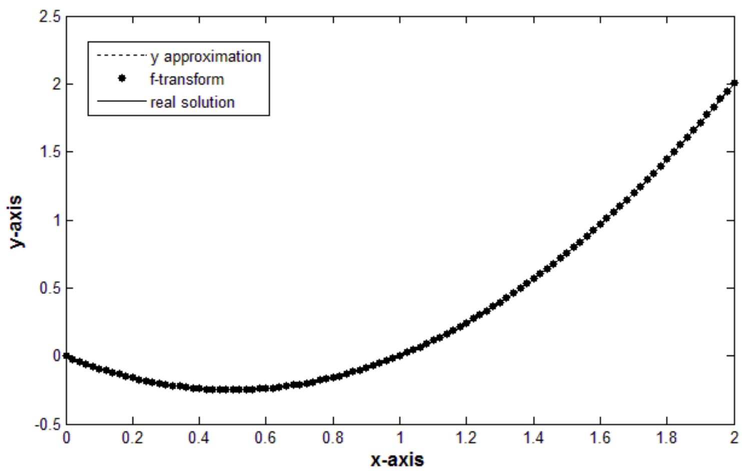

Figure 5. The approximated solution at $\alpha = 0.5$ with triangular shaped basic functions for $n = 10$.

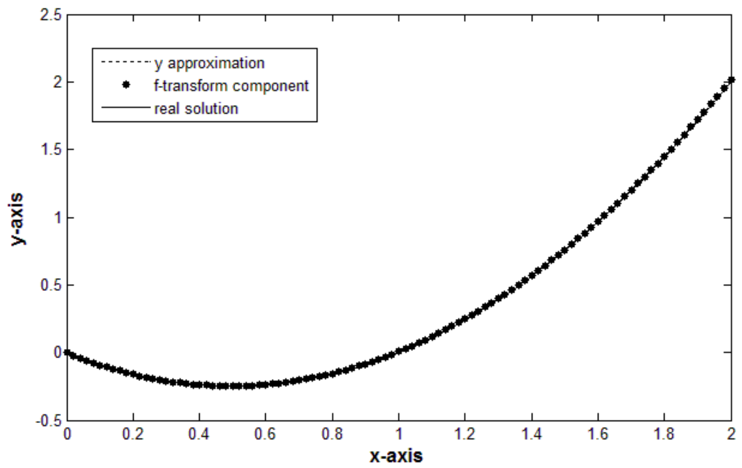

Figure 6. The approximated solution at $\alpha = 0.75$ with triangular shaped basic functions for $n = 100$.

5. Conclusions

In this study the F-transform technique is used to introduce a new a numerical algorithm for solving fractional order differential equations, the numerical algorithm is implemented as a user subroutine in the mathematical code MATLAB. The F-transform used to get a continuous approximation of the solution of fractional differential equations with constant coefficients, despite the difficulty of the computations, it's interesting to get a continuous solution and numerical approximations of fractional derivatives by using this new technique. The main idea consists in applying the fuzzy transform to both sides of the fractional order differential equation. This turns the fractional differential equation to algebraic one which is easy solvable by available numerical methods. Then the obtained numerical solution is transformed returned back to the space of continuous functions. It is remarkable to note that we have replaced the fractional derivative with first order approximation, but if we replace it with second or third approximation we will get more accurate results with less number of Fuzzy partitions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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