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Research Article

Characterisation of Semirings and Its Types by Using Ideals

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Abstract. We have introduced characterized semiring in this study. We have discussed the properties of ideals in semiring and characterization of semirings using ideals and some applications. We have also talked of bi-ideal and quasi-ideal in semiring and its significant classes. We have characterized three significant classes namely regular semiring, intra-regular semiring and weakly regular semiring using quasi ideals and bi-ideals.

Keywords. Semirings, Regular semiring, Intra-regular, Weakly regular semirings, Bi-ideal and quasi-ideal

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1. Introduction

H. S. Vandiver proposed the idea of semiring in 1934 [16]. Semiring is a natural topic in mathematics and it is broad and diverse mathematical topic, this issue has a significant impact on both mathematics and computer science. In this paper, we have characterized different classes using different ideals. Throughout this paper, triplet $(S,+,\cdot)$ for semiring and $(W,+,\cdot)$ represents several types of regular semiring that will be revealed when necessary. Here we have introduced quasi-ideals and bi-ideals to describe additional classes. We have proved some results using m-ideals and (m,n)-ideals with the help of Amla *et al*. [1], Munir and Habib [3], Munir and Shafiq [4], Shabir *et al*. [12] and Sulochana *et al*. [15]. Also, Munir and Habib [3] and Munir and Shafiq [4] have studied regular semirings using quasi-ideals and bi-ideals we have characterized three classes of semirings (regular semirings, intra-regular semirings and weakly regular semirings) using these ideals.

2. Preliminaries

Definition 2.1 ([8, 15])**.** A semiring $(S,+,\cdot)$ is an algebraic system with two binary operations $+$ and $\cdot$ such that $(S,+)$ is commutative semigroup and $(S,\cdot)$ is semigroup and following properties hold:

$$a(b+c)=ab+ac \text{ and } (b+c)a=ba+ca, \quad \text{for all } a,b\in S.$$

Definition 2.2 ([8, 15])**.** A non-empty subset C of a semiring $(S,+,\cdot)$ is said to be subsemiring if C itself is semiring under the operations of semiring S.

Definition 2.3 ([8, 15])**.** Let $(S,+,\cdot)$ be a semiring. A subsemigroup $(C,+)$ of $(S,+,\cdot)$ is said to be quasi ideal if $SC\cap CS\subseteq C$.

Definition 2.4 ([8])**.** A subsemiring C of a semiring S is called (m,n) quasi-ideal of S if $S^mC\cap CS^n\subseteq C$ where $m,n\in Z^+$.

Definition 2.5 ([12])**.** Let $(S,+,\cdot)$ be a semiring. A subsemiring B of $(S,+,\cdot)$ is said to be bi-ideal if $BSB\subseteq B$.

Definition 2.6 ([4])**.** Let $(S,+,\cdot)$ be a semiring. A subsemiring B of $(S,+,\cdot)$ is said to be m bi-ideal if $BS^mB\subseteq B$ where m is called the bipotency of bi-ideal B.

Example of bi-ideal is given below:

Example 2.1. Let $S=\left\{\begin{pmatrix} i & j\\ k & l\end{pmatrix} : i,j,k,l \text{ are in } Z^+\right\}$.
Than, clearly S is semiring under usual addition and multiplication of matrices.
$B_1=\left\{\begin{pmatrix} t & 0\\ 0 & 0\end{pmatrix} : t \text{ is in } Z^+\right\}$ is bi-ideal of S.

3. Characterizing Semirings

Theorem 3.1. *every (m,m)-quasi ideal Q of a semiring S is an m-bi ideal of S.*

Proof. Consider

$$QS^mQ\subseteq QS^mS=QS^{m+1}\subseteq QS^m.$$

Thus

$$QS^mQ\subseteq QS^m. \tag{3.1}$$

Similarly,

$$QS^mQ\subseteq S^mQ. \tag{3.2}$$

On combining (3.1) and (3.2), we obtain

$$QS^mQ\subseteq QS^m\cap S^mQ\subseteq Q.$$

Thus $QS^mQ \subseteq Q$, that Q is m-bi ideal. □

Theorem 3.2. *For every $m \geq 1$ every bi-ideal is an m-bi ideal.*

Proof. If B is bi-ideal of a semiring S than $BSB \subseteq B$ can be written as $BS^1B \subseteq B$ implies that B is bi-ideal with bipotency $m = 1$.
The converse of the above result is not true given by example:

$$S = \left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} : a,b,c,d,e,f \text{ are in } R^+ \right\} \text{ and } A = S^0 = S \cup \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Than $(A,+,\cdot)$ is a semiring under the usual operation of addition $+$ and multiplication $\cdot$ of matrices. Let

$$S = \left\{ \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} : a,f \text{ are in } R^+ \right\} \cup \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Than B is 2-bi-ideal of A as $BA^2B \subseteq B$ and $BAB \nsubseteq B$. □

Theorem 3.3. *Every quasi-ideal of a semiring S is subsemiring of S.*

Proof. Suppose N be a quasi-ideal of S. Than, by definition $(N,+)$ is subsemigroup of S.
Consider $N^2 = N \cdot N \subseteq SN$, i.e., $N^2 \subseteq SN$ and $N^2 \subseteq NS$.
So $N^2 \subseteq SN \cap NS \subseteq N$, i.e., $N^2 \subseteq N$, i.e., N is closed multiplicatively, implies that N is subsemiring of S. □

Theorem 3.4. *Any bi-ideal with bipotency* $\max\{m_1, m_2, \dots\}$ *is the intersection of family of bi-ideals of S with bipotencies $m_1, m_2, \dots$.*

Proof. Let $\{G_r : r \in \Omega\}$ be a family of bi-ideals of a semiring S than $G = \bigcap_{r\in\Omega} G_r$ is the sub-semiring of S which is the intersection of subsemirings of S (where Ω is the index set for r).
Since

$$G_r S^{m_r} G_r \subseteq G_r, \quad \text{for all } r \in \Omega$$

and

$$G \subseteq G_r, \quad \text{for all } r \in \Omega.$$

Therefore,

$$GS^{\max\{m_r : r\in\Omega\}} G \subseteq G_r S^{m_r} G_r \subseteq G_r, \quad \text{for all } r \in \Omega,$$

that is,

$$GS^{\max\{m_r : r\in\Omega\}} G \subseteq G_r, \quad \text{for all } r \in \Omega.$$

This gives

$$GS^{\max\{m_r : r\in\Omega\}} G \subseteq \bigcap_{r\in\Omega} G_r = G.$$

So

$$GS^{\max\{m_r : r \in \Omega\}} G \subseteq G.$$

Thus G is m-bi-ideal with bipotency $\max\{m_1, m_2, \ldots\}$. □

4. Characterizing Regular Semirings

Definition 4.1 ([3, 12]). Let $(S,+,\cdot)$ be a semiring an element a of a semiring $(S,+,\cdot)$ is considered to be regular if $a = axa$ for some $x \in S$ if every element of semiring is regular than it is considered to be regular semiring, e.g., Z_2 is regular semiring also R set of reals is regular semiring.

Theorem 4.1. *If W is regular semiring than $RL = R \cap L$ for every right-ideal R of A and left-ideal L of W.*

Proof. Suppose W is regular and assume that R and L be right-ideals of W than

$$RL \subseteq R \cap L. \tag{4.1}$$

For any $x \in W$ as W is regular $\exists$ some $y \in W$ such that we will have $x = xyx = (xy)$, $x \in RL$ because R is right-ideal implies that $x \in RL$.
Hence

$$R \cap L \subseteq RL. \tag{4.2}$$

On combining (4.1) and (4.2) we get $RL = R \cap L$. □

Theorem 4.2. *Let W be a semiring with multiplicative identity 1 than the following are equivalent:*

(i) *$U \cap V \subseteq VU$ for any right-ideal U and left-ideal V of W.*

(ii) *Every $\alpha \in W$ can be written as $\alpha = \sum_{i=1}^{n} x_i \alpha^2 y_i$, where $x_i, y_i \in W$.*

Proof. (i)⇒(ii): Let $\alpha \in W$. Let $U = \alpha W$ and $V = W\alpha$ be the right and the left-ideal generated by α, respectively.
Than by

$$U \cap V \subseteq VU, \quad \alpha \in U \cap V$$

implies that

$$\alpha \in VU.$$

So

$$\alpha = \sum_{i=1}^{n} x_i \alpha^2 y_i.$$

(ii)⇒(i): Let $\alpha \in U \subseteq V$ than $\alpha \in U$ and $\alpha \in V$.
Also, by (ii) $\alpha = \sum_{\text{finite}} x_i \alpha^2 y_i = \sum_{\text{finite}} (x_i \alpha)(\alpha y_i) \in UV$.
So $U \cap V \subseteq VU$. □

Theorem 4.3. *If W is a regular semiring than for all bi-ideals C, quasi-ideals D and any ideal I the given conditions holds:*

(i) $I \cap C = CIC$,

(ii) $I \cap D = DID$.

Proof. (i): Suppose W is regular semiring, I is any ideal of W and C is any bi-ideal of W than $CIC \subseteq I$ and $CWC \subseteq C$. Thus $CIC \subseteq I \cap C$.
Let $x \in I \cap C$ than $x = xyx$, for some $y \in W$.
Now $x = xyx = x(yxy)x \in CIC$. Thus, $I \cap C = CIC$.

(ii): Let I be any ideal of W and D be any quasi-ideal of W. As we know that every quasi-ideal is bi-ideal.
Therefore, by part (i), $I \cap D = DID$. □

5. Characterizing Intra-regular Semirings

Definition 5.1 ([12]). A semiring S with unity ($e = 1$) is called intra-regular if, for all $a \in S$ can be written as $a = \sum_{i=1}^{n} x_i a^2 y_i$, where $x_i, y_i \in S$. Therefore, if one of the criteria of Theorem 4.2 is satisfied, than a semiring S with multiplicative identity ($e = 1$) is said to be intra-regular.

Theorem 5.1. *For a regular and intra-regular semiring W which contains 1 than $B^2 = B$ for every ideal B of W.*

Proof. Let B be any bi-ideal of W than $B^2 \subseteq BWB$. Since W contains multiplicative identity 1. But $BWB \subseteq B$. Thus

$$B^2 \subseteq B. \tag{5.1}$$

Let $b \in B$ than

$$b = bxb, \quad \text{for some } x \in W.$$

Since W is regular and intra-regular therefore we can write

$$b = \sum_{\text{finite}} x_i b^2 y_i, \quad \text{for some } x_i, y_i \in W.$$

Thus

$$b = bxb = bxbxb = bx\left(\sum_{\text{finite}} x_i b^2 y_i\right)xb = \sum_{\text{finite}} (bxx_i b)(by_i xb).$$

Since $b \in B$, therefore

$$b(xx_i)b \in BAB \subseteq B \ \text{ and } \ b(y_i x)b \in BAB \subseteq B.$$

Thus

$$b = \sum_{\text{finite}} (bxx_i b)(by_i xb) \in BB = B^2.$$

Hence

$$B \subseteq B^2. \tag{5.2}$$

On combining (5.1) and (5.2) we get $B = B^2$. Hence result follows. □

Theorem 5.2. *If W is regular and intra-regular semiring having multiplicative unity 1 than $B \cap L \subseteq BLB$ for any left-ideal L and bi-ideal B of semiring W.*

Proof. Suppose W is regular and intra-regular and let $a \in B \cap L$, than $a \in B$ and $a \in L$. Since W is regular and intra-regular therefore we can write $a = axa$ and

$$a = \sum_{\text{finite}} x_i a^2 y_i, \quad \text{where } x_i, y_i \in S.$$

Now

$$\begin{aligned} a &= axa \\ &= axaxa \\ &= ax\left(\sum_{\text{finite}} x_i a^2 y_i\right) xa \\ &= a\left(\sum_{\text{finite}} xx_i a^2 y_i xa\right) \\ &= a\left(\sum_{\text{finite}} xx_i a\right)(ay_i xa) \in BLB \end{aligned}$$

$\Rightarrow \quad a \in BLB \quad$ (because $ay_i xa \in BWB \subseteq B$)

Since a is chosen arbitrarily which belongs to BLB,

$\Rightarrow \quad B \cap L \subseteq BLB.$ □

6. Characterizing Weakly Regular Semirings

Definition 6.1 ([4])**.** A semiring S is said to be right weakly-regular semiring if for each $a \in S$, $a \in (aS)^2$. Thus, if S is commutative than S is weakly regular iff S is regular.

Theorem 6.1. *The following are equivalent for a semiring W with multiplicative identity 1:*

(i) *W is weakly-regular,*

(ii) *$T^2 = T$, for all right-ideals T of W,*

(iii) *for every ideal I of W, $T \cap I = TI$.*

Proof. (i)⇒(ii): Since it is obvious that $T^2 \subseteq T$. Now for converse suppose $a \in T$ so $a \in (aT)^2$. Hence $a \in T^2$. So $T = T^2$.

(ii)⇒(iii): Let $a \in I$. Since

$$a \in (xW) = (xW)^2$$

$$\Rightarrow \quad a = ab, \quad \text{for some } b \in I.$$

For a right-ideal T of W obviously $TI \subseteq T \cap I$. Let $a \in T \cap I$ than $\exists\ b \in I : a = ab$. Thus $a \in TI$, i.e., $T \cap I \subseteq TI$. So $T \cap I = TI$.

(iii)⇒(i): Let $a \in W$, than

$$a \in (aW) \cap (WaW) = (aW)(WaW) \subseteq (aW^2)(aW) \subseteq (aW)(aW),$$

this implies $a \in (aW)^2$. Hence W is right weakly regular. □

Theorem 6.2. *If W is right weakly regular semiring with identity 1 than $B \cap I \cap R \subseteq BIR$ for bi-ideals B ideal I and right-ideal R of W.*

Proof. Suppose $x \in (B \cap I \cap R) \Rightarrow x \in B$, $x \in I$, $x \in R$.
Since $x \in W$ and it is given that W is right weakly regular implies that $x \in (xW)^2$, i.e.,

$$x = \sum_{\text{finite}} xr_i xs_i, \quad \text{for some } s_i \in W.$$

Now

$$\begin{aligned} x &= \sum_{\text{finite}} xs_i xr_i \\ &= \sum_{\text{finite}} xs_i \left(\sum_{\text{finite}} xs_i xr_i \right) r_i \\ &= \sum_{\text{finite}} xa_i xb_i xc_i, \end{aligned}$$

where $a_i, b_i, c_i \in W$.
Thus

$$x = \sum_{\text{finite}} x(a_i xb_i x)c_i \in BIR.$$

Hence $B \cap I \cap R \subseteq BIR$. □

Theorem 6.3. *If W is a semiring with unity than the following conditions are equivalent for all bi-ideals U, quasi-ideals X and two sided ideals V:*

(i) *W is right weakly regular,*

(ii) *$U \cap I \subseteq UI$,*

(iii) *$X \cap I \subseteq XI$.*

Proof. (i)⇒(ii): Assume $a \in U \cap V$ than $a \in U$ and $a \in V$.
Since $a \in W$ and W is right weakly regular it implies that $a \in (xW)^2$, i.e.,

$$a = \sum_{\text{finite}} xr_i as_i, \quad \text{for some } r_i, s_i \in W.$$

Now

$$a = \sum_{\text{finite}} x(r_i as_i) \in UV.$$

Thus

$$U \cap V \subseteq UV.$$

(ii)⇒(iii): Since a quasi-ideal is bi-ideal so by (ii) $X \cap I \subseteq XV$.
(iii)⇒(i): Since a one sided ideal is quasi-ideal, therefore by (iii) $X \cap V \subseteq XV$. But $XV \subseteq X \cap V$.
Therefore, $X \cap V = XV$.
So by the Theorem 6.1 W is right weakly regular. □

7. Few Semiring Applications in Electronics and Programming

More specifically, we look at how semirings are used in electronic logic gates and computer binary language to help us decide whether a result is strong or not using the 0 and 1 digits

Here, we will describe how to use the semiring set $B = (0,1)$ in various operations.

Logic gates are electronic circuits with several inputs and a single output. The three basic logic gates are AND, OR, and XOR gates.The NAND, NOR, EXCLUSIVE-OR, and EXCLUSIVE-NOR gates are additional logic gates that are derived from these fundamental gates.

Application of Logic Gates

Logic gates are the foundation of decision-making processes and task coordination in the digital world. It is a mathematical application of Boolean algebra and an electronic description of logical reasoning. We shall discuss the workings of the AND, OR, and XOR gates in relation to the set $B = (0,1)$ in terms of various semiring operations.
Let's construct two binary operations on B, namely (max and $\circ$), which are defined as for each a,b of B $\max(a,b) =$ greater of the two inputs and $\circ$ is defined as $a \circ b = a+b$, if $a \neq b$ otherwise 1. An OR gate is a gate that accepts two inputs simultaneously in the form of 0, and 1. The mechanism of the ring in the OR gate under the specified operation is shown in the table below.

OR Gate

0	0	0
0	1	1
1	0	1
1	1	1

The above table gets inputs from the first 2 columns and outputs them in the third column under the conditions of the operations (max) and ($\circ$) that we previously described.
This is the mechanism of the above ring under the above defined operations that is employed in OR gates of electronic studies to draw conclusions. In this table, the third column 1 and 0 values denote a strong and a weak result, respectively.

AND Gate

In contrast to the previous table, the current one's operations are (min) and ordinary multiplication, ($\cdot$) where (min) and ($\cdot$) are defined as the least input for each of the variables a,b in B and $a \cdot b = ab$, respectively. As an illustration, the semiring $(B, \min, \cdot)$ possesses additive identity as well as multiplicative identity.

The sole firm conclusion based on the gate mechanism makes this semiring an idempotent commutative semiring, which is why it is so beautiful.

0	0	0
0	1	0
1	0	0
1	1	1

XOR Gate

In this structure, the operations used are $+_2$ means addition modulo 2 defined as usual and $a\tau b$ defined as $a\tau b = a+b-2ab$, for all $a,b \in B$ with respect to these operations the mechanism follows ring structure.

0	0	$\circ$
0	1	1
1	0	1
1	1	0

8. Conclusion

This study introduces semirings, regular semirings and its kinds have been described using quasi-ideal and bi-ideal features, and some semiring applications in electronics and programming are demonstrated. The work can be extended to m-k ideals in gamma semirings and applications can be extended on different boolean structures.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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