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Research Article

Rough Identity-Summand Graph and Its Applications

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Abstract. This work aims to obtain various Gray code constructions from the structural properties of a rough identity-summand graph. To establish the nature of the rough identity-summand graph defined for the filters of a rough bi-Heyting algebra, the enumeration of distinct complete bipartite graphs from $G(F_X(T))$ will be presented in detail. Then, it will be proved that the union of these distinct complete bipartite graphs forms the subgraph of $G(F_X(T))$. This subgraph will be considered in identifying the various lengths of the Gray codes through two approximation transition sequences.

Keywords. Bi-Heyting algebra, Identity-summand graph, Complete bipartite graph, Gray code

Mathematics Subject Classification (2020). 06D20

1. Introduction

The mathematical approach of rough set theory addresses ambiguity and uncertainty in data. Z. Pawlak [3] introduced rough set theory in the early 1980s, offering a framework for estimating sets with inaccurate or insufficient information. The key concept of rough set theory is an information system, in general, an approximation space $I = (U,R)$, where the object set U is the universal set and R is an arbitrary equivalence relation that induces partition on U. For each subset X of U, T denotes the set of all rough sets defined through the lower and upper approximation. Two binary operations Praba Δ and Praba ∇ defined between the subsets of U, to find its least upper bound and greatest lower bound used to prove that $(T,\leq)$ is a lattice called a rough lattice. Later, the distributive property over T with the binary operations Δ and ∇ proved (T, Δ, ∇) to be a semiring known as a rough semiring (Praba *et al*. [5]). Considering this rough semiring structure, weaker notions of complements are verified over T to obtain the rough bi-Heyting algebra. In addition, establishing a filter (Praba and Freeda [4]) for the subsets of U in a rough bi-Heyting algebra helps to build the rough identity summand graph. In addition, various filters are characterized and their rough identity-summand graphs are obtained. The significance of the rough identity-summand graph is further extended to defining Gray code through its distinct complete bipartite graphs.

Suparta and Zanten [6] highlight the importance of consecutive numbers in the sequence and describe how to modify a Gray code's transition sequence to create a Gray code that produces a complete graph. The transition sequence of the Gray code is represented as a graph, with edges denoting subsequent bit flips in the Gray code sequence and vertices representing bit positions. This method aims to create a complete graph K_n. Suparta [7] has investigated techniques for creating complete and other structured bipartite graphs, Gray codes that induce particular bipartite graphs. While Arora [1] focuses on the generation of Gray codes using Hamiltonian networks and cycles, highlighting its importance in discrete mathematics and applications across a wide range of domains. A direct method based on Hamiltonian cycles and an iterative approach that can be computationally implemented are compared in the work in constructing Gray codes. The idea of strongly compatible codes and the investigation of how well Gray codes work with various graph configurations, such as paths and grids, are discussed by Wilmer and Ernst [8]. This work explores the existence of Gray codes compatible with different graph types and refutes a conjecture concerning the diameter of trees induced by cyclic Gray codes. Additionally, super-composed Gray codes are introduced and their properties are examined. To obtain *Generalized Gray Codes* (GGC) more easily, the idea of a GGC generation tree is presented by Lee and Lee [2]. Here each level of tree represents a distinct bit-length of GGC and the recursive link between the number of nodes at various tree levels is also covered in this study.

The literature discusses modifying the transition sequence so that it is used to create Gray code which produces a complete graph, particularly a complete bipartite graph. This paper introduces a new approach to establishing Gray code from the transition sequence with the help of distinct complete bipartite graphs obtained from the rough identity-summand graph. Analyzing the distinct complete bipartite graphs generated from the rough identity-summand graph, which contains all the vertices of the rough identity-summand graph but not the edges, plays a major role in the Gray code generation and aligns with the approach of identifying Gray codes of varying lengths. This applies to any filter $F_X(T)$ for which the rough identity-summand graph exists. Hence this work set a new strategy for constructing more robust Gray codes through the rough identity-summand graph.

The structure of this paper is as follows. The terminologies and notions related to filters and rough identity-summand graphs required for our study are given in Section 2. The primary outcome is discussed in Section 3 by establishing distinct complete bipartite graphs from the rough identity-summand graph and their vertex cardinality. Section 4 details the various Gray code generation through the subgraph of the rough identity-summand graph. The conclusion and the suggested future direction of this research are provided in Section 5.

2. Preliminaries

The primary notions of filter and rough identity-summand graph of a rough bi-Heyting algebra are as follows:

Definition 1 ([4]). For any $X \subseteq U$, the set $S \subseteq T$ is a filter of a rough bi-Heyting algebra. If

(1) S is closed under ∇,

(2) For $RS(Y) \in S$ and $RS(Z) \in T$, $RS(Y \Delta Z) \in S$.

Definition 2 ([4]). For any $X \subseteq U$, define $F_X(T)$ by $F_X(T) = \{RS(Y) \mid Y = X\Delta V \Delta W, V \in P(E \setminus E_X), W \in P(B \setminus B_X)\}$, where E is the set of equivalence classes in U and E_X be the set of equivalence classes in X, B is the pivot set of representative elements of equivalence classes and B_X is the pivot set of representative elements in X of equivalence classes, whose cardinality is greater than 1.

Theorem 1 ([4]). *$F_X(T)$ is a filter.*

Definition 3 ([4]). Let $(T, \Delta, \nabla, ^*, ^+, \rightarrow, \leftarrow, RS(\emptyset), RS(U))$ be a rough bi-Heyting algebra and $X \subseteq U$. A rough identity-summand graph of filter $F_X(T)$ is denoted by $G(F_X(T))$, whose vertex set is $V(F_X(T)) = \{RS(Y) \in T \setminus F_X(T) |$ for some $RS(Z) \in T \setminus F_X(T), RS(Y)\Delta RS(Z) = RS(Y \Delta Z) \in F_X(T)\}$ and the edge between the vertices of $RS(Y)$ and $RS(Z)$ exists if and only if $RS(Y\Delta Z) \in F_X(T)$.

3. Analyzing Distinct Complete Bipartite Graphs and Its Vertex Enumeration From Rough Identity-Summand Graph

This section deals with the enumeration of distinct complete bipartite graphs and their vertices from the rough identity-summand graph:

Theorem 2. *For any $X \subseteq U$, the cardinality of $F_X(T)$ is $2^r 3^{m-(t+r)} 2^{n-(m+k)}$.*

Proof. It is proved that $F_X(T) = \{RS(Y) | Y = X\Delta V \Delta W, V \in P(E \setminus E_X), W \in P(B \setminus B_X)\}$. Note that X contains pivot elements, equivalence classes with cardinality greater than one, and/(or) equivalence classes with cardinality equal to one. Let $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$ where $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\} \in B, |X_{s_1}|, |X_{s_2}|, \dots |X_{s_t}| > 1$ and $|X_{j_1}|, |X_{j_2}|, \dots, |X_{j_k}| = 1$ with $\{i_1, i_2, \dots, i_r, s_1, s_2, \dots, s_t\} \subseteq \{1, 2, \dots, m\}$ and $\{j_1, j_2, \dots, j_k\} \subseteq \{m+1, m+2, \dots, n\}$.
If $RS(Y) \in F_X(T)$ and when $Z = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$ then $Z \subseteq Y$ which can accommodate 2^r elements. And when $Z = \{X_{s_1}, X_{s_2}, \dots, X_{s_t}\}, Z \subseteq Y$, therefore the remaining $m-(t+r)$ equivalence classes whose cardinality is greater than one will have 3 choices namely $\emptyset$ (or) x_i (or) X_i.
Suppose when $Z = \{X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$ and $Z \subseteq Y$ then the remaining $n-(m+k)$ equivalence classes whose cardinality is equal to one will have 2 choices either $\emptyset$ (or) X_j. Hence the total number of such elements will be $2^r 3^{m-(t+r)} 2^{n-(m+k)}$. □

Lemma 1. *The number of distinct complete bipartite graphs generated from $G(F_X(T))$ when $X = \{z_1, z_2, \dots, z_\rho\}$ where $z_1, z_2, \dots, z_\rho$ are x_i (or) X_j for $i, j \in \{1, 2, \dots, m, m+1, \dots, n\}$ and $2 \le \rho \le m$,* *n is*

$$\sum_{p=1}^{v} {}^{r}\mathrm{C}_p + \sum_{p=1}^{v} {}^{t}\mathrm{C}_p + \sum_{p=1}^{v} {}^{k}\mathrm{C}_p + \sum_{p=2}^{v}\sum_{q=1}^{p-1}\{{}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q}\}$$
$$+ \sum_{p=3}^{v}\sum_{q=1}^{p-2}\sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)},$$

where $0 \le r,\ t \le m,\ 0 \le k \le n-m$ *and* $v = \frac{(r+t+k)-1}{2}$ *(or)* $v = \frac{(r+t+k)}{2}$.

Proof. In the proof, the number of distinct complete bipartite graphs generated from $G(F_X(T))$ will be given in different cases for various X

Case 1. Suppose $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$ for $\{i_1, i_2, \dots, i_r\} \in \{1,2,\dots,m\}$, the vertex set $V(F_X(T)) \subseteq T \setminus F_X(T)$ of $G(F_X(T))$ is partitioned into ${}^{r}\mathrm{C}_p$ ways for $p = 1,2,\dots,v$. When $p = 1$, the vertex set is partitioned as follows

$$V_{\{x_{i_1}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_1}\}\Delta P(E \setminus \{X_{i_2}, X_{i_3}, \dots, X_{i_r}\})\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$
$$V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_2}, x_{i_3}, \dots, x_{i_r}\}\Delta P(E \setminus \{X_{i_1}\}))\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$

where

$$V_{\{x_{i_1}\}}(F_X(T)) \cup V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) = V_{\{x_{i_1},x_{i_2}x_{i_3}\dots x_{i_r}\}}(F_X(T)) \subseteq V(F_X(T)),$$
$$V_{\{x_{i_1}\}}(F_X(T)) \cap V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) = \emptyset$$

and

$$V_{\{x_{i_2}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_2}\}\Delta P(E \setminus \{X_{i_1}, X_{i_3}, \dots, X_{i_r}\})\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$
$$V_{\{x_{i_1},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_1}, x_{i_3}, \dots, x_{i_r}\}\Delta P(E \setminus \{X_{i_2}\}))\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$

where

$$V_{\{x_{i_2}\}}(F_X(T)) \cup V_{\{x_{i_1},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) = V_{\{x_{i_2},x_{i_1}x_{i_3}\dots x_{i_r}\}}(F_X(T)) \subseteq V(F_X(T)),$$
$$V_{\{x_{i_2}\}}(F_X(T)) \cap V_{\{x_{i_1},x_{i_3},\dots,x_{i_r}\}}(F_X(T)) = \emptyset$$

and so on,

$$V_{\{x_{i_r}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_r}\}\Delta P(E \setminus \{X_{i_1}, X_{i_2}, \dots, X_{i_{r-1}}\})\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}, \dots, x_{i_{r-1}}\}\Delta P(E \setminus \{X_{i_r}\}))\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$

where

$$V_{\{x_{i_r}\}}(F_X(T)) \cup V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}}\}}(F_X(T)) = V_{\{x_{i_r},x_{i_1}x_{i_2}\dots x_{i_{r-1}}\}}(F_X(T)) \subseteq V(F_X(T)),$$
$$V_{\{x_{i_r}\}}(F_X(T)) \cap V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}}\}}(F_X(T)) = \emptyset.$$

So for $p = 1$, ${}^{r}\mathrm{C}_1$ ways vertex set are partitioned into generating ${}^{r}\mathrm{C}_1$ complete bipartite graphs. Now partitioning the vertex set for $p = 2$,

$$V_{\{x_{i_1},x_{i_2}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}\}\Delta P(E \setminus \{X_{i_3}, X_{i_4}, \dots, X_{i_r}\})\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$
$$V_{\{x_{i_3},x_{i_4},\dots,x_{i_r}\}}(F_X(T)) = \{RS(Z) | Z \in \{x_{i_3}, x_{i_4}, \dots, x_{i_r}\}\Delta P(E \setminus \{X_{i_1}, X_{i_2}\})\Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},$$

where

$$V_{\{x_{i_1},x_{i_2}\}}(F_X(T)) \cup V_{\{x_{i_3},x_{i_4},\dots,x_{i_r}\}}(F_X(T)) = V_{\{x_{i_1}x_{i_2},x_{i_3}\dots x_{i_r}\}}(F_X(T)) \subseteq V(F_X(T)),$$
$$V_{\{x_{i_1},x_{i_2}\}}(F_X(T)) \cap V_{\{x_{i_3},x_{i_4},\dots,x_{i_r}\}}(F_X(T)) = \emptyset$$

and so on,

$$\begin{aligned}&V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T))\cup V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-2}}\}}(F_X(T))\\&\quad=\{RS(Z)|Z\in\{x_{i_{r-1}},x_{i_r}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\dots,X_{i_{r-2}}\})\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\}\\&\qquad\cup\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_{r-2}}\}\Delta P(E\setminus\{X_{i_{r-1}},X_{i_r}\})\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},\end{aligned}$$

where

$$V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T))\cup V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-2}}\}}(F_X(T))=V_{\{x_{i_{r-1}}x_{i_r},x_{i_1}\dots x_{i_{r-2}}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T))\cap V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-2}}\}}(F_X(T))=\emptyset.$$

So ${}^r\mathrm{C}_2$ ways vertex set are partitioned into generating ${}^r\mathrm{C}_2$ complete bipartite graphs. Continuing this way for $p=3,4,\dots,(r-2)$, the vertex set partitioning of ${}^r\mathrm{C}_3,{}^r\mathrm{C}_4,\dots,{}^r\mathrm{C}_{(r-2)}$ ways enable to generate ${}^r\mathrm{C}_3,{}^r\mathrm{C}_4,\dots,{}^r\mathrm{C}_{(r-2)}$ complete bipartite graphs. Now partitioning the vertex set for $p=r-1$,

$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}}\}}(F_X(T))=\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}}\}\Delta P(E\setminus\{X_{i_r}\})\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$
$$V_{\{x_{i_r}\}}(F_X(T))=\{RS(Z)|Z\in\{x_{i_r}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\dots,X_{i_{r-1}}\})\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$

where

$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}}\}}(F_X(T))\cup V_{\{x_{i_r}\}}(F_X(T))=V_{\{x_{i_1}x_{i_2}\dots x_{i_{r-1}},x_{i_r}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}}\}}(F_X(T))\cap V_{\{x_{i_r}\}}(F_X(T))=\emptyset$$

and so on,

$$V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T))=\{RS(Z)|Z\in\{x_{i_2},x_{i_3},\dots,x_{i_r}\}\Delta P(E\setminus\{X_{i_1}\})\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$
$$V_{\{x_{i_1}\}}(F_X(T))=\{RS(Z)|Z\in\{x_{i_1}\}\Delta P(E\setminus\{X_{i_2},X_{i_3},\dots,X_{i_r}\})\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$

where

$$V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T))\cup V_{\{x_{i_1}\}}(F_X(T))=V_{\{x_{i_2}x_{i_3}\dots x_{i_r},x_{i_1}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{x_{i_2},x_{i_3},\dots,x_{i_r}\}}(F_X(T))\cap V_{\{x_{i_1}\}}(F_X(T))=\emptyset.$$

Then ${}^r\mathrm{C}_{(r-1)}$ complete bipartite graphs are obtained. But from the above arguments, the complete bipartite graph obtained while partitioning the vertex set for $p=1$ and $p=r-1$ are isomorphic, $p=2$ and $p=r-2$ are isomorphic, and so on. To avoid this repetition, assume r is odd. Then $(r-1)$ even number of ways vertex set are partitioned into generating ${}^r\mathrm{C}_{r-1}$ complete bipartite graphs. By using the previous argument, $(r-1)$ even terms are taken into $\frac{(r-1)}{2}$ terms excluding the repetition. Hence $\sum_{p=1}^{\frac{(r-1)}{2}}{}^r\mathrm{C}_p$ complete bipartite graphs are generated, when r is odd.

Suppose if r is even, then $(r-1)$ odd number ways vertex set are partitioned into generating ${}^r\mathrm{C}_{r-1}$ complete bipartite graphs. Then using the earlier argument to avoid the repetition, $(r-1)$ odd terms are divided into 2 equal parts by the middle term $\frac{r}{2}$. Hence $\sum_{p=1}^{\frac{r}{2}}{}^r\mathrm{C}_p$ complete bipartite graphs are generated. But if r is even and $\frac{r}{2}$ dividing $(r-1)$ terms equally is even, leading to the repetition again at the stage of $p=\frac{r}{2}$. Hence when $p=\frac{r}{2}$, ${}^r\mathrm{C}_p=\frac{{}^r\mathrm{C}_p}{2}$.

Case 2. Suppose $X = \{X_{s_1}, X_{s_2}, \ldots, X_{s_t}\}$, for $\{s_1, s_2, \ldots, s_t\} \in \{1, 2, \ldots, m\}$, the vertex set $V(F_X(T)) \subseteq T \setminus F_X(T)$ of $G(F_X(T))$ is partitioned into ${}^t\mathrm{C}_p$ ways for $p = 1, 2, \ldots, v$. When $p = 1$, the vertex set is partitioned as follows:

$$V_{\{X_{s_1}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{s_1}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}\}) \Delta P(B \setminus \{x_{s_1}\})\},$$

$$V_{\{X_{s_2}, X_{s_3}, \ldots, X_{s_t}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{s_2}, X_{s_3}, \ldots, X_{s_t}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}\})) \Delta P(B \setminus \{x_{s_2}, x_{s_3}, \ldots, x_{s_t}\})\}$$

where

$$V_{\{X_{s_1}\}}(F_X(T)) \cup V_{\{X_{s_2}, X_{s_3}, \ldots, X_{s_t}\}}(F_X(T)) = V_{\{X_{s_1}, X_{s_2} X_{s_3} \ldots X_{s_t}\}}(F_X(T)) \subseteq V(F_X(T)) V_{\{X_{s_1}\}}(F_X(T)) \cap V_{\{X_{s_2}, X_{s_3}, \ldots, X_{s_t}\}}(F_X(T)) = \emptyset$$

and

$$V_{\{X_{s_2}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{s_2}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_r}\}) \Delta P(B \setminus \{x_{s_2}\})\},$$

$$V_{\{X_{s_1}, X_{s_3}, \ldots, X_{s_t}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{s_1}, X_{s_3}, \ldots, X_{s_t}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}\})) \Delta P(B \setminus \{x_{s_1}, x_{s_3}, \ldots, x_{s_t}\})\},$$

where

$$V_{\{X_{s_2}\}}(F_X(T)) \cup V_{\{X_{s_1}, X_{s_3}, \ldots, X_{s_t}\}}(F_X(T)) = V_{\{X_{s_2}, X_{s_1} X_{s_3} \ldots X_{s_t}\}}(F_X(T)) \subseteq V(F_X(T)) V_{\{X_{s_2}\}}(F_X(T)) \cap V_{\{X_{s_1}, X_{s_3}, \ldots, X_{s_t}\}}(F_X(T)) = \emptyset$$

and so on,

$$V_{\{X_{s_t}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{s_t}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}\}) \Delta P(B \setminus \{x_{s_t}\})\}$$

$$V_{\{X_{s_1}, X_{s_2}, \ldots, X_{s_{t-1}}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{s_1}, X_{s_2}, \ldots, X_{s_{t-1}}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}\})) \Delta P(B \setminus \{x_{s_1}, x_{s_2}, \ldots, x_{s_{t-1}}\})\}$$

with

$$V_{\{X_{s_t}\}}(F_X(T)) \cup V_{\{X_{s_1}, X_{s_2}, \ldots, X_{s_{t-1}}\}}(F_X(T)) = V_{\{X_{s_t}, X_{s_1} X_{s_2} \ldots X_{s_{t-1}}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{s_t}\}}(F_X(T)) \cap V_{\{X_{s_1}, X_{s_2}, \ldots, X_{s_{t-1}}\}}(F_X(T)) = \emptyset.$$

So for $p = 1$, ${}^t\mathrm{C}_1$ ways vertex set are partitioned into generating ${}^t\mathrm{C}_1$ complete bipartite graphs. Similarly, partitioning the vertex set for $p = 2, 3, \ldots, (t-1)$, leads to generate ${}^t\mathrm{C}_2, {}^t\mathrm{C}_3, \ldots, {}^t\mathrm{C}_{(t-1)}$ complete bipartite graphs. Then using the arguments of r in Case 1, if t is odd, $\sum_{p=1}^{\frac{(t-1)}{2}} {}^t\mathrm{C}_p$ complete bipartite graphs are generated. But if t is even, $\sum_{p=1}^{\frac{t}{2}} {}^t\mathrm{C}_p$ complete bipartite graphs are generated, at $p = \frac{t}{2}$ replace ${}^t\mathrm{C}_p$ by $\frac{{}^t\mathrm{C}_p}{2}$.

Case 3. Suppose $X = \{X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}$ for $\{j_1, j_2, \ldots, j_k\} \in \{m+1, m+2, \ldots, n\}$. When $p = 1$, the vertex set $V(F_X(T)) \subseteq T \setminus F_X(T)$ of $G(F_X(T))$ is partitioned into ${}^k\mathrm{C}_1$ ways as follows

$$V_{\{X_{j_1}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{j_1}\} \Delta P(E \setminus \{X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\}) \Delta P(B)\},$$

$$V_{\{X_{j_2}, X_{j_3}, \ldots, X_{j_k}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{j_2}, X_{j_3}, \ldots, X_{j_k}\} \Delta P(E \setminus \{X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\}) \Delta P(B)\},$$

where

$$V_{\{X_{j_1}\}}(F_X(T)) \cup V_{\{X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = V_{\{X_{j_1},X_{j_2}X_{j_3}\dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{j_1}\}}(F_X(T)) \cap V_{\{X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = \emptyset,$$

$$V_{\{X_{j_2}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{j_2}\}\Delta P(E \setminus \{X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\})\Delta P(B)\},$$

$$V_{\{X_{j_1},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{j_1},X_{j_3},\dots,X_{j_k}\}\Delta P(E \setminus \{X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\}))\Delta P(B)\},$$

where

$$V_{\{X_{j_2}\}}(F_X(T)) \cup V_{\{X_{j_1},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = V_{\{X_{j_2},X_{j_1}X_{j_3}\dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{j_2}\}}(F_X(T)) \cap V_{\{X_{j_1},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = \emptyset$$

and so on,

$$V_{\{X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{j_k}\}\Delta P(E \setminus \{X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\})\Delta P(B)\},$$

$$V_{\{X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}\Delta P(E \setminus \{X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\}))\Delta P(B)\}$$

with

$$V_{\{X_{j_k}\}}(F_X(T)) \cup V_{\{X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T)) = V_{\{X_{j_k},X_{j_1}X_{j_2}\dots X_{j_{k-1}}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{j_k}\}}(F_X(T)) \cap V_{\{X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T)) = \emptyset.$$

This generates ${}^k\mathrm{C}_1$ complete bipartite graphs. The remaining ${}^k\mathrm{C}_2, {}^k\mathrm{C}_3, \dots, {}^k\mathrm{C}_{k-1}$ complete bipartite graphs are generated for $p = 2,3,\dots(k-1)$. Then using similar arguments from Case 1, if k is odd, $\sum_{p=1}^{\frac{(k-1)}{2}} {}^k\mathrm{C}_p$ complete bipartite graphs are generated. But if k is even, $\sum_{p=1}^{\frac{k}{2}} {}^k\mathrm{C}_p$ complete bipartite graphs are generated and at $p = \frac{k}{2}$, ${}^k\mathrm{C}_p = \frac{{}^k\mathrm{C}_p}{2}$.

Case 4. Suppose $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}\}$ for $\{i_1, i_2, \dots, i_r, s_1, s_2, \dots, s_t\} \in \{1,2,\dots,m\}$, the vertex set $V(F_X(T)) \subseteq T \setminus F_X(T)$ of $G(F_X(T))$ is partitioned for $p = 1,2,\dots,v$. When $p = 1$, ${}^r\mathrm{C}_1$ ways vertex set are partitioned into

$$V_{\{x_{i_1}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{x_{i_1}\}\Delta P(E \setminus \{X_{i_2},X_{i_3},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\})\Delta P(B \setminus \{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$

$$V_{\{x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}\Delta P(E \setminus \{X_{i_1},X_{s_1},X_{s_2},\dots,X_{s_t}\}))$$
$$\Delta P(B \setminus \{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_t}\})\},$$

where

$$V_{\{x_{i_1}\}}(F_X(T)) \cup V_{\{x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T)) = V_{\{x_{i_1},x_{i_2}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_t}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{x_{i_1}\}}(F_X(T)) \cap V_{\{x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T)) = \emptyset$$

and so on,

$$V_{\{x_{i_r}\}}(F_X(T)) \cup V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{x_{i_r}\}\Delta P(E \setminus \{X_{i_1},X_{i_2},\dots,X_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_t}\})\Delta P(B \setminus \{x_{i_1},x_{i_2},\dots,x_{i_r}\})\}$$

$$\cup\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_t}\}\Delta P(E\setminus\{X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}) \\ \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_t}\})\},$$

where

$$V_{\{x_{i_r}\}}(F_X(T))\cup V_{\{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T))=V_{\{x_{i_r},x_{i_1}\dots x_{i_{r-1}}X_{s_1}X_{s_2}\dots X_{s_t}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{x_{i_r}\}}(F_X(T))\cap V_{\{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T))=\emptyset$$

generating ${}^rC_1$ complete bipartite graphs. Similarly for $p=1$, ${}^tC_1$ ways vertex set are partitioned into

$$V_{\{X_{s_1}\}}(F_X(T))=\{RS(Z)|Z\in\{X_{s_1}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}) \\ \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1}\})\},$$

$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_2},X_{s_3},\dots,X_{s_t}\}}(F_X(T)) \\ =\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_2},X_{s_3},\dots,X_{s_t}\}\Delta P(E\setminus\{X_{s_1},X_{s_2},\dots,X_{s_t}\}) \\ \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_2},x_{s_3},\dots,x_{s_t}\})\},$$

where

$$V_{\{X_{s_1}\}}(F_X(T))\cup V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t}\}}(F_X(T))=V_{\{X_{s_1},x_{i_1}\dots x_{i_r}X_{s_2}X_{s_3}\dots X_{s_t}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{s_1}\}}(F_X(T))\cap V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t}\}}(F_X(T))=\emptyset$$

and so on,

$$V_{\{X_{s_t}\}}(F_X(T))\cup V_{\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_{t-1}}\}}(F_X(T)) \\ =\{RS(Z)|Z\in\{X_{s_t}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}) \\ \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_t}\})\}\cup\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_{t-1}}\} \\ \Delta P(E\setminus\{X_{s_1},X_{s_2},\dots,X_{s_t}\})\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_{t-1}}\})\}$$

with

$$V_{\{X_{s_t}\}}(F_X(T))\cup V_{\{x_{i_1},\dots,x_{i_r},X_{s_1},\dots,X_{s_{t-1}}\}}(F_X(T))=V_{\{X_{s_t},x_{i_1}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_{t-1}}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{s_t}\}}(F_X(T))\cap V_{\{x_{i_1},\dots,x_{i_r},X_{s_1},\dots,X_{s_{t-1}}\}}(F_X(T))=\emptyset$$

generating ${}^tC_1$ complete bipartite graphs. Hence the total number of complete bipartite graphs generated for $p=1$ is ${}^rC_1+{}^tC_1$. Now for $p=2$, ${}^rC_2$ ways vertex set are partitioned into

$$V_{\{x_{i_1},x_{i_2}\}}(F_X(T))=\{RS(Z)|Z\in\{x_{i_1},x_{i_2}\}\Delta P(E\setminus\{X_{i_3},X_{i_4},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}) \\ \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$

$$V_{\{x_{i_3},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T))=\{RS(Z)|Z\in\{x_{i_3},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\} \\ \Delta P(E\setminus\{X_{i_1},X_{i_2},X_{s_1},X_{s_2},\dots,X_{s_t}\}) \\ \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_t}\})\}$$

with

$$V_{\{x_{i_1},x_{i_2}\}}(F_X(T))\cup V_{\{x_{i_3},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T))=V_{\{x_{i_1}x_{i_2},x_{i_3}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_t}\}}(F_X(T)) \\ \subseteq V(F_X(T)),$$
$$V_{\{x_{i_1},x_{i_2}\}}(F_X(T))\cap V_{\{x_{i_3},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}}(F_X(T))=\emptyset$$

and so on

$$V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{x_{i_{r-1}},x_{i_r}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\ldots,X_{i_{r-2}},X_{s_1},X_{s_2},\ldots,X_{s_t}\})$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\ldots,x_{i_r}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\ldots,x_{i_{r-2}},X_{s_1},X_{s_2},\ldots,X_{s_t}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{x_{i_1},x_{i_2},\ldots,x_{i_{r-2}},X_{s_1},X_{s_2},\ldots,X_{s_t}\}$$
$$\Delta P(E\setminus\{X_{i_{r-1}},X_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_t}\}))\Delta P(B\setminus\{x_{i_1},x_{i_2},\ldots,x_{i_r},x_{s_1},x_{s_2},\ldots,x_{s_t}\})\}$$

with

$$V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T))\cup V_{\{x_{i_1},\ldots,x_{i_{r-2}},X_{s_1},X_{s_2},\ldots,X_{s_t}\}}(F_X(T))$$
$$= V_{\{x_{i_{r-1}}x_{i_r},x_{i_1}\ldots x_{i_{r-2}}X_{s_1}X_{s_2}\ldots X_{s_t}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T))\cap V_{\{x_{i_1},\ldots,x_{i_{r-2}},X_{s_1},X_{s_2},\ldots,X_{s_t}\}}(F_X(T)) = \emptyset$$

generating ${}^{r}\mathrm{C}_2$ complete bipartite graphs, and ${}^{t}\mathrm{C}_2$ ways vertex set are partitioned into

$$V_{\{X_{s_1},X_{s_2}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{s_1},X_{s_2}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\ldots,X_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_t}\})$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\ldots,x_{i_r},x_{s_1},x_{s_2}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_3},X_{s_4},\ldots,X_{s_t}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_3},X_{s_4},\ldots,X_{s_t}\}\Delta P(E\setminus\{X_{s_1},X_{s_2},\ldots,X_{s_t}\}))$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\ldots,x_{i_r},x_{s_3},x_{s_4},\ldots,x_{s_t}\})\}$$

with

$$V_{\{X_{s_1},X_{s_2}\}}(F_X(T))\cup V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_3},X_{s_4},\ldots,X_{s_t}\}}(F_X(T))$$
$$= V_{\{X_{s_1}X_{s_2},x_{i_1}\ldots x_{i_r}X_{s_3}\ldots X_{s_t}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{s_1},X_{s_2}\}}(F_X(T))\cap V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_3},X_{s_4},\ldots,X_{s_t}\}}(F_X(T)) = \emptyset$$

and so on,

$$V_{\{X_{s_{t-1}},X_{s_t}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{s_{t-1}},X_{s_t}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\ldots,X_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_t}\})$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\ldots,x_{i_r},x_{s_{t-1}},x_{s_t}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_{t-2}}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_{t-2}}\}\Delta P(E\setminus\{X_{s_1},X_{s_2},\ldots,X_{s_t}\}))$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\ldots,x_{i_r},x_{s_1},x_{s_2},\ldots,x_{s_{t-2}}\})\}$$

with

$$V_{\{X_{s_{t-1}},X_{s_t}\}}(F_X(T))\cup V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_{t-2}}\}}(F_X(T))$$
$$= V_{\{X_{s_{t-1}}X_{s_t},x_{i_1}\ldots x_{i_r}X_{s_1}\ldots X_{s_{t-2}}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{s_{t-1}},X_{s_t}\}}(F_X(T))\cap V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{s_1},X_{s_2},\ldots,X_{s_{t-2}}\}}(F_X(T)) = \emptyset$$

generating ${}^{t}C_2$ complete bipartite graphs. Also, for $p=2$, there is another possible way of partitioning the vertex set is ${}^{r}C_1{}^{t}C_1$ way as

$$V_{\{x_{i_1},X_{s_1}\}}(F_X(T)) = \{RS(Z)|Z \in \{x_{i_1},X_{s_1}\}\Delta P(E\setminus\{X_{i_2},X_{i_3},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\}) \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1}\})\},$$

$$V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t}\}}(F_X(T)) = \{RS(Z)|Z \in \{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t}\}\Delta P(E\setminus\{X_{i_1},X_{s_1},X_{s_2},\dots,X_{s_t}\})) \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_2},\dots,x_{s_t}\})\}$$

with

$$V_{\{x_{i_1},X_{s_1}\}}(F_X(T)) \cup V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t}\}}(F_X(T)) = V_{\{x_{i_1}X_{s_1},x_{i_2}\dots x_{i_r}X_{s_2}\dots X_{s_t}\}}(F_X(T)) \subseteq V(F_X(T))$$
$$V_{\{x_{i_1},X_{s_1}\}}(F_X(T)) \cap V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t}\}}(F_X(T)) = \emptyset$$

and so on,

$$V_{\{x_{i_r},X_{s_t}\}}(F_X(T)) = \{RS(Z)|Z \in \{x_{i_r},X_{s_t}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\dots,X_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_t}\}) \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_t}\})\}$$

$$V_{\{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},\dots,X_{s_{t-1}}\}}(F_X(T)) = \{RS(Z)|Z \in \{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},\dots,X_{s_{t-1}}\}\Delta P(E\setminus\{X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t}\})) \Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},\dots,x_{s_{t-1}}\})\}$$

with

$$V_{\{x_{i_r},X_{s_t}\}}(F_X(T)) \cup V_{\{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},\dots,X_{s_{t-1}}\}}(F_X(T)) = V_{\{x_{i_r}X_{s_t},x_{i_1}\dots x_{i_{r-1}}X_{s_1}\dots X_{s_{t-1}}\}}(F_X(T)) \subseteq V(F_X(T))$$
$$V_{\{x_{i_r},X_{s_t}\}}(F_X(T)) \cap V_{\{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},\dots,X_{s_{t-1}}\}}(F_X(T)) = \emptyset$$

generating ${}^{r}C_1{}^{t}C_1$ complete bipartite graphs. Hence for $p=2$, the total number of complete bipartite graphs generated are ${}^{r}C_2 + {}^{t}C_2 + {}^{r}C_1{}^{t}C_1$. Repeating this procedure for $p=3$, ${}^{r}C_3 + {}^{t}C_3 + {}^{r}C_1{}^{t}C_2 + {}^{r}C_2{}^{t}C_1$ complete bipartite graphs can be generated and so on. This procedure terminates at the finite step for the choices of r and t. Suppose if $r+t$ is odd, then using the arguments from the earlier cases $\frac{(r+t)-1}{2}$ ways vertex set can be partitioned, and if $r<t$ then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{\frac{(r+t)-1}{2}} {}^{t}C_p + \sum_{p=2}^{\frac{(r+t)-1}{2}} \sum_{q=1}^{p-1} {}^{r}C_q{}^{t}C_{p-q}$$

complete bipartite graphs can be generated. If $r>t$, then

$$\sum_{p=1}^{\frac{(r+t)-1}{2}} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=2}^{\frac{(r+t)-1}{2}} \sum_{q=1}^{p-1} {}^{r}C_q{}^{t}C_{p-q}$$

complete bipartite graphs can be generated. On the other hand, if $r+t$ is even then $\frac{(r+t)}{2}$ ways vertex set can be partitioned and if $r < t$ then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t)}{2}} {}^{t}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t)}{2}} \sum_{q=1}^{p-1} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q}$$

complete bipartite graphs are generated with when $p = \frac{(r+t)}{2}$, ${}^{t}\mathrm{C}_p = \frac{{}^{t}\mathrm{C}_p}{2}$, ${}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} = \frac{{}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q}}{2}$.
If $r > t$, then

$$\sum_{p=1}^{\frac{(r+t)}{2}} {}^{r}\mathrm{C}_p + \sum_{p=1}^{t} {}^{t}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t)}{2}} \sum_{q=1}^{p-1} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q}$$

complete bipartite graphs are generated with when $p = \frac{(r+t)}{2}$, ${}^{r}\mathrm{C}_p = \frac{{}^{r}\mathrm{C}_p}{2}$, ${}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} = \frac{{}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q}}{2}$.
Suppose if $r = t$, then

$$\sum_{p=1}^{\frac{(r+t)}{2}} {}^{r}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t)}{2}} {}^{t}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t)}{2}} \sum_{q=1}^{p-1} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q}$$

complete bipartite graphs are generated with at $p = \frac{(r+t)}{2}$, ${}^{r}\mathrm{C}_p = \frac{{}^{r}\mathrm{C}_p}{2}$, ${}^{t}\mathrm{C}_p = \frac{{}^{r}\mathrm{C}_p}{2}$, ${}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} = \frac{{}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q}}{2}$.

Case 5. Suppose $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$ for $\{i_1, i_2, \dots, i_r\} \in \{1, 2, \dots, m\}$ and $\{j_1, j_2, \dots, j_k\} \in \{m+1, m+2, \dots, n\}$, the vertex set partitioning of $V(F_X(T)) \subseteq T \setminus F_X(T)$ of $G(F_X(T))$ for $p = 1, 2, \dots, v$ is as follows. When $p = 1$, ${}^{r}\mathrm{C}_1$ ways vertex set are partitioned into

$$\begin{aligned}
&V_{\{x_{i_1}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{x_{i_1}\} \Delta P(E \setminus \{X_{i_2}, X_{i_3}, \dots, X_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\}, \\
&V_{\{x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{i_1}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})) \\
&\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},
\end{aligned}$$

where

$$\begin{aligned}
&V_{\{x_{i_1}\}}(F_X(T)) \cup V_{\{x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) = V_{\{x_{i_1}, x_{i_2} \dots x_{i_r} X_{j_1} X_{j_2} \dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)), \\
&V_{\{x_{i_1}\}}(F_X(T)) \cap V_{\{x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) = \emptyset
\end{aligned}$$

and so on,

$$\begin{aligned}
&V_{\{x_{i_r}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{x_{i_r}\} \Delta P(E \setminus \{X_{i_1}, X_{i_2}, \dots, X_{i_{r-1}}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\}, \\
&V_{\{x_{i_1}, x_{i_2}, \dots, x_{i_{r-1}}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}, \dots, x_{i_{r-1}}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})) \\
&\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},
\end{aligned}$$

where

$$\begin{aligned}
V_{\{x_{i_r}\}}(F_X(T)) \cup V_{\{x_{i_1}, \dots, x_{i_{r-1}}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) &= V_{\{x_{i_r}, x_{i_1} \dots x_{i_{r-1}} X_{j_1} X_{j_2} \dots X_{j_k}\}}(F_X(T)) \\
&\subseteq V(F_X(T)),
\end{aligned}$$

$$V_{\{x_{i_r}\}}(F_X(T)) \cap V_{\{x_{i_1},\ldots,x_{i_{r-1}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) = \emptyset$$

generating ${}^{r}\mathrm{C}_1$ complete bipartite graphs. Then partitioning the vertex set for $p = 1$ in ${}^{k}\mathrm{C}_1$ ways

$$\begin{aligned} &V_{\{X_{j_1}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{j_1}\}\Delta P(E \setminus \{X_{i_1}, X_{i_2}, \ldots, X_{i_r}, X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\}, \\ &V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_2},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\}\Delta P(E \setminus \{X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\}, \end{aligned}$$

where

$$\begin{aligned} V_{\{X_{j_1}\}}(F_X(T)) \cup V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_2},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) &= V_{\{X_{j_1},x_{i_1}\ldots x_{i_r}X_{j_2}X_{j_3}\ldots X_{j_k}\}}(F_X(T)) \\ &\subseteq V(F_X(T)), \end{aligned}$$

$$V_{\{X_{j_1}\}}(F_X(T)) \cap V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_2},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) = \emptyset$$

and so on,

$$\begin{aligned} &V_{\{X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{X_{j_k}\}\Delta P(E \setminus \{X_{i_1}, X_{i_2}, \ldots, X_{i_r}, X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\}, \\ &V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_{k-1}}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}, X_{j_1}, X_{j_2}, \ldots, X_{j_{k-1}}\}\Delta P(E \setminus \{X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\}, \end{aligned}$$

where

$$\begin{aligned} V_{\{X_{j_k}\}}(F_X(T)) \cup V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_{k-1}}\}}(F_X(T)) &= V_{\{X_{j_k},x_{i_1}\ldots x_{i_r}X_{j_1}X_{j_2}\ldots X_{j_{k-1}}\}}(F_X(T)) \\ &\subseteq V(F_X(T)), \end{aligned}$$

$$V_{\{X_{j_k}\}}(F_X(T)) \cap V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_{k-1}}\}}(F_X(T)) = \emptyset$$

generating ${}^{k}\mathrm{C}_1$ complete bipartite graphs. Hence the total number of complete bipartite graph generated when $p = 1$ is ${}^{r}\mathrm{C}_1 + {}^{k}\mathrm{C}_1$. Now for $p = 2$, ${}^{r}\mathrm{C}_2$ ways vertex set are partitioned into

$$\begin{aligned} &V_{\{x_{i_1},x_{i_2}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}\}\Delta P(E \setminus \{X_{i_3}, X_{i_4}, \ldots, X_{i_r}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\}, \\ &V_{\{x_{i_3},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z) | Z \in \{x_{i_3}, \ldots, x_{i_r}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}\Delta P(E \setminus \{X_{i_1}, X_{i_2}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\})\}, \end{aligned}$$

where

$$\begin{aligned} V_{\{x_{i_1},x_{i_2}\}}(F_X(T)) \cup V_{\{x_{i_3},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) &= V_{\{x_{i_1}x_{i_2},x_{i_3}\ldots x_{i_r}X_{j_1}X_{j_2}\ldots X_{j_k}\}}(F_X(T)) \\ &\subseteq V(F_X(T)), \end{aligned}$$

$$V_{\{x_{i_1},x_{i_2}\}}(F_X(T)) \cap V_{\{x_{i_3},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) = \emptyset$$

and so on,

$$\begin{aligned} &V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T)) \\ &\quad = \{RS(Z)|Z \in \{x_{i_{r-1}},x_{i_r}\}\Delta P(E \setminus \{X_{i_1},X_{i_2},\ldots,X_{i_{r-2}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1},x_{i_2},\ldots,x_{i_r}\})\}, \\ &V_{\{x_{i_1},x_{i_2},\ldots,x_{i_{r-2}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z)|Z \in \{x_{i_1},x_{i_2},\ldots,x_{i_{r-2}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}\Delta P(E \setminus \{X_{i_{r-1}},X_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_k}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1},x_{i_2},\ldots,x_{i_r}\})\}, \end{aligned}$$

where

$$\begin{aligned} V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T)) \cup V_{\{x_{i_1},\ldots,x_{i_{r-2}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) &= V_{\{x_{i_{r-1}}x_{i_r},x_{i_1}\ldots x_{i_{r-2}}X_{j_1}X_{j_2}\ldots X_{j_k}\}}(F_X(T)) \\ &\subseteq V(F_X(T)), \end{aligned}$$

$$V_{\{x_{i_{r-1}},x_{i_r}\}}(F_X(T)) \cap V_{\{x_{i_1},\ldots,x_{i_{r-2}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) = \emptyset$$

generating ${}^{r}\mathrm{C}_2$ complete bipartite graphs. Then ${}^{k}\mathrm{C}_2$ ways partitioning the vertex set as

$$\begin{aligned} &V_{\{X_{j_1},X_{j_2}\}}(F_X(T)) \\ &\quad = \{RS(Z)|Z \in \{X_{j_1},X_{j_2}\}\Delta P(E \setminus \{X_{i_1},X_{i_2},\ldots,X_{i_r},X_{j_1},X_{j_2},X_{j_3},\ldots,X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1},x_{i_2},\ldots,x_{i_r}\})\}, \\ &V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z)|Z \in \{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_3},X_{j_4},\ldots,X_{j_k}\}\Delta P(E \setminus \{X_{j_1},X_{j_2},X_{j_3},\ldots,X_{j_k}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1},x_{i_2},\ldots,x_{i_r}\})\}, \end{aligned}$$

where

$$\begin{aligned} V_{\{X_{j_1},X_{j_2}\}}(F_X(T)) \cup V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) &= V_{\{X_{j_1}X_{j_2},x_{i_1}\ldots x_{i_r}X_{j_3}\ldots X_{j_k}\}}(F_X(T)) \\ &\subseteq V(F_X(T)), \end{aligned}$$

$$V_{\{X_{j_1},X_{j_2}\}}(F_X(T)) \cap V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) = \emptyset$$

and so on. Then

$$\begin{aligned} &V_{\{X_{j_{k-1}},X_{j_k}\}}(F_X(T)) \\ &\quad = \{RS(Z)|Z \in \{X_{j_{k-1}},X_{j_k}\}\Delta P(E \setminus \{X_{i_1},X_{i_2},\ldots,X_{i_r},X_{j_1},X_{j_2},X_{j_3},\ldots,X_{j_k}\}) \\ &\qquad \Delta P(B \setminus \{x_{i_1},x_{i_2},\ldots,x_{i_r}\})\}, \\ &V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_{k-2}}\}}(F_X(T)) \\ &\quad = \{RS(Z)|Z \in \{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_1},X_{j_2},\ldots,X_{j_{k-2}}\}\Delta P(E \setminus \{X_{j_1},X_{j_2},X_{j_3},\ldots,X_{j_k}\})) \\ &\qquad \Delta P(B \setminus \{x_{i_1},x_{i_2},\ldots,x_{i_r}\})\}, \end{aligned}$$

where

$$\begin{aligned} V_{\{X_{j_{k-1}},X_{j_k}\}}(F_X(T)) \cup V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_3},\ldots,X_{j_{k-2}}\}}(F_X(T)) &= V_{\{X_{j_{k-1}}X_{j_k},x_{i_1}\ldots x_{i_r}X_{j_3}\ldots X_{j_{k-2}}\}}(F_X(T)) \\ &\subseteq V(F_X(T)), \end{aligned}$$

$$V_{\{X_{j_{k-1}},X_{j_k}\}}(F_X(T)) \cap V_{\{x_{i_1},x_{i_2},\ldots,x_{i_r},X_{j_3},\ldots,X_{j_{k-2}}\}}(F_X(T)) = \emptyset$$

generating ${}^{k}C_2$ complete bipartite graphs. Also for $p=2$, ${}^{r}C_1{}^{k}C_1$ ways partitioning the vertex set

$$\begin{aligned}&V_{\{x_{i_1},X_{j_1}\}}(F_X(T))\\&\quad=\{RS(Z)|Z\in\{x_{i_1},X_{j_1}\}\Delta P(E\setminus\{X_{i_2},X_{i_3},\dots,X_{i_r},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\})\\&\qquad\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},\\&V_{\{x_{i_2},x_{i_3},\dots,x_{i_r},X_{j_2},\dots,X_{j_k}\}}(F_X(T))\\&\quad=\{RS(Z)|Z\in\{x_{i_2},x_{i_3},\dots,x_{i_r},X_{j_2},X_{j_3},\dots,X_{j_k}\}\Delta P(E\setminus\{X_{i_1},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\}))\\&\qquad\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},\end{aligned}$$

where

$$\begin{aligned}V_{\{x_{i_1},X_{j_1}\}}(F_X(T))\cup V_{\{x_{i_2},x_{i_3},\dots,x_{i_r},X_{j_2},\dots,X_{j_k}\}}(F_X(T))&=V_{\{x_{i_1}X_{j_1},x_{i_2}\dots x_{i_r}X_{j_2}\dots X_{j_k}\}}(F_X(T))\\&\subseteq V(F_X(T)),\\V_{\{x_{i_1},X_{j_1}\}}(F_X(T))\cap V_{\{x_{i_2},x_{i_3},\dots,x_{i_r},X_{j_2},\dots,X_{j_k}\}}(F_X(T))&=\emptyset\end{aligned}$$

and so on. Then

$$\begin{aligned}&V_{\{x_{i_r},X_{j_k}\}}(F_X(T))\\&\quad=\{RS(Z)|Z\in\{x_{i_r},X_{j_k}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\dots,X_{i_{r-1}},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\})\\&\qquad\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},\\&V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T))\\&\quad=\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}\Delta P(E\setminus\{X_{i_r},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\}))\\&\qquad\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\}\end{aligned}$$

where

$$\begin{aligned}V_{\{x_{i_r},X_{j_k}\}}(F_X(T))\cup V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{j_1},\dots,X_{j_{k-1}}\}}(F_X(T))&=V_{\{x_{i_r}X_{j_k},x_{i_1}\dots x_{i_{r-1}}X_{j_1}\dots X_{j_{k-1}}\}}(F_X(T))\\&\subseteq V(F_X(T)),\\V_{\{x_{i_r},X_{j_k}\}}(F_X(T))\cap V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{j_1},\dots,X_{j_{k-1}}\}}(F_X(T))&=\emptyset\end{aligned}$$

generating ${}^{r}C_1{}^{k}C_1$ complete bipartite graphs. Hence the total number of complete bipartite graphs generated when $p=2$ is ${}^{r}C_2+{}^{k}C_2+{}^{r}C_1{}^{k}C_1$. Repeating this procedure for $p=3$, ${}^{r}C_3+{}^{k}C_3+{}^{r}C_1{}^{k}C_2+{}^{r}C_2{}^{k}C_1$ complete bipartite graphs can be generated and so on. Using the argument from Case 4 and, for the choices of r and k, the number of distinct complete bipartite graphs generated can be identified.

Case 6. Suppose $X=\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}$ for $\{s_1,s_2,\dots,s_t\}\in\{1,2,\dots,m\}$, $\{j_1,j_2,\dots,j_k\}\in\{m+1,m+2,\dots,n\}$, then the vertex set partitioning of $V(F_X(T))\subseteq T\setminus F_X(T)$ of $G(F_X(T))$ for $p=1,2,\dots,v$ is as follows. When $p=1$, ${}^{t}C_1$ ways vertex set are partitioned into

$$\begin{aligned}&V_{\{X_{s_1}\}}(F_X(T))\\&\quad=\{RS(Z)|Z\in\{X_{s_1}\}\Delta P(E\setminus\{X_{s_1},X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\})\\&\qquad\Delta P(B\setminus\{x_{s_1}\})\},\\&V_{\{X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T))\end{aligned}$$

$$= \{RS(Z)|Z \in \{X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}) \Delta P(B \setminus \{x_{s_2}, x_{s_3}, \dots, x_{s_t}\})\},$$

where

$$V_{\{X_{s_1}\}}(F_X(T)) \cup V_{\{X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T)) = V_{\{X_{s_1},X_{s_2}\dots X_{s_t}X_{j_1}X_{j_2}\dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{s_1}\}}(F_X(T)) \cap V_{\{X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_t}\}}(F_X(T)) = \emptyset$$

and so on. Then

$$V_{\{X_{s_t}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{s_t}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}) \Delta P(B \setminus \{x_{s_t}\})\},$$

$$V_{\{X_{s_1},X_{s_2},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{s_1}, X_{s_2}, \dots, X_{s_{t-1}}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}) \Delta P(B \setminus \{x_{s_1}, x_{s_2}, \dots, x_{s_{t-1}}\})\},$$

where

$$V_{\{X_{s_t}\}}(F_X(T)) \cup V_{\{X_{s_1},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T)) = V_{\{X_{s_t},X_{s_1}\dots X_{s_{t-1}}X_{j_1}X_{j_2}\dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{s_t}\}}(F_X(T)) \cap V_{\{X_{s_1},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_t}\}}(F_X(T)) = \emptyset$$

generating ${}^t\mathrm{C}_1$ complete bipartite graphs. And ${}^k\mathrm{C}_1$ ways vertex set are partitioned into

$$V_{\{X_{j_1}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{j_1}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}) \Delta P(B)\},$$

$$V_{\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}) \Delta P(B)\},$$

where

$$V_{\{X_{j_1}\}}(F_X(T)) \cup V_{\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = V_{\{X_{j_1},X_{s_1}\dots X_{s_t}X_{j_2}\dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{j_1}\}}(F_X(T)) \cap V_{\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T)) = \emptyset$$

and so on. Then

$$V_{\{X_{j_k}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{j_k}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}) \Delta P(B)\},$$

$$V_{\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T)) = \{RS(Z)|Z \in \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_{k-1}}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}) \Delta P(B)\},$$

where

$$V_{\{X_{j_k}\}}(F_X(T)) \cup V_{\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T)) = V_{\{X_{j_k},X_{s_1}\dots X_{s_t}X_{j_1}\dots X_{j_{k-1}}\}}(F_X(T)) \subseteq V(F_X(T)),$$

$$V_{\{X_{j_k}\}}(F_X(T)) \cap V_{\{X_{s_1},X_{s_2},\ldots,X_{s_t},X_{j_1},X_{j_2},\ldots,X_{j_{k-1}}\}}(F_X(T)) = \emptyset$$

generating ${}^kC_1$ complete bipartite graphs. Hence ${}^tC_1 + {}^kC_1$ complete bipartite graphs are generated when $p = 1$. Now for $p = 2$, ${}^tC_2$ ways partitioning the vertex set into

$$\begin{aligned}
&V_{\{X_{s_1},X_{s_2}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{X_{s_1}, X_{s_2}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}) \Delta P(B \setminus \{x_{s_1}, x_{s_2}\})\},\\
&V_{\{X_{s_3},\ldots,X_{s_t},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{X_{s_3}, X_{s_4}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\} \\
&\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\})) \Delta P(B \setminus \{x_{s_3}, x_{s_4}, \ldots, x_{s_t}\})\},
\end{aligned}$$

where

$$\begin{aligned}
V_{\{X_{s_1},X_{s_2}\}}(F_X(T)) \cup V_{\{X_{s_3},\ldots,X_{s_t},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) &= V_{\{X_{s_1}X_{s_2},X_{s_3}\ldots X_{s_t}X_{j_1}X_{j_2}\ldots X_{j_k}\}}(F_X(T)) \\
&\subseteq V(F_X(T)),
\end{aligned}$$

$$V_{\{X_{s_1},X_{s_2}\}}(F_X(T)) \cap V_{\{X_{s_3},\ldots,X_{s_t},X_{j_1},X_{j_2},\ldots,X_{j_t}\}}(F_X(T)) = \emptyset$$

and so on. Then

$$\begin{aligned}
&V_{\{X_{s_{t-1}},X_{s_t}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{X_{s_{t-1}}, X_{s_t}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}) \\
&\qquad \Delta P(B \setminus \{x_{s_{t-1}}, x_{s_t}\})\},\\
&V_{\{X_{s_1},X_{s_2},\ldots,X_{s_{t-2}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{X_{s_1}, X_{s_2}, \ldots, X_{s_{t-2}}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\} \\
&\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, \ldots, X_{j_k}\}) \Delta P(B \setminus \{x_{s_1}, x_{s_2}, \ldots, x_{s_{t-2}}\})\},
\end{aligned}$$

where

$$\begin{aligned}
&V_{\{X_{s_t},X_{s_{t-1}}\}}(F_X(T)) \cup V_{\{X_{s_1},\ldots,X_{s_{t-2}},X_{j_1},X_{j_2},\ldots,X_{j_k}\}}(F_X(T)) \\
&\quad = V_{\{X_{s_t}X_{s_{t-1}},X_{s_1}\ldots X_{s_{t-1}}X_{j_1}X_{j_2}\ldots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),\\
&V_{\{X_{s_t},X_{s_{t-1}}\}}(F_X(T)) \cap V_{\{X_{s_1},\ldots,X_{s_{t-2}},X_{j_1},X_{j_2},\ldots,X_{j_t}\}}(F_X(T)) = \emptyset
\end{aligned}$$

generating ${}^tC_2$ complete bipartite graphs. And ${}^kC_2$ ways partitioning the vertex set into

$$\begin{aligned}
&V_{\{X_{j_1},X_{j_2}\}}(F_X(T)) = \{RS(Z) | Z \in \{X_{j_1}\} \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\}) \Delta P(B)\},\\
&V_{\{X_{s_1},X_{s_2},\ldots,X_{s_t},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) \\
&\quad = \{RS(Z) | Z \in \{X_{s_1}, X_{s_2}, \ldots, X_{s_t}, X_{j_3}, \ldots, X_{j_k}\} \\
&\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, \ldots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \ldots, X_{j_k}\})) \Delta P(B)\},
\end{aligned}$$

where

$$\begin{aligned}
V_{\{X_{j_1},X_{j_2}\}}(F_X(T)) \cup V_{\{X_{s_1},X_{s_2},\ldots,X_{s_t},X_{j_3},\ldots,X_{j_k}\}}(F_X(T)) &= V_{\{X_{j_1}X_{j_2},X_{s_1}\ldots X_{s_t}X_{j_3}\ldots X_{j_k}\}}(F_X(T)) \\
&\subseteq V(F_X(T)),
\end{aligned}$$

$$V_{\{X_{j_1},X_{j_2}\}}(F_X(T)) \cap V_{\{X_{s_1},X_{s_2},\ldots,X_{s_t},X_{j_3},X_{j_4},\ldots,X_{j_k}\}}(F_X(T)) = \emptyset$$

and so on. Then

$$V_{\{X_{j_{k-1}},X_{j_k}\}}(F_X(T))$$

$$= \{RS(Z)|Z \in \{X_{j_{k-1}}, X_{j_k}\}\Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\})\Delta P(B)\},$$

$$V_{\{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_{k-2}}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_{k-2}}\}$$
$$\Delta P(E \setminus \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}))\Delta P(B)\},$$

where

$$V_{\{X_{j_k}, X_{j_{k-1}}\}}(F_X(T)) \cup V_{\{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_{k-2}}\}}(F_X(T))$$
$$= V_{\{X_{j_k} X_{j_{k-1}}, X_{s_1} \dots X_{s_t} X_{j_1} \dots X_{j_{k-2}}\}}(F_X(T)) \subseteq V(F_X(T)),$$
$$V_{\{X_{j_k}, X_{j_{k-1}}\}}(F_X(T)) \cap V_{\{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_{k-2}}\}}(F_X(T)) = \emptyset$$

generating ${}^kC_2$ complete bipartite graphs. Also ${}^tC_1{}^kC_1$ ways vertex set partitioning for $p = 2$,

$$V_{\{X_{s_1}, X_{j_1}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{s_1}, X_{j_1}\}\Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})\Delta P(B \setminus \{x_{s_1}\})\},$$

$$V_{\{X_{s_2}, \dots, X_{s_t}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{s_2}, X_{s_4}, \dots, X_{s_t}, X_{j_2}, X_{j_3}, \dots, X_{j_k}\}$$
$$\Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}))\Delta P(B \setminus \{x_{s_2}, x_{s_3}, \dots, x_{s_t}\})\},$$

where

$$V_{\{X_{s_1}, X_{j_1}\}}(F_X(T)) \cup V_{\{X_{s_2}, \dots, X_{s_t}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) = V_{\{X_{s_1} X_{j_1}, X_{s_2} \dots X_{s_t} X_{j_2} \dots X_{j_k}\}}(F_X(T))$$
$$\subseteq V(F_X(T)),$$
$$V_{\{X_{s_1}, X_{j_1}\}}(F_X(T)) \cap V_{\{X_{s_2}, \dots, X_{s_t}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) = \emptyset$$

and so on. Then

$$V_{\{X_{s_t}, X_{j_k}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{s_t}, X_{j_k}\}\Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})\Delta P(B \setminus \{x_{s_t}\})\},$$

$$V_{\{X_{s_1}, X_{s_2}, \dots, X_{s_{t-1}}, X_{j_1}, X_{j_2}, \dots, X_{j_{k-1}}\}}(F_X(T))$$
$$= \{RS(Z)|Z \in \{X_{s_1}, X_{s_2}, \dots, X_{s_{t-1}}, X_{j_1}, X_{j_2}, \dots, X_{j_{k-1}}\}$$
$$\Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})\Delta P(B \setminus \{x_{s_1}, x_{s_2}, \dots, x_{s_{t-1}}\})\},$$

where

$$V_{\{X_{s_t}, X_{j_k}\}}(F_X(T)) \cup V_{\{X_{s_1}, \dots, X_{s_{t-1}}, X_{j_1}, \dots, X_{j_{k-1}}\}}(F_X(T)) = V_{\{X_{s_t} X_{j_k}, X_{s_1} \dots X_{s_{t-1}} X_{j_1} \dots X_{j_{k-1}}\}}(F_X(T))$$
$$\subseteq V(F_X(T)),$$
$$V_{\{X_{s_t}, X_{j_k}\}}(F_X(T)) \cap V_{\{X_{s_1}, \dots, X_{s_{t-1}}, X_{j_1}, \dots, X_{j_{k-1}}\}}(F_X(T)) = \emptyset$$

generates ${}^tC_1{}^kC_1$ complete bipartite graphs. So the total number of complete bipartite graphs generated when $p = 2$ is ${}^tC_2 + {}^kC_2 + {}^tC_1{}^kC_1$. Continuing in this manner for $p = 3$, ${}^tC_3 + {}^kC_3 + {}^tC_1{}^kC_2 + {}^tC_2{}^kC_1$ complete bipartite graphs can be generated and so on. The number of distinct complete bipartite graphs generated can be identified using the argument from Case 4 and Case 5, for the choices of t and k.

Case 7. Suppose $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$ for $\{i_1, i_2, \dots, i_r, s_1, s_2, \dots, s_t\} \in \{1, 2, \dots, m\}$, $\{j_1, j_2, \dots, j_k\} \in \{m+1, m+2, \dots, n\}$, then the vertex set partitioning of $V(F_X(T)) \subseteq T \setminus F_X(T)$ of $G(F_X(T))$ for $p = 1, 2, \dots, v$ is as follows. When $p = 1$, ${}^r\mathrm{C}_1$ ways vertex set are partitioned into

$$\begin{aligned}
&V_{\{x_{i_1}\}}(F_X(T))\\
&\quad = \{RS(Z) | Z \in \{x_{i_1}\} \Delta P(E \setminus \{X_{i_2}, X_{i_3}, \dots, X_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})\\
&\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},\\
&V_{\{x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)),\\
&\quad = \{RS(Z) | Z \in \{x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}\\
&\qquad \Delta P(E \setminus \{X_{i_1}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}))\\
&\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_{s_1}, x_{s_2}, \dots, x_{s_t}\})\},
\end{aligned}$$

where

$$\begin{aligned}
&V_{\{x_{i_1}\}}(F_X(T)) \cup V_{\{x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T))\\
&\quad = V_{\{x_{i_1}, x_{i_2} \dots x_{i_r} X_{s_1} X_{s_2} \dots X_{s_t} X_{j_1} X_{j_2} \dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),\\
&V_{\{x_{i_1}\}}(F_X(T)) \cap V_{\{x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) = \emptyset
\end{aligned}$$

and so on. Then

$$\begin{aligned}
&V_{\{x_{i_r}\}}(F_X(T))\\
&\quad = \{RS(Z) | Z \in \{x_{i_r}\} \Delta P(E \setminus \{X_{i_1}, X_{i_2}, \dots, X_{i_{r-1}}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})\\
&\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\})\},\\
&V_{\{x_{i_1}, x_{i_2}, \dots, x_{i_{r-1}}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T))\\
&\quad = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}, \dots, x_{i_{r-1}}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}\\
&\qquad \Delta P(E \setminus \{X_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})) \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_{s_1}, x_{s_2}, \dots, x_{s_t}\})\},
\end{aligned}$$

where

$$\begin{aligned}
&V_{\{x_{i_r}\}}(F_X(T)) \cup V_{\{x_{i_1}, \dots, x_{i_{r-1}}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T))\\
&\quad = V_{\{x_{i_r}, x_{i_1} \dots x_{i_{r-1}} X_{s_1} X_{s_2} \dots X_{s_t} X_{j_1} X_{j_2} \dots X_{j_k}\}}(F_X(T)) \subseteq V(F_X(T)),\\
&V_{\{x_{i_r}\}}(F_X(T)) \cap V_{\{x_{i_1}, \dots, x_{i_{r-1}}, X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T)) = \emptyset
\end{aligned}$$

generating ${}^r\mathrm{C}_1$ complete bipartite graphs. Then partitioning the vertex set for ${}^t\mathrm{C}_1$ ways into

$$\begin{aligned}
&V_{\{X_{s_1}\}}(F_X(T))\\
&\quad = \{RS(Z) | Z \in \{X_{s_1}\} \Delta P(E \setminus \{X_{i_1}, X_{i_2}, X_{i_3}, \dots, X_{i_r}, X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\})\\
&\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_{s_1}\})\},\\
&V_{\{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}}(F_X(T))\\
&\quad = \{RS(Z) | Z \in \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}\\
&\qquad \Delta P(E \setminus \{X_{s_1}, X_{s_2}, X_{s_3}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}))\\
&\qquad \Delta P(B \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_{s_2}, x_{s_3}, \dots, x_{s_t}\})\},
\end{aligned}$$

where

$$V_{\{X_{s_1}\}}(F_X(T))\cup V_{\{x_{i_1},\dots,x_{i_r},X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T))$$
$$=V_{\{X_{s_1},x_{i_1}\dots x_{i_r}X_{s_2}X_{s_3}\dots X_{s_t}X_{j_1}X_{j_2}\dots X_{j_k}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{s_1}\}}(F_X(T))\cap V_{\{x_{i_1},\dots,x_{i_r},X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T))=\emptyset$$

and so on. Then

$$V_{\{X_{s_t}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{X_{s_t}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},X_{i_3},\dots,X_{i_r},X_{s_1},X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\})$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_t}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_k}\}$$
$$\Delta P(E\setminus\{X_{s_1},X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}))$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_{t-1}}\})\},$$

where

$$V_{\{X_{s_t}\}}(F_X(T))\cup V_{\{x_{i_1},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T))$$
$$=V_{\{X_{s_t},x_{i_1}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_{t-1}}X_{j_1}X_{j_2}\dots X_{j_k}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{s_t}\}}(F_X(T))\cap V_{\{x_{i_1},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_k}\}}(F_X(T))=\emptyset$$

generating ${}^tC_1$ complete bipartite graphs. Also partitioning the vertex set for ${}^kC_1$ ways into

$$V_{\{X_{j_1}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{X_{j_1}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},X_{i_3},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\})$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_2},X_{j_3},\dots,X_{j_k}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_2},X_{j_3},\dots,X_{j_k}\}$$
$$\Delta P(E\setminus\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\}))\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_t}\})\}$$

where

$$V_{\{X_{j_1}\}}(F_X(T))\cup V_{\{x_{i_1},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_2},\dots,X_{j_k}\}}(F_X(T))$$
$$=V_{\{X_{j_1},x_{i_1}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_t}X_{j_2}\dots X_{j_k}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{j_1}\}}(F_X(T))\cap V_{\{x_{i_1},\dots,x_{i_r},X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_2},\dots,X_{j_k}\}}(F_X(T))=\emptyset$$

and so on. Then

$$V_{\{X_{j_k}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{X_{j_k}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},X_{i_3},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\})$$
$$\Delta P(B\{x_{i_1},x_{i_2},\dots,x_{i_r}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}$$

$$\Delta P(E\setminus\{X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},X_{j_3},\dots,X_{j_k}\}))\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_t}\})\},$$

where

$$V_{\{X_{j_k}\}}(F_X(T))\cup V_{\{x_{i_1},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},\dots,X_{j_{k-1}}\}}(F_X(T))$$
$$=V_{\{X_{j_k},x_{i_1}\dots x_{i_r}X_{s_1}X_{s_2}\dots X_{s_t}X_{j_1}\dots X_{j_{k-1}}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{X_{j_k}\}}(F_X(T))\cap V_{\{x_{i_1},\dots,x_{i_r},X_{s_2},X_{s_3},\dots,X_{s_t},X_{j_1},\dots,X_{j_{k-1}}\}}(F_X(T))=\emptyset$$

generating ${}^k\mathrm{C}_1$ complete bipartite graphs. So the total number of complete bipartite graphs generated when $p=1$ is ${}^r\mathrm{C}_1+{}^t\mathrm{C}_1+{}^k\mathrm{C}_1$. Following from *Case* 4 to *Case* 6, when $p=2$, the total number of complete bipartite graphs can be generated by partitioning the vertex set will be ${}^r\mathrm{C}_2+{}^t\mathrm{C}_2+{}^k\mathrm{C}_2+{}^r\mathrm{C}_1{}^t\mathrm{C}_1+{}^r\mathrm{C}_1{}^k\mathrm{C}_1+{}^t\mathrm{C}_1{}^k\mathrm{C}_1$. While for $p=3$, there will be ${}^r\mathrm{C}_3+{}^t\mathrm{C}_3+{}^k\mathrm{C}_3+{}^r\mathrm{C}_1{}^t\mathrm{C}_2+{}^r\mathrm{C}_2{}^t\mathrm{C}_1+{}^r\mathrm{C}_1{}^k\mathrm{C}_2+{}^r\mathrm{C}_2{}^k\mathrm{C}_1+{}^t\mathrm{C}_1{}^k\mathrm{C}_2+{}^t\mathrm{C}_2{}^k\mathrm{C}_1+{}^r\mathrm{C}_1{}^t\mathrm{C}_1{}^k\mathrm{C}_1$ complete bipartite graphs generated. The ${}^r\mathrm{C}_1{}^t\mathrm{C}_1{}^k\mathrm{C}_1$ ways of partitioning the vertex set is as follows

$$V_{\{x_{i_1},X_{s_1},X_{j_1}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{x_{i_1},X_{s_1},X_{j_1}\}\Delta P(E\setminus\{X_{i_2},X_{i_3},\dots,X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\})$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1}\})\},$$
$$V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t},X_{j_2},\dots,X_{j_k}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t},X_{j_2},\dots,X_{j_k}\}$$
$$\Delta P(E\setminus\{X_{i_1},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}))\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_2},\dots,x_{s_t}\})\},$$

where

$$V_{\{x_{i_1},X_{s_1},X_{j_1}\}}(F_X(T))\cup V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t},X_{j_2},\dots,X_{j_k}\}}(F_X(T))$$
$$=V_{\{x_{i_1}X_{s_1}X_{j_1},x_{i_2}\dots x_{i_r}X_{s_2}\dots X_{s_t}X_{j_2}\dots X_{j_k}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{x_{i_1},X_{s_1},X_{j_1}\}}(F_X(T))\cap V_{\{x_{i_2},\dots,x_{i_r},X_{s_2},\dots,X_{s_t},X_{j_2},\dots,X_{j_k}\}}(F_X(T))=\emptyset$$

and so on.

$$V_{\{x_{i_r},X_{s_t},X_{j_k}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{x_{i_r},X_{s_t},X_{j_k}\}\Delta P(E\setminus\{X_{i_1},X_{i_2},\dots,X_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\})$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_t}\})\},$$
$$V_{\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}}(F_X(T))$$
$$=\{RS(Z)|Z\in\{x_{i_1},x_{i_2},\dots,x_{i_{r-1}},X_{s_1},X_{s_2},\dots,X_{s_{t-1}},X_{j_1},X_{j_2},\dots,X_{j_{k-1}}\}$$
$$\Delta P(E\setminus\{X_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}))$$
$$\Delta P(B\setminus\{x_{i_1},x_{i_2},\dots,x_{i_r},x_{s_1},x_{s_2},\dots,x_{s_{t-1}}\})\},$$

where

$$V_{\{x_{i_r},X_{s_t},X_{j_k}\}}(F_X(T))\cup V_{\{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},\dots,X_{s_{t-1}},X_{j_1},\dots,X_{j_{k-1}}\}}(F_X(T))$$
$$=V_{\{x_{i_r}X_{s_t}X_{j_k},x_{i_1}\dots x_{i_{r-1}}X_{s_1}\dots X_{s_{t-1}}X_{j_1}\dots X_{j_{k-1}}\}}(F_X(T))\subseteq V(F_X(T)),$$
$$V_{\{x_{i_r},X_{s_t},X_{j_k}\}}(F_X(T))\cap V_{\{x_{i_1},\dots,x_{i_{r-1}},X_{s_1},\dots,X_{s_{t-1}},X_{j_1},\dots,X_{j_{k-1}}\}}(F_X(T))=\emptyset.$$

Continuing the process for $p=4,5,\dots,v$, termination at $p=v$ is determined by the choices of r,t, and k. Suppose if $r+t+k$ is odd, then using the arguments from the earlier cases $\frac{(r+t+k)-1}{2}$

ways vertex set can be partitioned. The number of complete bipartite graphs generated from $G(F_X(T))$ for the various choices of r, t, and k is shown from (i) to (xix) with the comparison made between them

(i) If $r = t = k$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q {}^{t}C_{p-q} + {}^{r}C_q {}^{k}C_{p-q} + {}^{t}C_q {}^{k}C_{p-q}\}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q {}^{t}C_u {}^{k}C_{p-(q+u)}.$$

(ii) If $r < t < k$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q {}^{t}C_{p-q} + {}^{r}C_q {}^{k}C_{p-q} + {}^{t}C_q {}^{k}C_{p-q}\}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q {}^{t}C_u {}^{k}C_{p-(q+u)}.$$

(iii) If $r > t > k$, then

$$\sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q {}^{t}C_{p-q} + {}^{r}C_q {}^{k}C_{p-q} + {}^{t}C_q {}^{k}C_{p-q}\}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q {}^{t}C_u {}^{k}C_{p-(q+u)}.$$

(iv) If $r < t$ and $t = k$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q {}^{t}C_{p-q} + {}^{r}C_q {}^{k}C_{p-q} + {}^{t}C_q {}^{k}C_{p-q}\}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q {}^{t}C_u {}^{k}C_{p-(q+u)}.$$

(v) If $r < t$ and $t > k$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q {}^{t}C_{p-q} + {}^{r}C_q {}^{k}C_{p-q} + {}^{t}C_q {}^{k}C_{p-q}\}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q {}^{t}C_u {}^{k}C_{p-(q+u)}.$$

(vi) If $r > t$ and $t = k$, then

$$\sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q {}^{t}C_{p-q} + {}^{r}C_q {}^{k}C_{p-q} + {}^{t}C_q {}^{k}C_{p-q}\}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q {}^{t}C_u {}^{k}C_{p-(q+u)}.$$

(vii) If $r > t$ and $t < k$ and $r > k$, then

$$\sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q\,{}^{t}C_{p-q} + {}^{r}C_q\,{}^{k}C_{p-q} + {}^{t}C_q\,{}^{k}C_{p-q}\} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q\,{}^{t}C_u\,{}^{k}C_{p-(q+u)}.$$

If $r > t$ and $t < k$ and $r < k$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q\,{}^{t}C_{p-q} + {}^{r}C_q\,{}^{k}C_{p-q} + {}^{t}C_q\,{}^{k}C_{p-q}\} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q\,{}^{t}C_u\,{}^{k}C_{p-(q+u)}.$$

(viii) If $r = t$ and $t < k$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q\,{}^{t}C_{p-q} + {}^{r}C_q\,{}^{k}C_{p-q} + {}^{t}C_q\,{}^{k}C_{p-q}\} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q\,{}^{t}C_u\,{}^{k}C_{p-(q+u)}.$$

(ix) If $r = t$ and $t > k$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q\,{}^{t}C_{p-q} + {}^{r}C_q\,{}^{k}C_{p-q} + {}^{t}C_q\,{}^{k}C_{p-q}\} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q\,{}^{t}C_u\,{}^{k}C_{p-(q+u)}.$$

(x) If $r = k$ and $k > t$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{t} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q\,{}^{t}C_{p-q} + {}^{r}C_q\,{}^{k}C_{p-q} + {}^{t}C_q\,{}^{k}C_{p-q}\} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q\,{}^{t}C_u\,{}^{k}C_{p-(q+u)}.$$

(xi) If $r = k$ and $k < t$, then

$$\sum_{p=1}^{r} {}^{r}C_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{t}C_p + \sum_{p=1}^{k} {}^{k}C_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{{}^{r}C_q\,{}^{t}C_{p-q} + {}^{r}C_q\,{}^{k}C_{p-q} + {}^{t}C_q\,{}^{k}C_{p-q}\} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}C_q\,{}^{t}C_u\,{}^{k}C_{p-(q+u)}.$$

(xii) If $r > k > t$, then

$$\sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{r}\mathrm{C}_p + \sum_{p=1}^{t} {}^{t}\mathrm{C}_p + \sum_{p=1}^{k} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

(xiii) If $r < k < t$, then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{t}\mathrm{C}_p + \sum_{p=1}^{k} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

(xiv) If $r < k$ and $k > t$, then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{t} {}^{t}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

(xv) If $r > k$ and $k < t$ and $r < t$, then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{t}\mathrm{C}_p + \sum_{p=1}^{k} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

If $r > k$ and $k < t$ and $r > t$, then

$$\sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{r}\mathrm{C}_p + \sum_{p=1}^{t} {}^{t}\mathrm{C}_p + \sum_{p=1}^{k} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

(xvi) If $t > r > k$, then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{t}\mathrm{C}_p + \sum_{p=1}^{k} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \}$$
$$+ \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

(xvii) If $t<r<k$, then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{t} {}^{t}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

(xviii) If $t<r$ and $r>k$, then

$$\sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{r}\mathrm{C}_p + \sum_{p=1}^{t} {}^{t}\mathrm{C}_p + \sum_{p=1}^{k} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

(xix) If $t>r$ and $r<k$ and $t>k$, then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{t}\mathrm{C}_p + \sum_{p=1}^{k} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

If $t>r$ and $r<k$ and $t<k$, then

$$\sum_{p=1}^{r} {}^{r}\mathrm{C}_p + \sum_{p=1}^{t} {}^{t}\mathrm{C}_p + \sum_{p=1}^{\frac{(r+t+k)-1}{2}} {}^{k}\mathrm{C}_p + \sum_{p=2}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-1} \{ {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_{p-q} + {}^{r}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} + {}^{t}\mathrm{C}_q {}^{k}\mathrm{C}_{p-q} \} + \sum_{p=3}^{\frac{(r+t+k)-1}{2}} \sum_{q=1}^{p-2} \sum_{u=1}^{p-(q+1)} {}^{r}\mathrm{C}_q {}^{t}\mathrm{C}_u {}^{k}\mathrm{C}_{p-(q+u)}.$$

Suppose if $r+t+k$ is even, then using the arguments from the earlier cases $\frac{(r+t+k)}{2}$ ways vertex set can be partitioned. The number of complete bipartite graphs generated from $G(F_X(T))$ for the various choices of $r,t,$ and k is similar from (i) to (xix) with $\frac{(r+t+k)-1}{2}$ is replaced by $\frac{(r+t+k)}{2}$. In addition, at $p=\frac{(r+t+k)}{2}$, ${}^{r}\mathrm{C}_p=\frac{{}^{r}\mathrm{C}_p}{2}$, ${}^{t}\mathrm{C}_p=\frac{{}^{t}\mathrm{C}_p}{2}$, ${}^{k}\mathrm{C}_p=\frac{{}^{k}\mathrm{C}_p}{2}$, ${}^{r}\mathrm{C}_q{}^{t}\mathrm{C}_{p-q}=\frac{{}^{r}\mathrm{C}_q{}^{t}\mathrm{C}_{p-q}}{2}$, ${}^{r}\mathrm{C}_q{}^{k}\mathrm{C}_{p-q}=\frac{{}^{r}\mathrm{C}_q{}^{k}\mathrm{C}_{p-q}}{2}$, ${}^{t}\mathrm{C}_q{}^{k}\mathrm{C}_{p-q}=\frac{{}^{t}\mathrm{C}_q{}^{k}\mathrm{C}_{p-q}}{2}$ and ${}^{r}\mathrm{C}_q{}^{t}\mathrm{C}_u{}^{k}\mathrm{C}_{p-(q+u)}=\frac{{}^{r}\mathrm{C}_q{}^{t}\mathrm{C}_u{}^{k}\mathrm{C}_{p-(q+u)}}{2}$. □

Remark 1. Lemma 1, shows the number of distinct complete bipartite graphs generated from $G(F_X(T))$. The union of these distinct complete bipartite graphs forms the subgraph of $G(F_X(T))$.

Remark 2. The following lemma will show the correspondence between the cardinality of vertices of $G(F_X(T))$ and its subgraph.

Lemma 2. *The number of vertices in $G(F_X(T))$ (or) the subgraph of $G(F_X(T))$ when $X=\{z_1,z_2,\dots,z_\rho\}$ where $z_1,z_2,\dots,z_\rho$ are x_i (or) X_j for $i,j\in\{1,2,\dots,m,m+1,\dots,n\}$, $2\le\rho\le m,n$*

is

$$\sum_{p=1}^{v}[{}^{r}C_p(2^{t+p}+2^{r-p})+{}^{t}C_p(2^{t-p}+2^{r+p})+{}^{k}C_p(2^{t}+2^{r})]3^{m-(t+r)}2^{n-(m+k)}$$
$$+\sum_{p=2}^{v}\sum_{q=1}^{p-1}[{}^{r}C_q\,{}^{t}C_{p-q}(2^{q}2^{t-(p-q)}+2^{r-q}2^{p-q})+{}^{r}C_q\,{}^{k}C_{p-q}(2^{t+q}+2^{r-q})$$
$$+{}^{t}C_q\,{}^{k}C_{p-q}(2^{t-q}+2^{r}2^{t-(t-q)})]3^{m-(t+r)}2^{n-(m+k)}$$
$$+\sum_{p=3}^{v}\sum_{q=1}^{p-2}\sum_{u=1}^{p-(q+1)}{}^{r}C_q\,{}^{t}C_u\,{}^{k}C_{p-(q+u)}(2^{q}2^{t-u}+2^{r-q}2^{t-(t-u)})3^{m-(t+r)}2^{n-(m+k)},$$

where, $0\le r,\ t\le m,\ 0\le k\le n-m,\ v=\frac{(r+t+k)-1}{2}$ *(or)* $v=\frac{(r+t+k)}{2}$.

Proof. The cardinality of vertices in $G(F_X(T))$ for various X is discussed in the proof.

Case 1. When $X=\{x_{i_1},x_{i_2},\dots,x_{i_r}\}$ for $\{i_1,i_2,\dots,i_r\}\in\{1,2,\dots,m\}$, then using Lemma 1, the vertex set of $G(F_X(T))$ is partitioned for $p=1,2,\dots,\frac{(r-1)}{2}$ (or) $\frac{r}{2}$ accordingly r is odd (or) even, respectively.

When $p=1$, ${}^{r}C_1$ ways vertex set are partitioned into V_1 containing $(2)3^{m-r}2^{n-m}$ choices of vertices and V_2 containing $2^{r-1}3^{m-r}2^{n-m}$ choices of vertices. When $p=2$, ${}^{r}C_2$ ways vertex set are partitioned into V_1 containing $2^{2}3^{m-r}2^{n-m}$ choices of vertices and V_2 containing $2^{r-2}3^{m-r}2^{n-m}$ choices of vertices and so on. So the cardinality of the vertex set is

$$|V(F_X(T))|=\sum_{p=1}^{v}{}^{r}C_p(2^{p}+2^{r-p})3^{m-r}2^{n-m}.$$

When r is odd, $v=\frac{(r-1)}{2}$. Suppose if r is even, $v=\frac{r}{2}$ and provided at $p=\frac{r}{2}$, ${}^{r}C_p=\frac{{}^{r}C_p}{2}$.

Case 2. When $X=\{X_{s_1},X_{s_2},\dots,X_{s_t}\}$ for $\{s_1,s_2,\dots,s_t\}\in\{1,2,\dots,m\}$, then using Lemma 1, the vertex set of $G(F_X(T))$ is partitioned for $p=1,2,\dots,\frac{(t-1)}{2}$ (or) $\frac{t}{2}$ accordingly t is odd (or) even, respectively.

When $p=1$, ${}^{t}C_1$ ways vertex set are partitioned into V_1 containing $2^{t-1}3^{m-t}2^{n-m}$ choices of vertices and V_2 containing $2^{t-(t-1)}3^{m-t}2^{n-m}$ choices of vertices. When $p=2$, ${}^{t}C_2$ ways vertex set are partitioned into V_1 containing $2^{t-2}3^{m-t}2^{n-m}$ choices of vertices and V_2 containing $2^{t-(t-2)}3^{m-t}2^{n-m}$ choices of vertices and so on. The cardinality of the vertex set is

$$|V(F_X(T))|=\sum_{p=1}^{v}{}^{t}C_p(2^{t-p}+2^{t-(t-p)})3^{m-t}2^{n-m}.$$

When t is odd, $v=\frac{(t-1)}{2}$. Suppose if t is even, $v=\frac{t}{2}$ and provided at $p=\frac{t}{2}$, ${}^{t}C_p=\frac{{}^{t}C_p}{2}$.

Case 3. When $X=\{X_{j_1},X_{j_2},\dots,X_{j_k}\}$ for $\{j_1,j_2,\dots,j_k\}\in\{m+1,m+2,\dots,n\}$, then using Lemma 1, the vertex set of $G(F_X(T))$ is partitioned for $p=1,2,\dots,\frac{(k-1)}{2}$ (or) $\frac{k}{2}$ accordingly k is odd (or) even, respectively.

When $p=1$, ${}^{k}C_1$ ways vertex set are partitioned into V_1 containing $3^{m}2^{n-(m+k)}$ choices of vertices and V_2 containing $3^{m}2^{n-(m+k)}$ choices of vertices. When $p=2$, ${}^{k}C_2$ ways vertex set are partitioned into V_1 containing $3^{m}2^{n-(m+k)}$ choices of vertices and V_2 containing $3^{m}2^{n-(m+k)}$

choices of vertices and so on. When k is odd, the cardinality of the vertex set is

$$|V(F_X(T))| = 2\sum_{p=1}^{v} {}^k\mathrm{C}_p 3^m 2^{n-m-k}.$$

When k is odd, $v = \frac{(k-1)}{2}$. Suppose if k is even, $v = \frac{k}{2}$ and provided at $p = \frac{k}{2}$, ${}^k\mathrm{C}_p = \frac{{}^k\mathrm{C}_p}{2}$.

Case 4. When $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{s_1}, X_{s_2}, \dots, X_{s_t}\}$ for $\{i_1, i_2, \dots, i_r, s_1, s_2, \dots, s_t\} \in \{1,2,\dots,m\}$ then using Lemma 1, the vertex set of $G(F_X(T))$ is partitioned for $p = 1,2,\dots,\frac{(r+t)-1}{2}$ (or) $\frac{(r+t)}{2}$ accordingly $(r+t)$ is odd (or) even, respectively.

When $p = 1$, ${}^r\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t+1}3^{m-(t+r)}2^{n-m}$ choices of vertices and V_2 containing $2^{r-1}3^{m-(t+r)}2^{n-m}$ choices of vertices, and ${}^t\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t-1}3^{m-(t+r)}2^{n-m}$ choices of vertices and V_2 containing $2^{r+1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices, and there will be

$${}^r\mathrm{C}_1(2^{t+1}+2^{r-1})3^{m-(t+r)}2^{n-m} + {}^t\mathrm{C}_1(2^{t-1}+2^{r+1})3^{m-(t+r)}2^{n-m}$$

vertices for $p = 1$. Suppose if $p = 2$, ${}^r\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^{t+2}3^{m-(t+r)}2^{n-m}$ choices of vertices and V_2 containing $2^{r-2}3^{m-(t+r)}2^{n-m}$ choices of vertices, ${}^t\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^{t-2}3^{m-(t+r)}2^{n-m}$ choices of vertices and V_2 containing $2^{r+2}3^{m-(t+r)}2^{n-m}$ choices of vertices, and ${}^r\mathrm{C}_1{}^t\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^t3^{m-(t+r)}2^{n-m}$ choices of vertices and V_2 containing $2^r3^{m-(t+r)}2^{n-m}$ choices of vertices. So for $p = 2$

$${}^r\mathrm{C}_2(2^{t+2}+2^{r-2})3^{m-(t+r)}2^{n-m} + {}^t\mathrm{C}_2(2^{t-2}+2^{r+2})3^{m-(t+r)}2^{n-m} + {}^r\mathrm{C}_1{}^t\mathrm{C}_1(2^t+2^r)3^{m-(t+r)}2^{n-m}$$

vertices and so on. The cardinality of the vertex set when $(r+t)$ odd is

$$|V(F_X(T))| = \sum_{p=1}^{v}[{}^r\mathrm{C}_p(2^{t+p}+2^{r-p}) + {}^t\mathrm{C}_p(2^{t-p}+2^{r+p})] + \sum_{p=2}^{\frac{(r+t)-1}{2}}\sum_{q=1}^{p-1}{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}(2^q2^{t-(p-q)}+2^{r-q}2^{p-q})]3^{m-(t+r)}2^{n-m}$$

with

(i) If $r < t$, then the summation terminates for p at $v = r$ in the first term and $v = \frac{(r+t)-1}{2}$ in the second term

(ii) If $r > t$, then the summation terminates for p at $v = \frac{(r+t)-1}{2}$ in the first term and $v = t$ in the second term

The cardinality of the vertex set when $(r+t)$ even is

$$|V(F_X(T))| = \sum_{p=1}^{v}[{}^r\mathrm{C}_p(2^{t+p}+2^{r-p}) + {}^t\mathrm{C}_p(2^{t-p}+2^{r+p})] + \sum_{p=2}^{\frac{(r+t)}{2}}\sum_{q=1}^{p-1}{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}(2^q2^{t-(p-q)}+2^{r-q}2^{p-q})]3^{m-(t+r)}2^{n-m}$$

with

(i) If $r < t$, then the summation terminates for p at $v = r$ in the first term and $v = \frac{(r+t)}{2}$ in the second term provided at $p = \frac{(r+t)}{2}$, ${}^t\mathrm{C}_p = \frac{{}^t\mathrm{C}_p}{2}$ and ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$.

(ii) If $r > t$, then the summation terminates for p at $v = \frac{(r+t)}{2}$ in the first term and $v = t$ in the second term provided at $p = \frac{(r+t)}{2}$, ${}^r\mathrm{C}_p = \frac{{}^r\mathrm{C}_p}{2}$ and ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$.

(iii) If $r = t$, then the summation terminates for p at $v = \frac{(r+t)}{2}$ in both first and the second term provided at $p = \frac{(r+t)}{2}$, ${}^r\mathrm{C}_p = \frac{{}^r\mathrm{C}_p}{2}$, ${}^t\mathrm{C}_p = \frac{{}^t\mathrm{C}_p}{2}$ and ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$.

Case 5. When $X = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$ for $\{i_1, i_2, \dots, i_r\} \in \{1, 2, \dots, m\}$ and $\{j_1, j_2, \dots, j_k\} \in \{m+1, m+2, \dots, n\}$, then using Lemma 1, the vertex set of $G(F_X(T))$ is partitioned for $p = 1, 2, \dots, \frac{(r+k)-1}{2}$ (or) $\frac{(r+k)}{2}$ accordingly $(r+k)$ is odd (or) even respectively.

When $p = 1$, ${}^r\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $(2)3^{m-r}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r-1}3^{m-r}2^{n-(m+k)}$ choices of vertices, and ${}^k\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $3^{m-r}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^r3^{m-r}2^{n-(m+k)}$ choices of vertices. Hence there will be

$$ {}^r\mathrm{C}_1(2+2^{r-1})3^{m-r}2^{n-(m+k)} + {}^k\mathrm{C}_1(1+2^r)3^{m-r}2^{n-(m+k)} $$

vertices for $p = 1$. Suppose if $p = 2$, ${}^r\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^2 3^{m-r}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r-2}3^{m-r}2^{n-(m+k)}$ choices of vertices, ${}^k\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $3^{m-r}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^r3^{m-r}2^{n-(m+k)}$ choices of vertices, and ${}^r\mathrm{C}_1{}^k\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $(2)3^{m-r}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r-1}3^{m-r}2^{n-(m+k)}$ choices of vertices. So for $p = 2$

$$ {}^r\mathrm{C}_2(2^2+2^{r-2})3^{m-r}2^{n-(m+k)} + {}^k\mathrm{C}_2(1+2^r)3^{m-r}2^{n-(m+k)} + {}^r\mathrm{C}_1{}^k\mathrm{C}_1(2+2^{r-1})3^{m-r}2^{n-(m+k)} $$

vertices and so on. The cardinality of the vertex set when $(r+k)$ odd (or) even is

$$ |V(F_X(T))| = \sum_{p=1}^{v}[{}^r\mathrm{C}_p(2^p+2^{r-p}) + {}^k\mathrm{C}_p(1+2^r)] + \sum_{p=2}^{\frac{(r+k)-1}{2}}\sum_{q=1}^{p-1}{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}(2^q+2^{r-q})]3^{m-r}2^{n-(m+k)}. $$

However, various comparisons for the choices of r and k are similar to Case 4.

Case 6. When $X = \{X_{s_1}, X_{s_2}, \dots, X_{s_t}, X_{j_1}, X_{j_2}, \dots, X_{j_k}\}$ for $\{s_1, s_2, \dots, s_t\} \in \{1, 2, \dots, m\}$ and $\{j_1, j_2, \dots, j_k\} \in \{m+1, m+2, \dots, n\}$, then using Lemma 1, the vertex set of $G(F_X(T))$ is partitioned for $p = 1, 2, \dots, \frac{(t+k)-1}{2}$ (or) $\frac{(t+k)}{2}$ accordingly $(t+k)$ is odd (or) even, respectively.

When $p = 1$, ${}^t\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t-1}3^{m-t}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^1 3^{m-t}2^{n-(m+k)}$ choices of vertices, and ${}^k\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^t3^{m-t}2^{n-(m+k)}$ choices of vertices and V_2 containing $3^{m-t}2^{n-(m+k)}$ choices of vertices. Hence there will be

$$ {}^t\mathrm{C}_1(2^{t-1}+2^1)3^{m-t}2^{n-(m+k)} + {}^k\mathrm{C}_1(2^t+1)3^{m-t}2^{n-(m+k)} $$

vertices for $p = 1$. Suppose if $p = 2$, ${}^t\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^{t-2}3^{m-t}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^2 3^{m-t}2^{n-(m+k)}$ choices of vertices, ${}^k\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^t3^{m-t}2^{n-(m+k)}$ choices of vertices and V_2 containing $3^{m-t}2^{n-(m+k)}$ choices of vertices, and ${}^t\mathrm{C}_1{}^k\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t-1}3^{m-t}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^1 3^{m-t}2^{n-(m+k)}$ choices of vertices. So for $p=2$

$$^{t}\mathrm{C}_2(2^{t-2}+2^2)3^{m-t}2^{n-(m+k)}+{}^{k}\mathrm{C}_2(2^t+1)3^{m-t}2^{n-(m+k)}+{}^{t}\mathrm{C}_1{}^{k}\mathrm{C}_1(2^{t-1}+2^1)3^{m-t}2^{n-(m+k)}$$

vertices and so on. The cardinality of the vertex set when $(t+k)$ odd (or) even is

$$|V(F_X(T))|=[\sum_{p=1}^{v}[{}^{t}\mathrm{C}_p(2^{t-p}+2^p)+{}^{k}\mathrm{C}_p(2^t+1)]+\sum_{p=2}^{\frac{(t+k)-1}{2}}\sum_{q=1}^{p-1}{}^{t}\mathrm{C}_q{}^{k}\mathrm{C}_{p-q}(2^{t-q}+2^q)]3^{m-t}2^{n-(m+k)}.$$

However, various comparisons for the choices of t and k are similar to *Case* 4.

Case 7. When $X=\{x_{i_1},x_{i_2},\dots,x_{i_r},X_{s_1},X_{s_2},\dots,X_{s_t},X_{j_1},X_{j_2},\dots,X_{j_k}\}$ for $\{i_1,i_2,\dots,i_r,s_1,s_2,\dots,s_t\}\in\{1,2,\dots,m\}$ and $\{j_1,j_2,\dots,j_k\}\in\{m+1,m+2,\dots,n\}$, then using Lemma 1, the vertex set of $G(F_X(T))$ is partitioned for $p=1,2,\dots,\frac{(r+t+k)-1}{2}$ (or) $\frac{(r+t+k)}{2}$ accordingly $(r+t+k)$ is odd (or) even, respectively.

When $p=1$, ${}^{r}\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t+1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r-1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices, ${}^{t}\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t-1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r+1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and ${}^{k}\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices. Hence there will be

$$\begin{aligned}&{}^{r}\mathrm{C}_1(2^{t+1}+2^{r-1})3^{m-(t+r)}2^{n-(m+k)}+{}^{t}\mathrm{C}_1(2^{t-1}+2^{r+1})3^{m-(t+r)}2^{n-(m+k)}\\&\quad+{}^{k}\mathrm{C}_1(2^t+2^r)3^{m-(t+r)}2^{n-(m+k)}\end{aligned}$$

vertices for $p=1$. For $p=2$, ${}^{r}\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^{t+2}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r-2}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices, ${}^{t}\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^{t-2}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r+2}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices, ${}^{k}\mathrm{C}_2$ ways vertex set are partitioned into V_1 containing $2^{t}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices. Also, ${}^{r}\mathrm{C}_1{}^{t}\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices, ${}^{r}\mathrm{C}_1{}^{k}\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t+1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r-1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices, and ${}^{t}\mathrm{C}_1{}^{k}\mathrm{C}_1$ ways vertex set are partitioned into V_1 containing $2^{t-1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices and V_2 containing $2^{r+1}3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices. Hence there will be

$$\begin{aligned}&[{}^{r}\mathrm{C}_2(2^{t+2}+2^{r-2})+{}^{t}\mathrm{C}_2(2^{t-2}+2^{r+2})+{}^{k}\mathrm{C}_2(2^t+2^r)+{}^{r}\mathrm{C}_1{}^{t}\mathrm{C}_1(2^t+2^r)+{}^{r}\mathrm{C}_1{}^{k}\mathrm{C}_1(2^{t+1}+2^{r-1})\\&\quad+{}^{t}\mathrm{C}_1{}^{k}\mathrm{C}_1(2^{t-1}+2^{r+1})]3^{m-(t+r)}2^{n-(m+k)}\end{aligned}$$

vertices for $p=2$. Now for $p=3$, ${}^{r}\mathrm{C}_3$, ${}^{t}\mathrm{C}_3$ and ${}^{k}\mathrm{C}_3$ ways vertex set are partitioned into V_1 and V_2 containing $(2^{t+3}+2^{r-3})3^{m-(t+r)}2^{n-(m+k)}$, $(2^{t-3}+2^{r+3})3^{m-(t+r)}2^{n-(m+k)}$ and $(2^t+2^r)3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices respectively. Further, ${}^{r}\mathrm{C}_1{}^{t}\mathrm{C}_2$ and ${}^{r}\mathrm{C}_2{}^{t}\mathrm{C}_1$ ways vertex set are partitioned into V_1 and V_2 containing $(2^{t-1}+2^{r+1})3^{m-(t+r)}2^{n-(m+k)}$ and $(2^{t+1}+2^{r-1})3^{m-(t+r)}2^{n-(m+k)}$ vertices, ${}^{r}\mathrm{C}_1{}^{k}\mathrm{C}_2$ and ${}^{r}\mathrm{C}_2{}^{k}\mathrm{C}_1$ ways vertex set are partitioned into V_1 and V_2 containing $(2^{t+1}+2^{r-1})3^{m-(t+r)}2^{n-(m+k)}$ and $(2^{t+2}+2^{r-2})3^{m-(t+r)}2^{n-(m+k)}$ vertices, and ${}^{t}\mathrm{C}_1{}^{k}\mathrm{C}_2$ and ${}^{t}\mathrm{C}_2{}^{k}\mathrm{C}_1$ ways vertex set are partitioned into V_1 and V_2 containing $(2^{t-1}+2^{r+1})3^{m-(t+r)}2^{n-(m+k)}$ and $(2^{t-2}+2^{r+2})3^{m-(t+r)}2^{n-(m+k)}$ vertices. Also, for $p=3$, ${}^{r}\mathrm{C}_1{}^{t}\mathrm{C}_1{}^{k}\mathrm{C}_1$ ways vertex set are partitioned into V_1 and V_2 containing $(2^t+2^r)3^{m-(t+r)}2^{n-(m+k)}$ choices of vertices. There will be

$$[{}^r\mathrm{C}_3(2^{t+3}+2^{r-3})+{}^t\mathrm{C}_3(2^{t-3}+2^{r+3})+{}^k\mathrm{C}_3(2^t+2^r)+{}^r\mathrm{C}_1{}^t\mathrm{C}_2(2^{t-1}+2^{r+1})+{}^r\mathrm{C}_2{}^t\mathrm{C}_1(2^{t+1}+2^{r-1})$$
$$+{}^r\mathrm{C}_1{}^k\mathrm{C}_2(2^{t+1}+2^{r-1})+{}^r\mathrm{C}_2{}^k\mathrm{C}_1(2^{t+2}+2^{r-2})+{}^t\mathrm{C}_1{}^k\mathrm{C}_2(2^{t-1}+2^{r+1})+{}^t\mathrm{C}_2{}^k\mathrm{C}_1(2^{t-2}+2^{r+2})$$
$$+{}^r\mathrm{C}_1{}^t\mathrm{C}_1{}^k\mathrm{C}_1(2^t+2^r)]3^{m-(t+r)}2^{n-(m+k)}$$

vertices and so on. The cardinality of the vertex set when $(r+t+k)$ odd is

$$|V(F_X(T))|=\sum_{p=1}^{v}{}^r\mathrm{C}_p\{(2^{t+p}+2^{r-p})3^{m-(t+r)}2^{n-(m+k)}\}+\sum_{p=1}^{v}{}^t\mathrm{C}_p\{(2^{t-p}+2^{r+p})3^{m-(t+r)}2^{n-(m+k)}\}$$
$$+\sum_{p=1}^{v}{}^k\mathrm{C}_p\{(2^t+2^r)3^{m-(t+r)}2^{n-(m+k)}\}+\sum_{p=2}^{\frac{(r+t+k)-1}{2}}\sum_{q=1}^{p-1}[{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}(2^q2^{t-(p-q)}+2^{r-q}2^{p-q})$$
$$+{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}(2^{t+q}+2^{r-q})+{}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}(2^{t-q}+2^r2^{t-(t-q)})]3^{m-(t+r)}2^{n-(m+k)}$$
$$+\sum_{p=3}^{\frac{(r+t+k)-1}{2}}\sum_{q=1}^{p-2}\sum_{u=1}^{p-(q+1)}{}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}(2^q2^{t-u}+2^{r-q}2^{t-(t-u)})3^{m-(t+r)}2^{n-(m+k)}$$

with

(i) If $r=t=k$, then the summation terminates for p at $v=r$ in the first term, $v=t$ in the second term and $v=k$ in the third term.

(ii) If $r<t<k$, then the summation terminates for p at $v=r$ in the first term, $v=t$ in the second term and $v=\frac{(r+t+k)-1}{2}$ in the third term.

(iii) If $r>t>k$, then the summation terminates for p at $v=\frac{(r+t+k)-1}{2}$ in the first term, $v=t$ in the second term and $v=k$ in the third term.

(iv) If $r<t$ and $t=k$, then the summation terminates for p at $v=r$ in the first term, $v=t$ in the second term and $v=k$ in the third term.

(v) If $r<t$ and $t>k$, then the summation terminates for p at $v=r$ in the first term, $v=\frac{(r+t+k)-1}{2}$ in the second term and $v=k$ in the third term.

The cardinality of the vertex set when $(r+t+k)$ even is

$$|V(F_X(T))|=\sum_{p=1}^{v}{}^r\mathrm{C}_p\{(2^{t+p}+2^{r-p})3^{m-(t+r)}2^{n-(m+k)}\}+\sum_{p=1}^{v}{}^t\mathrm{C}_p\{(2^{t-p}+2^{r+p})3^{m-(t+r)}2^{n-(m+k)}\}$$
$$+\sum_{p=1}^{v}{}^k\mathrm{C}_p\{(2^t+2^r)3^{m-(t+r)}2^{n-(m+k)}\}+\sum_{p=2}^{\frac{(r+t+k)}{2}}\sum_{q=1}^{p-1}[{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}(2^q2^{t-(p-q)}+2^{r-q}2^{p-q})$$
$$+{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}(2^{t+q}+2^{r-q})+{}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}(2^{t-q}+2^r2^{t-(t-q)})]3^{m-(t+r)}2^{n-(m+k)}$$
$$+\sum_{p=3}^{\frac{(r+t+k)}{2}}\sum_{q=1}^{p-2}\sum_{u=1}^{p-(q+1)}{}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}(2^q2^{t-u}+2^{r-q}2^{t-(t-u)})3^{m-(t+r)}2^{n-(m+k)}$$

with

(i) If $r=t=k$, then the summation terminates for p at $v=r$ in the first term, $v=t$ in the second term and $v=k$ in the third term provided at $p=\frac{(r+t+k)}{2}$, ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}=\frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$, ${}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}=\frac{{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, ${}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}=\frac{{}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, and ${}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}=\frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}}{2}$.

(ii) If $r<t<k$, then the summation terminates for p at $v=r$ in the first term, $v=t$ in

the second term and $v = \frac{(r+t+k)}{2}$ in the third term provided at $p = \frac{(r+t+k)}{2}$, ${}^k\mathrm{C}_p = \frac{{}^k\mathrm{C}_p}{2}$, ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$, ${}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, ${}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, and ${}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}}{2}$.

(iii) If $r > t > k$, then the summation terminates for p at $v = \frac{(r+t+k)}{2}$ in the first term, $v = t$ in the second term and $v = k$ in the third term provided at $p = \frac{(r+t+k)}{2}$, ${}^r\mathrm{C}_p = \frac{{}^r\mathrm{C}_p}{2}$, ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$, ${}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, ${}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, and ${}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}}{2}$.

(iv) If $r < t$ and $t = k$, then the summation terminates for p at $v = r$ in the first term, $v = t$ in the second term and $v = k$ in the third term ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$, ${}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, ${}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, and ${}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}}{2}$.

(v) If $r < t$ and $t > k$, then the summation terminates for p at $v = r$ in the first term, $v = \frac{(r+t+k)}{2}$ in the second term and $v = k$ in the third term $p = \frac{(r+t+k)}{2}$, ${}^t\mathrm{C}_p = \frac{{}^t\mathrm{C}_p}{2}$, ${}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_{p-q}}{2}$, ${}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^r\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, ${}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q} = \frac{{}^t\mathrm{C}_q{}^k\mathrm{C}_{p-q}}{2}$, and ${}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)} = \frac{{}^r\mathrm{C}_q{}^t\mathrm{C}_u{}^k\mathrm{C}_{p-(q+u)}}{2}$. □

4. Gray Code Length Identification From Rough Identity-Summand Graph

This section discusses the scheme that the transition sequence obtained using the distinct complete bipartite graph generated from the rough identity-summand graph helps to obtain Gray codes of various lengths. However, the length of the gray code varies for each transition sequence corresponding to the vertex set cardinality. Then from [4], the cardinality of vertices in each distinct complete bipartite graph generates Gray codes of length $|V_{B_i}(F_X(T))|$ for $i = 1,2,\dots,l$, where l is the number of distinct complete bipartite graphs generated from $G(F_X(T))$ and $|V(F_X(T))| = \sum_{i}^{l} |V_{B_i}(F_X(T))|$. The final lower and upper approximation transition sequences are obtained using the initial lower and upper transition sequence for the given X. To begin with, let $I = (U,R)$ be an approximation space, where the universal set U is the sequence of n-bit binary numbers say $\{00\dots0, 10\dots0, 01\dots0, \dots, 11\dots1\}$. The cardinality of n-bit binary numbers in U is 2^n. The equivalence classes obtained by the partition on U is given by $\{X_1, X_2, \dots, X_n, X_{n+1}\}$. For $1 \le i \le n$, X_i is the set of i zero n-bit binary numbers and X_{n+1} is the equivalence class containing n-bit binary number of all 1's. The cardinality of $|T|$ is $2^2 3^{n-1}$.

For example, consider a 3-bit binary sequence as the universal set $U = \{000, 100, 010, 001, 110, 011, 101, 111\}$. Then, the partition on U forms the equivalence classes $\{X_1, X_2, X_3, X_4\}$ where $X_1 = \{110, 101, 011\}, X_2 = \{100, 010, 001\}, X_3 = \{000\}$ and $X_4 = \{111\}$.

Let $X = \{x_1, x_2, X_3, X_4\}$ and $|T| = 2^2 3^{3-1} = 36$, $|F_X(T)| = 2^2 = 4$, then the filter

$$F_{\{x_1,x_2,X_3,X_4\}}(T) = \{RS(x_1 \cup x_2 \cup X_3 \cup X_4), RS(x_1 \cup X_2 \cup X_3 \cup X_4), RS(X_1 \cup x_2 \cup X_3 \cup X_4), RS(X_1 \cup X_2 \cup X_3 \cup X_4)\}.$$

So the Rough identity-summand graph $G(F_{\{x_1,x_2,X_3,X_4\}}(T))$ exists [4] and by Lemma 1, the number of complete bipartite graphs generated from $G(F_{\{x_1,x_2,X_3,X_4\}}(T))$ for $r = k = 2$ and $t = 0$ is

$$\begin{aligned} l &= \sum_{p=1}^{\frac{(2+2)}{2}} {}^2\mathrm{C}_p + \sum_{p=1}^{\frac{(2+2)}{2}} {}^2\mathrm{C}_p + \sum_{p=2}^{\frac{(2+2)}{2}} \sum_{q=1}^{p-1} {}^2\mathrm{C}_q {}^2\mathrm{C}_{p-q} \\ &= {}^2\mathrm{C}_1 + \frac{{}^2\mathrm{C}_2}{2} + {}^2\mathrm{C}_1 + \frac{{}^2\mathrm{C}_2}{2} + \frac{{}^2\mathrm{C}_1 {}^2\mathrm{C}_1}{2} \\ &= 2+2+1+2 = 7. \end{aligned}$$

Therefore, 7 complete bipartite graphs are generated from $G(F_X(T))$. Now, to calculate the cardinality of vertices in $G(F_{\{x_1,x_2,X_3,X_4\}}(T))$

$$\begin{aligned} |V(F_{\{x_1,x_2,X_3,X_4\}}(T))| &= \sum_{p=1}^{\frac{(2+2)}{2}} {}^2\mathrm{C}_p\{(2^p + 2^{2-p})3^{2-(2)}2^{4-(2+2)}\} + \sum_{p=1}^{\frac{(2+2)}{2}} {}^2\mathrm{C}_p\{(2^0 + 2^2)3^{2-(2)}2^{4-(2+2)}\} \\ &\quad + \sum_{p=2}^{\frac{(2+2)}{2}} \sum_{q=1}^{p-1} {}^2\mathrm{C}_q {}^2\mathrm{C}_{p-q}(2^q + 2^{2-q})3^{2-(2)}2^{4-(2+2)} \\ &= {}^2\mathrm{C}_1(2^1 + 2^{2-1}) + \frac{{}^2\mathrm{C}_2}{2}(2^2 + 2^{2-2}) + {}^2\mathrm{C}_1(1+2^2) \\ &\quad + \frac{{}^2\mathrm{C}_2}{2}(1+2^2) + \frac{{}^2\mathrm{C}_1 {}^2\mathrm{C}_1}{2}(2+2) \\ &= 4+4+5+5+5+4+4 = 31. \end{aligned}$$

When $X = \{x_1, x_2, X_3, X_4\}$, $S_L(1) = 1$ and $S_U(0) = 0$. The 7 distinct complete bipartite graphs generated from $G(F_{\{x_1,x_2,X_3,X_4\}}(T))$ are denoted by $B(F_X(x_1, x_2X_3X_4))$, $B(F_X(x_2, x_1X_3X_4))$, $B(F_X(x_1x_2, X_3X_4))$, $B(F_X(X_3, x_1x_2X_4))$, $B(F_X(X_4, x_1x_2X_3))$, $B(F_X(x_1X_3, x_2X_4))$, $B(F_X(x_1X_4, x_2X_3))$. Here the Gray code of length 4 is generated for $B(F_X(x_1, x_2X_3X_4))$, $B(F_X(x_2, x_1X_3X_4))$, $B(F_X(x_1X_3, x_2X_4))$, $B(F_X(x_1X_4, x_2X_3))$ and length 5 is for $B(F_X(x_1x_2, X_3X_4))$, $B(F_X(X_3, x_1x_2X_4))$, $B(F_X(X_4, x_1x_2X_3))$, respectively.

Remark 3. From [4], generating Gray codes from $G(F_X(T))$ depends on finding the number of distinct complete bipartite graphs generated from $G(F_X(T))$ and the length of the Gray code corresponds to the vertex set cardinality of the respective complete bipartite graph. Hence for each complete bipartite graph Gray codes of the same length are obtained through its lower and upper approximation transition sequences.

5. Conclusion

This study defined to extract different Gray code lengths from the distinct complete bipartite graphs of the rough identity-summand graph $G(F_X(T))$. A detailed enumeration of distinct complete bipartite graphs from $G(F_X(T))$ was established to determine the nature of the rough identity-summand graph defined for the filters of a rough bi-Heyting algebra. Additionally, the vertex cardinality was determined. This resulted in two approximate transition sequences producing Gray codes of different lengths.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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