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Research Article

A Geometric View on the Surfaces of Rotation Generated by Killing Vector Field in Galilean 3-Space

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Abstract. In this paper, by using rotational matrix which is the subgroups of the manifold M corresponding to rotation, some certain results of describing the surface are given in detail obtaining Killing vector field in G_3, the rotational surfaces generated by the rotational matrices are given using an isotropic curve and isotropic matrices of rotation in G_3. Moreover, taking the Gaussian and mean curvatures of this special rotational surface, the conditions being linear Weingarten surfaces and HK-quadric surface are expressed harmonic surface.

Keywords. Galilean space, Rotated surface, Killing vector field, Weingarten surface, HK-quadric surface

Mathematics Subject Classification (2020). 53B30, 53B50, 53C80

1. Introduction

The geometry of rotational surfaces in E^3 has been studied widely. The rotational surfaces in three dimensional Euclidean space are generated by rotating an arbitrary axis. In particular, rotational surfaces spaces such as Galilean and Minowski. It has been studied for a long time and there are many examples of such surfaces have been introduced newly, see Ganchev and Milousheva [7], Goemans, [8], and Yoon [28].

A surface whose mean curvature is in functional relationship with its Gaussian curvature. Namely, a surface is said to be a Weingarten surface if there exists a relation, that does not depend on the parameters, between the mean curvature and the total curvature (or between the principal curvatures). Also, a surface is said to be a Weingarten if there is a smooth relation $U(k_1,k_2)=0$ between two principle curvatures k_1 and k_2. If K and H are the Gauss curvature and the mean curvatures, respectively, then $U(k_1,k_2)=0$ expresses a relation as $\Phi(K,H)=0$. The existence of a non-trivial relation $\Phi(K,H)=0$ on a surface, which is parameterized by a patch $x(w,v)$, is equivalent to the following Jacobian determinant $\frac{\partial(K,H)}{\partial(w,v)}=0$. Furthermore, if the equations $U=a_1k_1+a_2k_2-a_3$ (or $\Phi=a_1H+a_2K-a_3$) hold, the surfaces are called linear Weingarten surfaces, where a_i, $i\in\mathbb{R}$ with $a_1^2+a_2^2\neq 0$. On the other hand, if a surface satisfies the following equation:

$$a_1H^2+2a_2HK+a_3K^2=\text{constant},\quad a_1\neq 0, \tag{1.1}$$

then the surface is called as a HK-quadric surface (Kim and Yoon [14]). In Almaz and Külahcı [3], a brief description of rotational surfaces in 4-dimensional (4D) Galilean space is given using a curve and matrices and different types of rotational matrices also are given, which are the subgroups of by rotating a selected axis in Galilean 4-space. In Almaz and Külahcı [2], the tubular surface generated by rectifying curves are explored, using the Gaussian and mean curvatures of tubular surfaces, for the linear Weingarten surfaces and HK-quadric surfaces, harmonic surfaces some characterizations are given. In Abdel-Aziz and Saad [1], time-like tube surface around the space-like curve with time-like and space-like binormal vectors was studied in E_1^3 by the authors. Moreover, Weingarten and linear Weingarten conditions were given for this surface with respect to their curvatures. In Dede *et al*. [6] determined the connection between the curvatures of the parallel surfaces in G_3 and the definition of parallel surfaces was given by the authors in Galilean space. Consequently, the Gauss curvature and mean curvature of parallel surface as to those curvatures of the base surface were found by the author. The same studies and consequences about surfaces in G_3 were given by the various authors by Dede [5]. Ganchev and Milousheva [7] contemplated the resembling of these surfaces in the Minkowski 4-space and they studied general rotational surfaces with special unchangeable on space-like surfaces. Goemans [8] constructed a new type of surfaces in E^4 and Lorentz-Minkowski space by performing two simultaneous rotations on a planar curve. Furthermore, classification theorems of flat double rotational surfaces were proved by the author. Hoffmann and Zhou [10] discussed some matters of denoting two-dimensional surfaces in four-dimensional space, involving the treatment of surface normal under projection, the silhouette points due to the projection, and methods for object orientation and projection center description. In Lopez [15], the surfaces in Euclidean 3-space foliated by pieces of circles are studied and that satisfy a Weingarten condition of type $aH+bK=c$, $a,b,c\in\mathbb{R}$, H and K denote the mean curvature and the Gauss curvature respectively, by the author. In Ro and Yoon [21], a tube in Euclidean 3-space satisfying some equation in terms of the Gaussian curvature, the mean curvature and the second Gaussian curvature were studied by the authors. Furthermore, ruled Weingarten surfaces in the Galilean space were studied by Šipuš [23]. Also, the some consequences about surfaces and curves in different ambient spaces were investigated by Karacan and Yayli [13], Kim and Yoon [14], and Yaylı *et al*. [25]. In Yilmaz [26], the Frenet-Serret frame of a curve in G_4 were expressed by the author and the Frenet-Serret equations of mentioned curve were obtained. Furthermore, the surfaces in various ambient spaces have been classified by Yoon [27,28].

2. Preliminaries

The Galilean space is a 3D complex projective space P_3 in which the absolute figure $\{w, f, I_1, I_2\}$ originates of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute plane) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points). The study mechanics of plane-parallel movements drop to the study of a geometry of 3D space with coordinates $\{x, y, t\}$ are written using the movement formula (Pressley [20]). This geometry is said to be Galilean geometry, in [20], is represented that 4D Galilean geometry, which studies all features unchangeable under movements of objects in space, is as well more complex. In affine coordinates the Galilean, scalar product between two points $W_1 = (w_1^1, w_2^1, w_3^1)$ and $W_2 = (w_1^2, w_2^2, w_3^2)$ is defined by

$$g(W_1, W_2) = \begin{cases} \text{if } w_1^1 \neq w_1^2, & |w_1^1 - w_1^2|, \\ \text{if } w_1^1 = w_1^2, & \sqrt{(w_2^1 - w_2^2)^2 + (w_3^1 - w_3^2)^2}. \end{cases} \tag{2.1}$$

Hence, we can give following cases:

***Case* 1.** All unit isotropic vectors are written as the structure $W_1 = (1, w_2^1, w_3^1)$. For isotropic vectors, $w_1^1 = 0$ holds and the matrix is given as

$$ds^2 = (dw_2^1)^2 + (dw_3^1)^2 \ \text{ or } \ \eta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.2}$$

***Case* 2.** A vector $W_1 = (w_1^1, w_2^1, w_3^1)$ is called a non-isotropic if $w_1^1 \neq 0$, the matrix is given as

$$ds^2 = (dw_1^1)^2 \ \text{ or } \ \eta_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

Hence, for the vectors $W_i = (w_1^i, w_2^i, w_3^i)$, $i \in \{1,2\}$, the Galilean cross product in G_3 is given as follows:

$$W_1 \times W_2 = \begin{bmatrix} 0 & e_2 & e_3 \\ w_1^1 & w_2^1 & w_3^1 \\ w_1^2 & w_2^2 & w_3^2 \end{bmatrix}, \tag{2.4}$$

where e_i are the standard basic vectors.

The scalar product of two vectors $W_1 = (w_1^1, w_2^1, w_3^1)$ and $W_2 = (w_1^2, w_2^2, w_3^2)$ in G_3 is defined as

$$\langle W_1, W_2 \rangle_{G_3} = \begin{cases} \text{if } w_1^1 \neq 0 \ \textit{ or } \ w_1^2 \neq 0, & w_1^1 w_1^2, \\ \text{if } w_1^1 = 0, \ w_1^2 = 0, & w_2^1 w_2^2 + w_3^1 w_3^2. \end{cases} \tag{2.5}$$

Let $\alpha : I \subset \mathbb{R} \to G_3$, $\alpha(s) = (s, \varrho(s), \vartheta(s))$ be a curve with arc-length parameter s in G_3. The tangent vector of α is defined as

$$t = \alpha'(s) = (1, \varrho'(s), \vartheta'(s)) \tag{2.6}$$

and t is a unit vector. For all $s \in I$, we assume that $\kappa \neq 0$. The principal normal vector is described as

$$n(s)=\frac{t'(s)}{\kappa(s)}=\frac{1}{\kappa(s)}(0,\varrho''(s),\vartheta''(s)). \tag{2.7}$$

The binormal vector field of the curve α is defined as

$$b(s)=\frac{1}{\tau(s)}\left(0,\left(\frac{\varrho''(s)}{\kappa}\right)',\left(\frac{\vartheta''(s)}{\kappa}\right)'\right). \tag{2.8}$$

Since the vector $b(s)$ is perpendicular to both t and n, the vector

$$b(s)=\zeta t(s)\times n(s) \tag{2.9}$$

is written where the coefficient ζ can be taken ±1 to obtain the determinant of the $[t,n,b]$ matrix $+1$.

Then the curvatures $\kappa(s)$, $\tau(s)$ of the curve α can be written as follows:

$$\kappa(s)=\sqrt{(\varrho''(s))^2+(\vartheta''(s))^2},\quad \tau(s)=\|n'(s)\|_{G_4}.$$

Also, the Frenet formulas of the Frenet curve α are written as follows:

$$t'=\kappa n,\ n'=\tau b,\ b'=-\tau b, \tag{2.10}$$

here, the expressions t,n,b are mutually orthogonal vector fields supplying following equations:

$$\langle t,t\rangle_{G_4}=\langle n,n\rangle_{G_4}=\langle b,b\rangle_{G_4}=1,\ \ \langle t,n\rangle_{G_4}=\langle t,b\rangle_{G_4}=\langle n,b\rangle_{G_4}.$$

(Gordon [9], Lugo [17], Montiel and Ros [18], Murray *et al*. [19], and Shifrin [22])

Definition 1. Let M be a differentiable manifold and let φ be a parameterized group of transformations on M, which is a differentiable conversion from $M\times\mathbb{R}$ to M. So that for $\forall\ x\in M,t,s\in\mathbb{R}$, the following equation is written as:

$$\varphi(x,0)=x\ \text{ and }\ \varphi(\varphi(x,t),s)=\varphi(x,t+s).$$

(Castillo [4], Montiel and Ros [18], and Shifrin [22])

Definition 2. Let a parameterized group of diffeomorphisms of an M manifold be the diffeomorphism $\varphi_t:M\to M$ that is smooth transformation defined by

$$\begin{aligned}\varphi:M\times\mathbb{R}&\to M\\(v,t)&\to\varphi(v,t)=\varphi_t(v).\end{aligned}$$

So that $\frac{d}{dt}\varphi_t(v)=W(v)$ is the group of diffeomorphisms that combine with a vector field, diffeomorphism φ_t is called the flow of W (Castillo [4], Montiel and Ros [18], and Shifrin [22]).

Definition 3. Let W be a vector field on a smooth manifold M and let ψ_t be the local flow generated by W. For each $t\in\mathbb{R}$, the map ψ_t is diffeomorphism of M and for a function f given on M, we consider the Pull-back $\psi_t f$. Lie derivative of the function f as to W is define by

$$L_W f=\lim_{t\to0}\left(\frac{\psi_t f-f}{t}\right)=\frac{d\psi_t f}{dt}\bigg|_{t=0}. \tag{2.11}$$

(Castillo [4], Montiel and Ros [18], and Shifrin [22])

Definition 4. If an one-parameter group of isometries is generated by a vector field W, then this vector field is called as a Killing vector field (Hswley [11], and Lerner [16]).

Let $g_{\xi\varrho}$ be any pseudo-Riemannian matrix, then the derivative is given as

$$L_W g_{\xi\varrho} = g_{\xi\varrho,z} W^z + g_{\xi z} W^z_{,\varrho} + g_{z\varrho} W^z_{,\xi}.$$

In Cartesian coordinates in Euclidean spaces where $g_{\xi\varrho,z} = 0$, and the Lie derivative is given by

$$L_W g_{\xi\varrho} = g_{\xi z} W^z_{,\varrho} + g_{z\varrho} W^z_{,\xi}.$$

In [11, 16], if the vector W generates a Killing field if and if only

$$L_W g = 0.$$

If a parameter group of isometries is generated by a vector field W, this vector field is called as the Killing vector field. Moreover, Lie bracket of Killing vector fields W_1 and W_2 is given by

$$[W_1, W_2](h) = W_1(W_2(h)) - W_2(W_1(h)) = L_{W_1} W_2$$

and which is also a Killing vector field. Also, Lie operator for these vectors is given by

$$[W_1, W_2] = \sum_{i,j} \left(W_1^i \frac{\partial W_2^j}{\partial x^i} - W_2^i \frac{\partial W_1^j}{\partial x^i} \right) \partial_j. \quad \text{(Hswley [11], and Lerner [16])} \tag{2.12}$$

Definition 5. Let Lagrangian function be a real-valued function defined in TM. If a differentiable C curve in M is a solution of Euler-Lagrange equations solved according to L Lagrangian equation, the following equation is satisfied

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \bar{C}(t) \right) - \frac{\partial L}{\partial q^i} \bar{C}(t) = 0, \quad i = 1, 2, \dots, n, \tag{2.13}$$

where $\bar{C}$ that is $\bar{C}(t) = C'_t$ is a curve in TM. If $XL = 0$, the vector field X over M represents a symmetry of the Lagrangian equation. So, if X is a symmetry for L,

$$X^i(C(t)) \frac{\partial L}{\partial \dot{q}^i} \bar{C}(t)$$

is the constant of motion (Castillo [4]).

Proposition 1. *Δf Laplacian of the differentiable function given by $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is defined as*

$$\Delta f = \left(\frac{d^2 f}{du^2} \right) + \left(\frac{d^2 f}{dv^2} \right), \quad (u, v) \in U.$$

If $\Delta f = 0$ then the function f is harmonic in U (Kühnel [12]).

3. The Rotational Surfaces Generated by ξ-axis in G_3

In this section, one has used the matrix (2.2). Therefore, one will provide one type of matrix of rotation, which is the subgroup of M corresponding to rotation about a chosen axis in E^3. Hence, one will choose two parameter matrices group of rotation. In particular, one will has given according to a brief description of rotated surfaces in G_3, the rotational matrices corresponding to the appropriate subgroup of the Galilean space and rotational surface are generated. The rotation matrices are replaced by Galilean transformation as follows:

$$M^T \eta_i M = \eta_i, \quad i = 1, 2, \tag{3.1}$$

where η_i is the matrix of G_3. The Galilean group is a subgroup of the diffeomorphisms group in G_3. Also, the rotated matrices can be obtained using any axis. Thus, the rotational matrices with respect to coordinate axes can be given. Hence, we will need rotational axes and Killing vector field to constitute rotation matrices. In this cause, we examine the Killing vector field (G_3,η_i) handled on the manifold, that is, by taking the isotropic matrix η_i in G_3, Lie derivative for any W vector field in G_3 is written as

$$L_W\eta_{iab}=\eta_{i_{ab,c}}W^c+\eta_{i_{ac}}W^c_{,b}+\eta_{i_{cb}}W^c_{,a}=0.$$

Also, with $\eta_{i_{ab,c}}=0$, this last expression is is written

$$L_W\eta_{i_{ab}}=\eta_{i_{ac}}W^c_{,b}+\eta_{i_{cb}}W^c_{,a}=0. \tag{3.2}$$

Hence, for the functions $W^1(\xi,\varrho,\vartheta)$, $W^2(\xi,\varrho,\vartheta)$ and $W^3(\xi,\varrho,\vartheta)$ in G_3, a vector field can be written as

$$W=W^1(\xi,\varrho,\vartheta)d\xi+W^2(\xi,\varrho,\vartheta)d\vartheta+W^3(\xi,\varrho,\vartheta)d\varrho. \tag{3.3}$$

For the Killing vector field W, one has to find the functions $W^1(\xi,\varrho,\vartheta)$, $W^2(\xi,\varrho,\vartheta)$ and $W^3(\xi,\varrho,\vartheta)$. Thus, equation (3.2) is calculate to find a given Killing vector field with matrix $\eta_{i_{ij}}$. In this cause, using coordinates with $\eta_{i_{ab}}$, the equation

$$\eta_{i_{ac}}W^c_{;b}+\eta_{i_{cb}}W^c_{;a}=0 \implies L_W\eta_{i_{ab}}=\eta_i([W,a],b)+\eta_i([W,b],a) \tag{3.4}$$

is written.

Theorem 1. *Let the Galilean group be a subgroup of the diffeomorphisms group of G_3 and let W be the isotropic vector which generate the isometries. Then, the killing vector field associated with the matrix η_1 is given by*

$$W(0,\varrho,\vartheta)=b_1(\vartheta d\varrho-\varrho d\vartheta), \quad b_1\in\mathbb{R}^+_0. \tag{3.5}$$

Proof. Let W be an isotropic vector that constitutes isometries. Thus, for $j=1,2,3$, as a result of algebraic operations using (3.4), the W^j and hence Killing vector field W is found. □

Theorem 2. *For the killing vector field $W(0,\varrho,\vartheta)=b_1(\vartheta d\varrho-\varrho d\vartheta)$, a subgroup of parameters of the rotational matrix is given by the matrix*

$$\Delta_\xi(w)=\begin{vmatrix}1&0&0\\0&\cos w&\sin w\\0&-\sin w&\cos w\end{vmatrix}, \quad b_1\in\mathbb{R}^+_0 \tag{3.6}$$

and for the isotropic curve $\zeta(u)=(u,f_1(u),f_2(u))$ the rotational surface is given by

$$\Omega(u,w)=(u,f_1(u)\cos w+f_2(u)\sin w,-f_1(u)\sin w+f_2(u)\cos w). \tag{3.7}$$

Proof. Using killing vector field $W=b_1(\vartheta d\varrho-\varrho d\vartheta)$, one can find one parameter group of isometry. Thus, assuming $b_1=1$ and the equation (3.27) is written as

$$W=\vartheta d\varrho-\varrho d\vartheta \ \text{ or } \ W=(0,\vartheta,-\varrho).$$

Hence, with the help of the previous expression, the rotational matrix Δ with ξ-axis is given as

$$\Delta=\begin{vmatrix}0&0&0\\0&0&1\\0&-1&0\end{vmatrix}.$$

Also, a parameterized group of homomorphisms $\varphi_w(\xi,\varrho,\vartheta)$ given by

$$\dot{\varphi}_w(\xi)=\Delta\varphi_w(\xi)$$

can be considered. Therefore, the solution obtained from the previous equation is obtained as

$$\Delta_\xi(w)=\begin{vmatrix}1 & 0 & 0\\ 0 & \cos w & \sin w\\ 0 & -\sin w & \cos w\end{vmatrix}.$$

Hence, we mean to rotate a curve with the rotation matrix obtained by using the rotation $W=\vartheta d\varrho-\varrho d\vartheta$. Therefore, any point can be moved to the ξ-axis. So, a curve with u parameter that holds this point is given as

$$\zeta(u)=(u,f_1(u),f_2(u)).$$

So, the rotation surface Ω generated by using $\Delta_\xi(w)$ is written as

$$\Omega(u,w)=\Delta_\xi(w)\begin{bmatrix}u\\ f_1(u)\\ f_2(u)\end{bmatrix}=\begin{pmatrix}u,\\ f_1(u)\cos w+f_2(u)\sin w,\\ -f_1(u)\sin w+f_2(u)\cos w\end{pmatrix}. \qquad \square$$

Theorem 3. *Let the Galilean group be a subgroup of the diffeomorphisms group in G_3 and let W be the non-isotropic vector which generate the isometries on space. Then, the Killing vector field associated with the matrix η_2 is given by*

$$W(\xi,\varrho,\vartheta)=Bd\xi \ \ or \ \ W(\xi,\varrho,\vartheta)=Bd\xi+B^2d\varrho+B^3d\vartheta, \quad B=B^i\in\mathbb{R}_0^+. \tag{3.8}$$

Proof. Similar to the proof given for Theorem 1, the proof is found. □

Theorem 4. *Let $W(\xi,\varrho,\vartheta)=Bd\xi$ be a non-isotropic Killing vector field in G_3 and let $\alpha(s)=(f(s),0,0)$ be a non-isotropic curve. Then, the rotated surface is given by*

$$\Upsilon(\omega,s)=(f(s)e^{A\omega},0,0), \quad -\infty<\omega<\infty, \quad s\in I,B\in\mathbb{R}_0^+. \tag{3.9}$$

Proof. Similar to the proof given for Theorem 2, the proof is found. □

3.1 The Some Characterizations on Rotational Surface in Galilean 3-Space

In this section, one has done examinations on the rotated surface obtained with the help of a parameterized group transformation according to the ξ axis, which is the rotation axis. Also, we will talk about a rotation with ξ-axis by using the rotation matrix in (3.6). Thus, the curve containing any point in G_3 can be moved to the $\varrho\vartheta$-plane. So, $f_1(u),f_2(u)\in C^\infty$, let ζ be the curve given by

$$\zeta(u)=(u,f_1(u),f_2(u)). \tag{3.10}$$

Therefore, the rotational surface Ω obtained according to the ξ-rotation axis in G_3 is obtained as

$$\Omega(u,w)=\begin{vmatrix}1 & 0 & 0\\ 0 & \cos w & \sin w\\ 0 & -\sin w & \cos w\end{vmatrix}\begin{vmatrix}u\\ f_1(u)\\ f_2(u)\end{vmatrix}, \tag{3.11}$$

$$\Omega(u,w)=(u,f_1(u)\cos w+f_2(u)\sin w,-f_1(u)\sin w+f_2(u)\cos w). \tag{3.12}$$

Considering some basic concepts in Galilean space, and from (3.12), one gets

$$\Omega_u = (1, \dot{f}_1(u)\cos w + \dot{f}_2(u)\sin w, -\dot{f}_1(u)\sin w + \dot{f}_2(u)\cos w), \tag{3.13a}$$
$$\Omega_w = (0, -f_1(u)\sin w + f_2(u)\cos w, -f_1(u)\cos w - f_2(u)\sin w). \tag{3.13b}$$

Hence, one gets

$$\Omega_u \times \Omega_w = \begin{pmatrix} 0 & i_2 & i_3 \\ 1 & \dot{f}_1(u)\cos w + \dot{f}_2(u)\sin w & \dot{f}_2(u)\cos w - \dot{f}_1(u)\sin w \\ 0 & f_2(u)\cos w - f_1(u)\sin w & -f_1(u)\cos w - f_2(u)\sin w \end{pmatrix}. \tag{3.14}$$

Finally, one has

$$\Omega_u \times \Omega_w = (0, f_1(u)\cos w + f_2(u)\sin w, -f_1(u)\sin w + f_2(u)\cos w). \tag{3.15}$$

Therefore, one gets

$$\|\Omega_u \times \Omega_w\| = \sqrt{f_1^2(u) + f_2^2(u)} = \widehat{W}. \tag{3.16}$$

In this case, the isotropic unit normal vector field is obtained as

$$\eta = \frac{\Omega_u \times \Omega_w}{\|\Omega_u \times \Omega_w\|} = \frac{1}{\sqrt{f_1^2 + f_2^2}} \begin{pmatrix} 0, \\ f_1(u)\cos w + f_2(u)\sin w, \\ -f_1(u)\sin w + f_2(u)\cos w \end{pmatrix}. \tag{3.17}$$

Hence, to find the first fundamental form, using the following equations:

$$x(u,w) = u,\ x_u = 1 = g_1,\ x_v = 0 = g_2,\ g_{11} = 1,\ g_{12} = 0,\ g_{22} = 0, \tag{3.18a}$$
$$g^1 = \frac{x_{,w}}{\widehat{W}} = 0,\ g^2 = \frac{x_{,u}}{\widehat{W}} = \frac{1}{\sqrt{f_1^2 + f_2^2}} \tag{3.18b}$$

and

$$h_{11} = \langle \Omega_u, \Omega_u \rangle = 1,\ h_{12} = \langle \Omega_u, \Omega_w \rangle = 0,\ h_{22} = \langle \Omega_w, \Omega_w \rangle = f_1^2 + f_2^2. \tag{3.19}$$

We get

$$ds^2 = du^2 + \varepsilon(du^2 + (f_1^2 + f_2^2)dw^2). \tag{3.20}$$

Also, the first fundamental form according to isotropic vectors is written as

$$I = ds^2 = 2du^2 + (f_1^2 + f_2^2)dw^2. \tag{3.21}$$

Also,

$$\Omega_{ww} = (0, -f_1(u)\cos w - f_2(u)\sin w, f_1(u)\sin w - f_2(u)\cos w), \tag{3.22a}$$
$$\Omega_{uu} = (0, \ddot{f}_1(u)\cos w + \ddot{f}_2(u)\sin w, -\ddot{f}_1(u)\sin w + \ddot{f}_2(u)\cos w), \tag{3.22b}$$
$$\Omega_{uw} = (0, -\dot{f}_1(u)\sin w + \dot{f}_2(u)\cos w, -\dot{f}_1(u)\cos w - \dot{f}_2(u)\sin w). \tag{3.22c}$$

Components of the second fundamental form with the help of the equations in (3.22), one gets

$$L_{11} = \langle \Omega_{uu}, \eta \rangle = \frac{1}{\sqrt{f_1^2(u) + f_2^2(u)}}(f_1(u)f_1''(u) + f_2(u)f_2''(u)), \tag{3.23}$$

$$L_{12} = \langle \Omega_{uw}, \eta \rangle = \frac{1}{\sqrt{f^2(u) + f_2^2(u)}}(\dot{f}_2(u)f_1(u) - \dot{f}_1(u)f_2(u)), \tag{3.24a}$$
$$L_{22} = \langle \Omega_{ww}, \eta \rangle = -\frac{f_1^2(u) + f_2^2(u)}{\sqrt{f_1^2(u) + f_2^2(u)}} = -\sqrt{f_1^2(u) + f_2^2(u)}. \tag{3.24b}$$

Thus, the Gaussian curvature and mean curvature of the rotational surface are obtained by

$$K = -\frac{(f_1(u)\ddot{f}_1(u) + f_2(u)\ddot{f}_2(u)) + (\dot{f}_2(u)f_1(u) - \dot{f}_1(u)f_2(u))^2}{f_1^2(u) + f_2^2(u)}, \tag{3.25a}$$

$$H = f_1^2(u) + f_2^2(u), \tag{3.25b}$$

respectively. Moreover, for the condition that this rotational surface is a minimal surface, and

$$f_1^2(u) + f_2^2(u) = 0 \implies f_1(u) = 0 \text{ and } f_2(u) = 0. \tag{3.26}$$

The previous equation is satisfied. So that, $f_1^2(u) + f_2^2(u) \neq 0$, hence there is no rotational surface generated by ξ-axis that is minimal in G_3. Also,

$$\Omega_w = (0, -f_1(u)\sin w + f_2(u)\cos w, -f_1(u)\cos w - f_2(u)\sin w), \tag{3.27a}$$

$$\Omega_u = (1, \dot{f}_1(u)\cos w + \dot{f}_2(u)\sin w, -\dot{f}_1(u)\sin w + \dot{f}_2(u)\cos w). \tag{3.27b}$$

From the above equations, we can write that Ω_w is the vector field along the parallels, while Ω_u is the vector field along the meridians. Thus, as in Euclidean space, we can talk about an orthonormal basis, since $g(\Omega_u, \Omega_w) = 0$. Therefore, let T be the tangent vector on the surface and φ is the angle between T and Ω_u, and hence we can write

$$T = \Omega_u \cos\varphi + \Omega_w \sin\varphi. \tag{3.28}$$

Moreover, let us consider the Lagrangian equation $L = 2du^2 + (f_1^2 + f_2^2)dw^2$ with the first fundamental form and

$$\frac{d}{ds}\left(\frac{\partial L}{\frac{\partial u}{\partial s}}\right) = \frac{\partial L}{\partial u}, \quad \frac{d}{ds}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w}. \tag{3.29}$$

Let us try to examine the geodesics with the help of Lagrangian equations. In this case, for a geodesic curve ζ on the surface, one writes

$$\dot{\zeta} = \dot{u}\Omega_u + \dot{w}\Omega_w = \Omega_u \cos\varphi + \Omega_w \sin\varphi, \tag{3.30}$$

where the φ is the angle between $\frac{d\zeta}{ds}$ and Ω_u and with the help of the equation $L = 2\dot{u}^2 + (f_1^2(u) + f_2^2(u))\dot{w}^2$ given in (3.21) and hence for Lagrangian equations, we can say that

(1): For $\frac{d}{ds}\left(\frac{\partial L}{\frac{\partial u}{\partial s}}\right) = \frac{\partial L}{\partial u}$, one gets

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\partial L}{\frac{\partial u}{\partial s}}\right) = \frac{\partial L}{\partial u} &\implies \frac{d}{ds}(4\dot{u}) = 2(f_1\dot{f}_1 + \dot{f}_2 f_2)\dot{w}^2 \\
&\implies 2\dot{u} = \left(f_1\frac{df_1}{du} + f_2\frac{df_2}{du}\right)\dot{w}^2 ds \\
&\implies \dot{u}^2 = (f_1 df_1 + f_2 df_2)\dot{w}^2 \\
&\implies u = \int \sqrt{f_1 df_1 + f_2 df_2} dw + c_7.
\end{aligned} \tag{3.31}$$

(2): For $\frac{d}{ds}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w}$, one gets

$$\begin{aligned}
\frac{d}{ds}\left(\frac{\partial L}{\frac{\partial w}{\partial s}}\right) = \frac{\partial L}{\partial w} &\implies \frac{d}{ds}(2(f_1^2(u) + f_2^2(u))\dot{w}) = 0 \\
&\implies \dot{w} = \frac{c_3}{f_1^2(u) + f_2^2(u)}
\end{aligned}$$

$$\Longrightarrow w = c_3 \int \frac{1}{f_1^2(u)+f_2^2(u)} ds + c_4. \tag{3.32}$$

Hence, we can write

$$\frac{d\zeta}{ds} = \frac{du}{ds}\Omega_u + \frac{dw}{ds}\Omega_w = \frac{du}{ds}\cos\varphi + \frac{dw}{ds}\sin\varphi, \tag{3.33a}$$

$$\dot{\zeta} = \dot{u}\Omega_u + \dot{w}\Omega_w = \Omega_u\cos\varphi + \Omega_w\sin\varphi \tag{3.33b}$$

and from these equations and considering $2(f_1^2+f_2^2)\dot{w} = c_2 = \text{constant}$, $\sin\varphi = \dot{w}$. Therefore, if necessary arrangements are made, one has

$$2(f_1^2+f_2^2)\sin\varphi = 2(f_1^2+f_2^2)\dot{w} = \text{constant}. \tag{3.34}$$

Also, from the equation $\sin\varphi = \dot{w}$, one obtains

$$\sin\varphi = \dot{w} \quad \Longrightarrow \quad w = \int \sin\varphi ds + c_8. \tag{3.35}$$

Conversely, let ζ be a curve $2(f_1^2+f_2^2)\sin\varphi = 2(f_1^2+f_2^2)\dot{w} = \text{constant}$. Thus, the first Lagrangian equation is obtained by taking the derivative and the constant expression in the Lagrangian equation.

In summary, pne can give the following theorems and conclusions as a result of our application of Clairaut's theorem on the rotation surface $\Omega \in G_3$ that one obtainsed according to the ξ-axis in G_3.

Theorem 5. *Let ζ be a geodesic curve on rotational surface $\Omega \in G_3$ obtained using the ξ-axis in the G_3, and the distance from the axis of rotation to a point on the surface is $\sqrt{f_1^2(u)+f_2^2(u)}$. In addition, the equality $2(f_1^2(u)+f_2^2(u))\sin\varphi$ is constant along ζ, with the angle φ between the meridians of the surface and the geodesic ζ. Conversely, if $2(f_1^2(u)+f_2^2(u))\sin\varphi = \text{constant}$ along ζ, this curve becomes geodesic.*

Conclusion 1. Clairaut's theorem is expressed on the rotational surface given in Galilean 3-space, and it is shown that the expression $2(f_1^2(u)+f_2^2(u))\sin\varphi$ is constant along the geodesic. Thus, all meridians on the rotational surface become geodesic.

Conclusion 2. The first fundamental form of the rotational surface $\Omega \in G_3$ formed according to the ξ-axis in G_3 is given by

$$ds^2 = 2du^2 + (f_1^2(u)+f_2^2(u))dw^2.$$

In this case, the Gaussian curvature and mean curvature of the surface are given by

$$K = -\frac{(f_1(u)\ddot{f}_1(u)+f_2(u)\ddot{f}_2(u)) + (\dot{f}_2(u)f_1(u)-\dot{f}_1(u)f_2(u))^2}{f_1^2(u)+f_2^2(u)},$$

$$H = f_1^2(u)+f_2^2(u).$$

Conclusion 3. Since $f_1^2(u)+f_2^2(u) \neq 0$, there is no minimal rotational surface generated by ξ-axis in G_3.

Theorem 6. *The general equation of geodesics on the rotational surface Ω generated by ξ-axis in G_3, and for the parameters $u = \int \sqrt{f_1 df_1 + f_2 df_2} dw + c_7$, $w = c_3 \int \frac{1}{f_1^2(u)+f_2^2(u)} ds + c_4$*

(or $w = \int \sin\varphi ds + c_8$), are given by

$$\frac{du}{dw} = \frac{1}{\sqrt{2}\sin\xi}\sqrt{L-(f_1^2+f_2^2)\sin\varphi} \tag{3.36}$$

or

$$\frac{du}{dw} = \sqrt{c_9(f_1^2+f_2^2)^2L - \frac{(f_1^2+f_2^2)}{2}}. \tag{3.37}$$

Proof. One will use Lagrangian equations to find the general equation of geodesics on the rotational surface generated by ξ-axis in G_3. Accordingly, for the parameters u,w given as

$$u = \frac{1}{\sqrt{2}}\int\sqrt{f_1df_1+f_2df_2}\,dw + c_7, \tag{3.38a}$$

$$w = c_3\int\frac{1}{f_1^2+f_2^2}ds + c_4 \quad \left(\text{or}\ \ w = \int\sin\varphi ds + c_8\right) \tag{3.38b}$$

and which is equivalent to the parameter value w and hence we can write the following equation

$$\dot{w} = \frac{dw}{ds} = \frac{c_3}{f_1^2+f_2^2} \quad (\text{or}\ \ \dot{w} = \sin\varphi) \tag{3.39}$$

and considering (3.39) in equation $L = 2du^2+(f_1^2+f_2^2)dw^2$. Therefore, one writes

$$L = 2du^2+(f_1^2+f_2^2)dw^2 \implies L = 2(\dot{u})^2+(f_1^2+f_2^2)\dot{w}^2$$

$$= 2\left(\frac{du}{dw}\frac{dw}{ds}\right)^2+(f_1^2+f_2^2)\left(\frac{dw}{ds}\right)^2,$$

$$\left(\frac{du}{dw}\right)^2 = \frac{L-(f_1^2+f_2^2)\left(\frac{dw}{ds}\right)^2}{2\left(\frac{dw}{ds}\right)^2}. \tag{3.40}$$

Thus, if the values $\dot{w} = \frac{c_3}{f_1^2+f_2^2}$ and $\dot{w} = \sin\varphi$ are added in the last equation, respectively, one gets

$$\frac{du}{dw} = \sqrt{c_4(f_1^2+f_2^2)^2L - \frac{f_1^2+f_2^2}{2}} \quad \text{or} \quad \frac{du}{dw} = \frac{\sqrt{L-(f_1^2+f_2^2)(\sin\varphi)^2}}{\sqrt{2}\sin\varphi}. \tag{3.41}$$

□

Theorem 7. *The rotational surface Ω generated by the ξ-axis in G_3 is also a $\Phi(K,H)$-Weingarten surfaces.*

Proof. From the Weingarten surface definition, taking derivative of the rotational surface Ω according to the parameters u and w in the equations of Gaussian and mean curvatures, one gets

$$K_u = \frac{-\begin{pmatrix}(\dot{f}_1\ddot{f}_1+f_1\dddot{f}_1+\dot{f}_2\ddot{f}_2+f_2\dddot{f}_2+2(\dot{f}_2f_1-\dot{f}_1f_2)(\ddot{f}_2f_1-\ddot{f}_1f_2))(f_1^2+f_2^2)\\ +(2\dot{f}_1f_1+2\dot{f}_2f_2)((f_1\ddot{f}+f_2\ddot{f}_2)+(\dot{f}_2f_1-\dot{f}_1f_2)^2)\end{pmatrix}}{(f_1^2+f_2^2)^2}, \quad K_w = 0, \tag{3.42}$$

$$H_u = 2\dot{f}_1f_1+2\dot{f}_2f_2, \quad H_w = 0. \tag{3.43}$$

Therefore, the Jacobian equation is obtained from the definition of $\Phi(K,H)$-Weingarten surface. So that using (3.42) and (3.43) equations, with the help of equality $\frac{\partial(K,H)}{\partial(u,w)} = 0$ and $H_w = 0$,

$K_w = 0$, one writes

$$\frac{\partial(K,H)}{\partial(u,w)} = K_u H_w - K_w H_u = 0. \tag{3.44}$$

Thus, the rotational surface given in G_3 also becomes a $\Phi(K,H)$-Weingarten surface. □

Theorem 8. *For $\Phi(K,H)$-Weingarten rotational surface generated by the ξ-axis in G_3 and providing the equation $a_1K + a_2H = a_3$. Then, the following equation is satisfied*

$$\frac{a_3}{a_2} = f_1^2(u) + f_2^2(u), \quad a_1 = 0. \tag{3.45}$$

Thus, since the rotational surface given in G_3 is a linear Weingarten surface, the surface can be reduced to a surface with constant mean curvature. Also, if $a_1 \neq 0$, the following equality is satisfied

$$f_1(u)\ddot{f}_1(u) + f_2(u)\ddot{f}_2(u) + (\dot{f}_2(u)f_1(u) - \dot{f}_1(u)f_2(u))^2 = 0. \tag{3.46}$$

Proof. From the definition of Weingarten surface, for the surface Ω the equation

$$a_1K + a_2H = a_3 \tag{3.47}$$

can be written. In this case, if the expressions K and H are written instead of equation (3.47), one gets

$$\begin{aligned} 0 = &-a_1((f_1(u)\ddot{f}_1(u) + f_2(u)\ddot{f}_2(u)) + (\dot{f}_2(u)f_1(u) - \dot{f}_1(u)f_2(u))^2) \\ &+ (f_1^2(u) + f_2^2(u))(a_2(f_1^2(u) + f_2^2(u)) - a_3). \end{aligned} \tag{3.48}$$

In this last equation, using linear independence for vectors, one can write

$$-a_1((f_1(u)\ddot{f}_1(u) + f_2(u)\ddot{f}_2(u)) + (\dot{f}_2(u)f_1(u) - \dot{f}_1(u)f_2(u))^2) = 0, \tag{3.49}$$

$$(f_1^2(u) + f_2^2(u))(a_2(f_1^2(u) + f_2^2(u)) - a_3) = 0, \tag{3.50}$$

Here one can algebraically talk about two different situations:

(1): If $(f_1(u)\ddot{f}_1(u) + f_2(u)\ddot{f}_2(u)) + (\dot{f}_2(u)f_1(u) - \dot{f}_1(u)f_2(u))^2 \neq 0$ and $f_1^2(u) + f_2^2(u) \neq 0$, one gets

$$a_1 = 0; \ \ (a_2(f_1^2(u) + f_2^2(u)) - a_3) = 0. \tag{3.51}$$

So, from the Weingarten surface definition, since $a_1 = 0$, the linear Weingarten surface is reduced to a surface with constant mean curvature. Finally, for the linear Weingarten surface, the following equality is satisfied

$$(a_2(f_1^2(u) + f_2^2(u)) - a_3) = 0 \implies \frac{a_3}{a_2} = f_1^2(u) + f_2^2(u). \tag{3.52}$$

(2): If $a_1 \neq 0$ and $f_1^2(u) + f_2^2(u) \neq 0$, one writes

$$f_1(u)\ddot{f}_1(u) + f_2(u)\ddot{f}_2(u) + (\dot{f}_2(u)f_1(u) - \dot{f}_1(u)f_2(u))^2 = 0.$$

□

Theorem 9. *If the equality $a_2^2 + a_1a_3 = 0$ is satisfied for the rotational surface Ω formed by the ξ-axis in G_3, then the surface becomes HK-quadric surface.*

Proof. Let the rotational surface Ω be HK-quadric surface. From HK-quadric surface definition, and the equation

$$a_1H^2 + 2a_2HK + a_3K^2 = \text{constant}, \quad a_1 \neq 0 \tag{3.53}$$

is provided. If the values K and H are written instead of this last equation, one gets

$$2a_1HH_u + 2a_2(HK_u + H_uK) + 2a_3KK_u = 0,$$
$$(a_1H_u + a_2K_u)H + (a_3K_u + a_2H_u)K = 0. \tag{3.54}$$

From the last equation and since $H, K \neq 0$, one writes

$$a_1H_u + a_2K_u = 0, \quad a_3K_u + a_2H_u = 0 \tag{3.55}$$

and from here, one gets

$$\frac{a_1}{a_2} = -\frac{K_u}{H_u} \quad \text{and} \quad \frac{a_3}{a_2} = \frac{H_u}{K_u} \quad \left(\text{or } \frac{a_2}{a_3} = \frac{K_u}{H_u}\right). \tag{3.56}$$

In addition, also from two inequalities given in (3.56), one obtains

$$-\frac{a_1}{a_2} = \frac{a_2}{a_3} \quad \Longrightarrow \quad a_2^2 + a_1a_3 = 0. \tag{3.57}$$

□

Theorem 10. *The rotational surface Ω generated by the ξ-axis in G_3 is a harmonic $\Longleftrightarrow$ the following expression is provided*

$$w = \arctan\left(\frac{f_1 - \ddot{f}_1}{\ddot{f}_2 - f_2}\right), \quad w = \frac{\text{arccos}}{\text{arcsin}}\left(\frac{\ddot{f}_1 - f_1}{\ddot{f}_2 - f_2}\right),$$
$$\ddot{f}_2^2 + \ddot{f}_1^2 + f_2^2 + f_1^2 = 2f_1\ddot{f}_1 + 2f_2\ddot{f}_2.$$

Proof. Let the rotational surface Ω be formed by the ξ-axis in G_3. Also, in order for its Ω^i, $i = 1,2,3$ coordinates functions to be harmonic, it is necessary to provide $\Delta\Omega^i = 0$ equality from the definition. So, by making the necessary calculations in the following expression, one gets

$$\Omega(u,w) = (u, f_1(u)\cos w + f_2(u)\sin w, -f_1(u)\sin w + f_2(u)\cos w)$$
$$= (\Omega^1, \Omega^2, \Omega^3),$$

making necessary calculations in previous equations, one writes

$$\Delta\Omega_u^1 = \frac{\partial^2\Omega^1}{\partial u^2} = 0, \quad \Delta\Omega_w^1 = \frac{\partial^2\Omega^1}{\partial w^2} = 0, \quad \Delta\Omega^1 = 0,$$
$$\Delta\Omega_u^2 = \frac{\partial^2\Omega^2}{\partial u^2} = \ddot{f}_1\cos w + \ddot{f}_2\sin w, \quad \Delta\Omega_w^2 = \frac{\partial^2\Omega^2}{\partial w^2} = -f_1\cos w - f_2\sin w,$$
$$\Delta\Omega^2 = (\ddot{f}_1 - f_1)\cos w + (\ddot{f}_2 - f_2)\sin w = 0, \tag{3.58}$$
$$\Delta\Omega_u^3 = \frac{\partial^2\Omega^3}{\partial u^2} = -\ddot{f}_1\sin w + \ddot{f}_2\cos w, \quad \Delta\Omega_w^3 = \frac{\partial^2\Omega^3}{\partial w^2} = f_1\sin w - f_2\cos w, \tag{3.59}$$
$$\Delta\Omega^3 = (-\ddot{f}_1 + f_1)\sin w + (\ddot{f}_2 - f_2)\cos w = 0. \tag{3.60}$$

Therefore, from equations (3.58) and (3.60) respectively, one obtains

$$\tan w = \frac{f_1 - \ddot{f}_1}{\ddot{f}_2 - f_2} \quad \text{or} \quad \cot w = \frac{\ddot{f}_1 - f_1}{\ddot{f}_2 - f_2}.$$

Hence, considering equality of two expressions in the last equation, one gets

$$\frac{f_1 - \ddot{f}_1}{\ddot{f}_2 - f_2} = \frac{\ddot{f}_2 - f_2}{\ddot{f}_1 - f_1} \Longrightarrow \ddot{f}_2^2 + \ddot{f}_1^2 + f_2^2 + f_1^2 = 2f_1\ddot{f}_1 + 2f_2\ddot{f}_2.$$

□

4. Conclusion

In this study, using rotation matrix which is the subgroups of the manifold M corresponding to rotation about a chosen axis in E^3, some certain results of describing the surface are presented in detail obtaining Killing vector field in G_3 and a theorem according to the rotation matrix corresponding to the appropriate subgroup of the Galilean space is given and a theorem about rotational surface generated by the rotation matrix in G_3 is also given using a curve and matrix η_1 in G_3. Moreover, using the Gaussian and mean curvatures of rotational surface, the conditions being linear Weingarten surfaces and HK-quadric surface, harmonic surface are tried to express. The authors are currently working on the properties of the rotational surfaces with a view to devising suitable matrix in different spaces by adapting the type of conservation laws considered in the paper.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

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