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A New Operational Method for Solving Nonlinear Volterra Integro-differential Equations with Fractional Order *

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Abstract. In this paper we introduce a new operational method, which is based on the generalized Taylor formula and Caputo fractional derivative. We employ this method to solve a fractional Volterra integro-differential equation with supplementary conditions. Some examples are included to demonstrate the validity and applicability of the technique. Numerical results confirm the efficiency of the method.

1. Introduction

The aim of this work is to present an operational method for approximating the solution of a nonlinear fractional integro-differential equation of the second kind:

$$D_*^\beta f(x) + h(x)f(x) - \lambda \int_0^x k(x,t)F(f(t))\,\mathrm{dt} = g(x), \quad 0 \le t \le x \le X, \quad X \in I = [0,a], \tag{1.1}$$

with these supplementary conditions:

$$\frac{d^i f(0)}{dx^i} = \xi_i, \quad \frac{d^i f(a)}{dx^i} = \eta_i, \quad i = 0,\dots,m, \tag{1.2}$$

where the function f is unknown, the functions $h, g : I \to \mathbb{R}$ and $k : S \to \mathbb{R}$ (with $S = \{(x,t) : 0 \le t \le x \le a\}$) are given analytical functions and D_*^β denotes the fractional differential operator of order $\beta \notin \mathbb{N}$ in the sense of Caputo and $F(f(x))$ is a polynomial of $f(x)$ with constant coefficients. For convenience, we assume that $F(f(x)) = [f(x)]^q$ where $1 \le q \in \mathbb{N}$. This kind of equations arises in the mathematical modeling of various physical phenomena, such as heat conduction

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in materials. Moreover, these equations are encountered in combined conduction, convection and radiation problems [5, 10, 18]. Local and global existence and uniqueness solution of the integro-differential equation given by (1.1) and (1.2) is given in [11]. In recent years, fractional integro-differential equations have been investigated by many authors [20, 3, 14, 2, 13, 4]. Most of the methods have been utilized in linear problems and a few number of works have considered nonlinear problems.

In this paper, we introduce a new operational method to solve nonlinear Volterra integro-differential equations of fractional order. The method is based on reducing the equation to a system of algebraic equations by expanding the solution as the generalized Taylor formula with unknown coefficients. The main characteristic of our operational method is to convert a fractional Volterra integro-differential equation into an algebraic one. This method not only simplifies the problem but also speeds up the processes. It is considerable that, for $\beta \in \mathbb{N}$, Eqs. (1.1) and (1.2) are ordinary Volterra integro-differential equations and the method can be easily applied for them.

2. **Preliminaries**

2.1. *Fractional Calculus*

There are several definitions of a fractional derivative of order $\alpha > 0$. The two most commonly used definitions are the Riemann-Liouville and Caputo. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. First let us define the following useful funcion spaces:

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Riemann-Liouville fractional integration of order α is defined as:

Definition 2.2. Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f(x) \in C_\mu$, $\mu \geq -1$, with $x \in \mathbb{R}^+$, is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(x)\,\mathrm{dt}, \quad x > 0,\ J^0 f(x) = f(x), \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function which has this property: $\Gamma(x+1) = x\Gamma(x)$, $x > 0$.

Properties of the operator J^α can be found in [19]. We mention only the following:
For $f(x) \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, $\gamma \geq -1$:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \quad J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \quad J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Here we introduce Caputo fractional differential operator, D_*^β, which is proposed by Caputo [19]:

Definition 2.3. Fractional derivative of $f(x)$ for $f \in C_{-1}^\mu$, in the Caputo sense is defined by $D_*^\beta f(x) = J^{m-\beta} D^m f(x)$, where D^m is usual integer differential operator of order m and J^α is the Riemann-Liouville integral operator of order $\alpha > 0$ and $m-1 < \beta \leq m$.

It is clear that:

$$D_*^\beta f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\beta+1-m}} \mathrm{dt}, & m-1 < \beta < m; \\ \dfrac{d^m f(x)}{dx^m}, & \beta = m. \end{cases} \tag{2.2}$$

The relation between the Riemann-Liouville operator and the Caputo operator is given by the following lemma:

Lemma 2.1. *If $m-1 < \beta \leq m$, $m \in \mathbb{N}$ and $f \in C_m^\mu$, $\mu \geq -1$, then $D_*^\beta J^\beta f(x) = f(x)$, and*

$$J^\beta D_*^\beta f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{2.3}$$

Also Caputo's fractional differentiation is a linear operator, that is:

$$D_*^\beta [af(x) + bg(x)] = aD_*^\beta f(x) + bD_*^\beta g(x),$$

where a and b are two constants.

Proof. The proof is a straight forward of definitions. □

2.2. *Generalized Taylor Formula*

Theorem 2.1 (*Generalized Taylor formula* [17])**.** *Suppose $(D_*^\alpha)^i f(x) \in C[a,b]$ for $i = 0, 1, \ldots, N+1$, and $0 < \alpha \leq 1$, then:*

$$f(x) = \sum_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} [(D_*^\alpha)^i f(x)]_{x=a} + R_N^\alpha(x,a), \tag{2.4}$$

where

$$R_n^\alpha(x,a) = \frac{(x-a)^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} [(D_*^\alpha)^{N+1} f(x)]_{x=\xi}, \quad \xi \in [a,x], \ \forall\, x \in (a,b), \tag{2.5}$$

$$(D_*^\alpha)^i = \underbrace{D_*^\alpha . D_*^\alpha \ldots D_*^\alpha}_{i \text{ times}}. \tag{2.6}$$

According to the generalized Taylor formula let us expand the analytical and continuous function $f(x)$ in terms of a fractional power series as follows [12]:

$$f(x) = \sum_{i=0}^{\infty} F_\alpha(i)(x-a)^{i\alpha}. \tag{2.7}$$

In [17, 1, 12], it has been also shown that the above series can be written as:

$$(2.8) \qquad f(x)=\sum_{i=0}^{\infty}F_{\alpha_0}(i)(x-a)^{\frac{i}{\alpha_0}},$$

with $\alpha_0=\frac{1}{\alpha}$, i.e. $\alpha_0\geq 1$. In (2.8) we call $F_{\alpha_0}(i)$ as the i-th coefficient of generalized Taylor expansion, for $i=0,1,\ldots$.

Here we state the following theorem:

Theorem 2.2. *Suppose that the k-th coefficients of generalized Taylor expansion of $f(x)$, $g(x)$ and $h(x)$ are $F_\alpha(k)$, $G_\alpha(k)$ and $H_\alpha(k)$, respectively:*

(i) *If $f(x)=g(x)\pm h(x)$, then $F_\alpha(k)=G_\alpha(k)\pm H_\alpha(k)$.*

(ii) *If $f(x)=g(x)h(x)$, then $F_\alpha(k)=\sum_{l=0}^{k}G_\alpha(l)H_\alpha(k-l)$.*

(iii) *If $f(x)=D_a^\beta[g(x)]$, then $F_\alpha(k)=\frac{\Gamma(\beta+1+\frac{k}{\alpha})}{\Gamma(1+\frac{k}{\alpha})}G_\alpha(k+\alpha\beta)$.*

(iv) *If $f(x)=\int_a^x g(t)\,dt$, then $F_\alpha(k)=\alpha\frac{G_\alpha(k-\alpha)}{k}$, where $k\geq\alpha$.*

(v) *If $f(x)=g(x)\int_a^x h(t)\,dt$, then $F_\alpha(k)=\alpha\sum_{k_1=\alpha}^{k}\frac{H_\alpha(k_1-\alpha)}{k_1}G_\alpha(k-k_1)$, where $k\geq\alpha$.*

3. **Description of Method**

According to Eq. (2.8), we consider the approximate solution of Eqs (1.1) and (1.2) as:

$$(3.1) \qquad f(x)=\sum_{i=0}^{\infty}f_i x^{i/\alpha}\cong\sum_{i=0}^{N}f_i x^{i/\alpha}=\underline{\mathbf{f}}\,\underline{\mathbf{x}}=f_N(x),$$

where α is the order of fraction,

$$(3.2) \qquad \underline{\mathbf{f}}=[f_0,f_1,\ldots,f_N,0,0,\ldots],$$

and

$$\underline{\mathbf{x}}=[1,x^{1/\alpha},x^{2/\alpha},\ldots,x^{N/\alpha},\ldots].$$

Also assume that $g(x)$, in Eq. (1.1), can be written as:

$$(3.3) \qquad g(x)=\sum_{i=0}^{\infty}g_i x^{i/\alpha}\cong\sum_{i=0}^{N}g_i x^{i/\alpha}=\underline{\mathbf{g}}\,\underline{\mathbf{x}}=g_N(x),$$

It is considerable that the vectors $\underline{\mathbf{f}}$ and $\underline{\mathbf{g}}$ in (3.1) and (3.3) are the truncated generalized Taylor series of $f(x)$ and $g(x)$, respectively, and α should be chosen suitably.

Substituting Eq. (1.1) by (3.1) and (3.3), we have:

$$(3.4) \qquad D_*^\beta f_N(x)+h_N(x)f_N(x)+\lambda\int_a^x k(x,t)f_N^q(t)\,\mathrm{d}t\cong g_N(x).$$

Without loss of generality, we can assume that the kernel is separable i.e. $k(x,t) = p(x)r(t)$. The goal of this work is to convert Eq. (3.4) to a system of algebraic equations. To this end, we transform each part of Eq. (3.4) to a matrix form. By solving the obtained system, the unknown coefficients $\{f_0, f_1, \dots, f_N\}$ are calculated. Finally by these coefficients an approximate function $f_N(x)$ is obtained.

3.1. *Main Results*

In this section we will describe how to transform each part of Eq. (3.4) into a matrix representation. To this end we use the following infinity matrices:

$$\mu = \begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & 0 & \dots \\ & & 0 & 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \tag{3.5}$$

$$\gamma_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \dots & \\ 0 & 0 & 0 & \dots & \\ \frac{\Gamma(1+\beta)}{\Gamma(1)} & 0 & 0 & \dots & \\ 0 & \frac{\Gamma(1+\beta+1/\alpha)}{\Gamma(1+1/\alpha)} & 0 & \dots & \\ 0 & 0 & \frac{\Gamma(1+\beta+2/\alpha)}{\Gamma(1+2/\alpha)} & 0 & \\ 0 & 0 & 0 & \frac{\Gamma(1+\beta+3/\alpha)}{\Gamma(1+3/\alpha)} & \\ & & & & \ddots \end{pmatrix} \tag{3.6}$$

Note that $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{R}^+ \cup \{0\}$ should be chosen in such a way that $\alpha\beta \in \mathbb{N} \cup \{0\}$ and the $\alpha\beta$-th first rows of matrix $\gamma_{\alpha\beta}$ are zero.

Definition 3.1. Corresponding to each function $f_N(x)$ of the form (3.1), we define the comrade polynomial of $f_N(x)$, as the following:

$$F_N(x) = \sum_{i=0}^{N} f_i x^i. \tag{3.7}$$

Lemma 3.1. *If* $f_N(x) = \sum_{i=0}^{N} f_i x^{i/\alpha} = \underline{\boldsymbol{f}}\,\underline{\boldsymbol{x}}$ *and* $h_N(x) = \sum_{i=0}^{N} h_i x^{i/\alpha} = \underline{\boldsymbol{h}}\,\underline{\boldsymbol{x}}$, *then:*

$$f_N(x)h_N(x) = \underline{\boldsymbol{f}}\,H(\mu)\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{h}}\,F(\mu)\,\underline{\boldsymbol{x}}\,, \tag{3.8}$$

where $F(t)$ and $H(t)$ are the comrade polynomials of $f(x)$ and $h(x)$, respectively.

Proof.

$$\begin{aligned} f_N(x)h_N(x) &= f_0h_0 + f_0h_1x^{1/\alpha} + f_0h_2x^{2/\alpha} + f_0h_3x^{3/\alpha} + \dots \\ &= f_1h_0x^{1/\alpha} + f_1h_1x^{2/\alpha} + f_1h_3x^{3/\alpha} + \dots \end{aligned}$$

$$
\begin{aligned}
&= f_2 h_0 x^{2/\alpha} + f_2 h_1 x^{3/\alpha} + \dots \\
&= f_0[h_0\ h_1\ \dots\ h_N]\underline{\mathbf{x}} + f_1[0\ h_0\ h_1\ \dots\ h_N]\underline{\mathbf{x}} + \dots \\
&= [f_0\ f_1\ \dots\ f_N]\begin{pmatrix} h_0 & h_1 & h_2 & \dots & h_N \\ 0 & h_0 & h_1 & \dots & h_N \\ 0 & 0 & h_0 & \dots & h_N \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}\underline{\mathbf{x}} \\
&= \underline{\mathbf{f}}(h_0 I + h_1\mu + h_2\mu^2 + \dots + h_N\mu^N)\underline{\mathbf{x}} \\
&= \underline{\mathbf{f}}\, H(\mu)\, \underline{\mathbf{x}}\,.
\end{aligned}
$$

The next equality is resulted similarly, by replacing $f_N(x)$ and $h_N(x)$. ☐

Lemma 3.2. *If $f_N(x) = \sum_{i=0}^{N} f_i x^{i/\alpha} = \underline{f}\,\underline{x}$, then:*

$$D_\alpha^\beta f_N(x) = \underline{f}\,\gamma_{\alpha\beta}\,\underline{\mathbf{x}}\,. \tag{3.9}$$

Proof. By parts of Theorem 2.2 we have:

$$
\begin{aligned}
D_\alpha^\beta f_N(x) &= \sum_{k=0}^{N+\alpha\beta} \frac{\Gamma(\beta+1+k/\alpha)}{\Gamma(1+k/\alpha)} f_{k+\alpha\beta}\ x^{k/\alpha} \\
&= \underline{\mathbf{f}}[\underbrace{0\ \dots\ 0}_{\alpha\beta\ \text{times}}\ \frac{\Gamma(1+\beta)}{\Gamma(1)}\ 0\ \dots]^T\ \underline{\mathbf{x}} \\
&\quad + \underline{\mathbf{f}}[\underbrace{0\ \dots\ 0}_{\alpha\beta+1\ \text{times}}\ \frac{\Gamma(1+\beta+1/\alpha)}{\Gamma(1+1/\alpha)}\ 0\ \dots]^T\ \underline{\mathbf{x}} + \dots \\
&= \underline{\mathbf{f}}\begin{pmatrix} 0 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \dots & \\ 0 & 0 & 0 & \dots & \\ \frac{\Gamma(1+\beta)}{\Gamma(1)} & 0 & 0 & \dots & \\ 0 & \frac{\Gamma(1+\beta+1/\alpha)}{\Gamma(1+1/\alpha)} & 0 & \dots & \\ 0 & 0 & \frac{\Gamma(1+\beta+2/\alpha)}{\Gamma(1+2/\alpha)} & 0 & \\ 0 & 0 & 0 & \frac{\Gamma(1+\beta+3/\alpha)}{\Gamma(1+3/\alpha)} & \\ & & & & \ddots \end{pmatrix}.\underline{\mathbf{x}} \\
&= \underline{\mathbf{f}}\,\gamma_{\alpha\beta}\,\underline{\mathbf{x}}\,.
\end{aligned}
$$
☐

Lemma 3.3. *If $f_N(x) = \sum_{i=0}^{N} f_i x^{i/\alpha} = \underline{f}\,\underline{x}$ and $r \in \mathbb{N}$, then:*

$$f_N^q(x) = \underline{f}\, F^{\,q-1}(\mu)\, \underline{x}\,. \tag{3.10}$$

Proof. The proof follows immediately by induction and Lemma 3.1. ☐

Lemma 3.4. *Suppose that $f_N(x) = \underline{f}\,\underline{\mathbf{x}}$ and the functions $p(x)$ and $r(x)$ can be shown as follows:*

$$p(x) = \sum_{k=0}^{\infty} p_k x^{k/\alpha}, \quad r(x) = \sum_{k=0}^{\infty} h_k x^{k/\alpha}, \tag{3.11}$$

then:

$$\int_0^x p(x) r(t) f(t)\, dt = \alpha\, \underline{f}\, M_\alpha\, \underline{\mathbf{x}}, \tag{3.12}$$

where the infinity matrix M_α is as the following:

$$M_\alpha = \begin{pmatrix} 0 & \dots & 0 & m_{0,0} & m_{1,0} & m_{2,0} & m_{3,0} & \dots \\ 0 & \dots & 0 & 0 & m_{0,1} & m_{1,1} & m_{2,1} & \dots \\ 0 & \dots & 0 & 0 & 0 & m_{0,2} & m_{1,2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix}, \tag{3.13}$$

such that:

$$m_{i,j} = \sum_{l=0}^{i} \frac{r_l p_{i-l}}{\alpha + l + j}, \quad \textit{for } i, j = 0, 1, 2, \dots . \tag{3.14}$$

Proof. By using Theorem 2.2 we have:

$$\begin{aligned} \int_0^x p(x) r(t) f(t)\, \mathrm{d}t &= \sum_{k=\alpha}^{\infty} \frac{\alpha}{k} \sum_{k_1=\alpha}^{k} \sum_{k_2=0}^{k_1-\alpha} f_{k_2}\, r_{k_1-k_2-\alpha}\, p_{k-k_1} x^{k/\alpha} \\ &= \alpha \Big[f_0 \begin{pmatrix} 0 & \dots & 0 & m_{0,0} & m_{1,0} & m_{2,0} & m_{3,0} & \dots \end{pmatrix} \underline{\mathbf{x}} \\ &\quad + f_1 \begin{pmatrix} 0 & \dots & 0 & 0 & m_{0,1} & m_{1,1} & m_{2,1} & \dots \end{pmatrix} \underline{\mathbf{x}} \\ &\quad + f_2 \begin{pmatrix} 0 & \dots & 0 & 0 & 0 & m_{0,2} & m_{1,2} & \dots \end{pmatrix} \underline{\mathbf{x}} + \dots \Big] \\ &= \alpha\, \underline{\mathbf{f}}\, M_\alpha\, \underline{\mathbf{x}}. \end{aligned}$$

□

Note that the first α column of matrix M_α are zero ($\alpha \in \mathbb{N}$). Now by using Lemma 3.3 and the above lemma we have the following corollary:

Corollary 3.1. *We have:*

$$\int_0^x p(x) r(t) f_N^q(t)\, dt = \alpha\, \underline{f}\, F^{\,q-1}(\mu)\, M_\alpha\, \underline{\mathbf{x}}. \tag{3.15}$$

Now, by using Lemmas 3.2 and 3.1, Corollary 3.1 and Eq. (3.3), we are able to transform the Eq. (3.1) to the following matrix representation:

$$\underline{\mathbf{f}}\, D_{\alpha\beta}\, \underline{\mathbf{x}} + \underline{\mathbf{f}}\, H(\mu)\, \underline{\mathbf{x}} - \lambda\alpha\, \underline{\mathbf{f}}\, F^{q-1}(\mu)\, M_\alpha\, \underline{\mathbf{x}} = \underline{\mathbf{g}}\, \underline{\mathbf{x}}. \tag{3.16}$$

Since $\{1, x^{1/\alpha}, x^{2/\alpha}, x^{3/\alpha}, \dots\}$ is a base vector, one can write the Eq. (3.16) as the following system of nonlinear equations:

$$\underline{\mathbf{f}}\, D_{\alpha\beta} + \underline{\mathbf{f}}\, H(\mu) - \lambda\alpha\, \underline{\mathbf{f}}\, F^{q-1}(\mu)\, M_\alpha = \underline{\mathbf{g}}. \tag{3.17}$$

Through triangularity of matrices μ and γ, this system of algebraic equations can be usually converted to a recursive formula which is solved easily. For calculating the approximate solution of this system, we need some first values of the vector solution (3.2). Since the initial conditions, Eq. (1.2), are implemented to the integer order derivatives, they can be computed as follows, see [1]:

$$f_k = \begin{cases} \dfrac{1}{(\frac{k}{\alpha})!}\left[\dfrac{d^{\frac{k}{\alpha}} f(x)}{d^{\frac{k}{\alpha}} x}\right]_{x=x_0}, & \text{if } \frac{k}{\alpha} \in \mathbb{Z}^+; \\ 0, & \text{if } \frac{k}{\alpha} \notin \mathbb{Z}^+; \end{cases} \quad \text{for } k = 0, 1, 2, \ldots, (\beta\alpha - 1), \tag{3.18}$$

where β is the fractional order of Eq. (1.1).

Remark 3.1. In the case of $q = 1$, system (3.17) is linear. Also in the case of $\beta \in \mathbb{N}$, Eq. (1.1) is an ordinary integro-differential equation and the method can be easily applied to it.

4. Error Analysis

We can easily check the accuracy of our method. Since we suppose that the truncated generalized Taylor series is an approximate solution of Eq. (1.1), when our method is used, the resulting equation, (3.16), must be satisfied approximately, that is for $x \in [0, a)$

$$R_N(x) = |\underline{\mathbf{f}}\, D_{\alpha\beta}\, \underline{\mathbf{x}} + \underline{\mathbf{f}}\, H(\mu)\, \underline{\mathbf{x}} - \lambda\alpha\, \underline{\mathbf{f}}\, F^{q-1}(\mu)\, M_\alpha\, \underline{\mathbf{x}} - \underline{\mathbf{g}}\, \underline{\mathbf{x}}| \cong 0\ , \tag{4.1}$$

If we set $x = x_i$, then our aim is to have $R_N(x_i) \le 10^{r_i}$, where r_i is any positive integer. If we prescribe, $\max\{r_i\} = 10^r$, then we increase N as long as the following inequality holds at each point x_i:

$$R_N(x_i) \le 10^r,$$

in other words, by increasing N the error function $R_N(x_i)$ approaches zero. If $R_N(x) \to 0$ when N is sufficiently large enough, then the error decreases.

Note that, the convergence of this method is such as in Taylor's series expansion.

5. Numerical Illustrations

Example 5.1. Let us consider the following fractional integro-differential equation [20]:

$$f^{(0.75)}(x) = \left(\frac{-x^2 e^x}{5}\right) f(x) + \frac{6x^{2.25}}{\Gamma(3.25)} + \int_0^x e^x t f(t)\, \mathrm{d}t, \tag{5.1}$$

with the initial condition:

$$f(0) = 0. \tag{5.2}$$

Considering the solution of (5.1) as (3.1) and selecting the order of fraction as $\alpha = 4$ and $N = 12$. By applying the method through using lemmas (3.1), (3.2) and (3.4) we have $Af' = g'$, where:

$$A = \begin{pmatrix}
0 & 0 & 0 & 0.919 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.103 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.278 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.446 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1.608 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 1.765 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.667 & 0 & 0 & 0 & 0 & 0 & 0 & -0.667 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.571 & 0 & 0 & 0 & 0 & 0 & 0 & 2.066 & 0 & 0 \\
-0.8 & 0 & 0 & 0 & -0.500 & 0 & 0 & 0 & 0 & 0 & 0 & 2.211 & 0 \\
0 & -0.6 & 0 & 0 & 0 & -0.444 & 0 & 0 & 0 & 0 & 0 & 0 & 2.353 \\
0 & 0 & -0.466 & 0 & 0 & 0 & -0.400 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.371 & 0 & 0 & 0 & -0.363 & 0 & 0 & 0 & 0 & 0 \\
-0.3 & 0 & 0 & 0 & -0.300 & 0 & 0 & 0 & -0.333 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$f' = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} & f_{11} & f_{12} \end{pmatrix}^T,$$

and

$$g' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{6}{\Gamma(3.25)} \end{pmatrix}.$$

The Eq. (3.18) and initial condition (5.2) result:

$$f_0 = 0, \quad f_1 = 0, \quad f_2 = 0\,. \tag{5.3}$$

Using (5.3) and forward substitution in system $Af' = g'$, $f_{12}(x)$ is evaluated as follows:

$$f_{12}(x) = \sum_{i=0}^{12} f_i x^{i/4} = x^3,$$

which is the exact solution of the problem.

Example 5.2. Consider the following fractional integro-differential equation [20]:

$$D^{1/2} f(x) = (\cos x - \sin x) f(x) + \int_0^x x \sin t f(t)\,\mathrm{d}t + g(x), \tag{5.4}$$

with the initial condition: $y(0) = 0$ and

$$g(t) = \frac{2}{\Gamma(2.5)} x^{1.5} + \frac{1}{\Gamma(1.5)} x^{0.5} + x(2 - 3\cos x - x\sin x + x^2 \cos x).$$

For $N = 4$ and $\alpha = 2$, the presented method gives the following system of linear equations:

$$\begin{aligned}
-f_0 + \frac{\Gamma(3/2)}{\Gamma(1)} f_1 &= 0, \\
-f1 + \frac{\Gamma(2)}{\Gamma(3/2)} f_2 &= \frac{\Gamma(2)}{\Gamma(3/2)}, \\
f_0 - f_2 + \frac{\Gamma(5/2)}{\Gamma(2)} f_3 &= -1,
\end{aligned}$$

$$f_1 - f_3 + \frac{\Gamma(3)}{\Gamma(5/2)} f_4 = \frac{2}{\Gamma(5/2)},$$
$$\frac{1}{2} f_0 + f_2 - f_4 + \frac{\Gamma(7/2)}{\Gamma(3)} f_4 = 0.$$

By using the initial condition, $y(0) = 0$, and Eq. (3.18) we have $f_0 = 0$. Hence the solution of the above system can be computed as the following:

$$f_1 = 0, \quad f_2 = 1, \quad f_3 = 0, \quad f_4 = 1.$$

Therefore we have:

$$f(x) \cong f_4(x) = x + x^2,$$

which is the exact solution of the problem.

Example 5.3. Let us consider the following fractional nonlinear integro-differential equation of order $\alpha = \frac{3}{4}$:

$$D_*^{\frac{3}{4}} f(x) - \int_0^x xt[f(t)]^4 \,\mathrm{d}t = g(x), \quad 0 \le x < 1,$$

such that

$$g(x) = \frac{1}{\Gamma(1/4)} \left(\frac{32}{5} x^{5/4} - 4x^{1/4} \right) - \frac{1}{10} x^{11} + \frac{4}{9} x^{10} - \frac{4}{3} x^9 + \frac{4}{7} x^8 - \frac{x^7}{6}$$

and

$$f(0) = f'(0) = 0.$$

Here $\beta = 3/4$, by applying the method for $N = 9$ and $\alpha = 4$ and substituting the initial values, $f_0 = f_1 = f_2 = f_3 = 0$, from (3.18) we have the following system of equations:

$$\frac{3280}{2973} f_4 + \frac{3280}{2973} = 0,$$
$$\frac{1359}{1063} f_5 = 0,$$
$$\frac{1309}{905} f_6 = 0,$$
$$\frac{1039}{646} f_7 = 0,$$
$$\frac{5451}{3088} f_8 - \frac{5451}{3088} = 0.$$

Hence $f(x) \cong f_8(x) = -x + x^2$ which is the exact solution of the equation.

Example 5.4. In the following we consider the fourth order equation [12]:

$$D_*^{\beta} f(x) - \int_0^x e^{-t} [f(t)]^2 \,\mathrm{d}t = 1, \quad 0 \le x < 1, \ 3 < \beta \le 4,$$

such that $f(0) = f'(0) = f''(0) = f'''(0) = 1$.

First we assume $\beta = 4$. In this case we choose $\alpha = 1$ and $N = 9$. By applying the above initial conditions to Eq. (3.18) we have:

$$f_0 = f_1 = 1, \quad f_2 = 1/2, \quad f_3 = 1/6.$$

Applying the method and substituting the above initial values to the obtained, system we have:

$$\begin{aligned} 24f_4 - 1 &= 0, \\ 120f_5 - 1 &= 0, \\ 260f_6 - 1/2 &= 0, \\ 840f_7 - 1/6 &= 0, \\ 1680f_8 - 1/24 &= 0, \\ 3024f_9 + 1/120 - 2/5f_4 &= 0. \end{aligned}$$

Hence:

$$f(x) \cong 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880},$$

which is the truncated Taylor series of e^x and this is the exact solution of the equation in this case. This shows the applicability and accuracy of our method in nonfractional cases.

Now, we apply our method for $\beta = 3.25$, 3.5 and $\beta = 3.75$. In these cases we choose $\alpha = 4$ and $N = 30$. Both Table 1 together with the graphical results in Figure 1, show excellent agreement with the solution of Adomian decomposition method in reference [12].

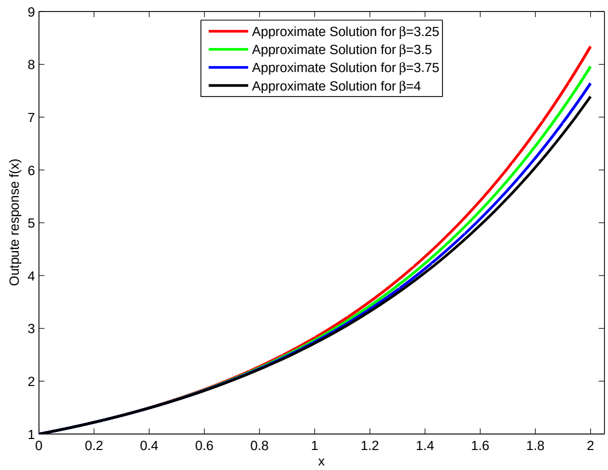

Figure 1. The Approximate solution of Example 5.4 for $N = 30$, $\alpha = 4$ and some $3 < \beta \leq 4$.

x_i	$\alpha = 3.25$		$\alpha = 3.5$		$\alpha = 3.75$	
	Ref. [12]	Our method	Ref. [12]	Our method	Ref. [12]	Our method
0	1.0	1.000000	1.0	1.000000	1.0	1.000000
0.1	1.106551	1.105236	1.106751	1.105194	1.106151	1.105178
0.2	1.223932	1.222011	1.224324	1.221655	1.223227	1.221484
0.3	1.353200	1.352092	1.353756	1.350861	1.352308	1.350204
0.4	1.495601	1.497435	1.496270	1.494480	1.494636	1.492783
0.5	1.652553	1.660168	1.653273	1.654360	1.651615	1.650830
0.6	1.825655	1.842596	1.826354	1.832532	1.824824	1.826127
0.7	2.016687	2.047215	2.017295	2.031222	2.016024	2.020638
0.8	2.227634	2.276721	2.228084	2.252860	2.227176	2.236527
0.9	2.460691	2.534030	2.460931	2.500096	2.460459	2.476173

Table 1. Numerical results for Example 5.4 with comparison to Ref. [12].

6. **Conclusion**

In this work we derive matrix representations of each parts of a class of nonlinear Volterra integro-differential equation of fractional (arbitrary) order when the approximate solution was expanded in the generalized Taylor formula and we use them to solve the problem. Several examples are given to demonstrate the powerfulness of the proposed method. The solution is convergent, even though the size of increment may be large. Also this method can be used to obtain the numerical solutions of ordinary nonlinear integro-differential equations.

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