Journal of Informatics and Mathematical Sciences
Volume 1 (2009), Number 1, pp. 1–13

(*Invited paper*)

General KKM Theorems for Abstract Convex Spaces

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Abstract. A general KKM type theorem for abstract convex spaces is obtained. This is applied to G-convex spaces and ϕ_A-spaces. Further comments on related works are given.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or others of the corresponding equilibrium problem under consideration. One of the remarkable results on the nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 [1], which is concerned with certain types of multimaps later called the KKM maps.

The KKM theory, first named by the author [2], is the study of applications of equivalent formulations or generalizations of the KKM theorem. At the beginning, the theory was mainly devoted by Ky Fan to the study on convex subsets of topological vector spaces; see [3]. Later, it has been extended to convex spaces by Lassonde [4], and to C-spaces (or H-spaces) by Horvath [5, 6], and others. In the last decade, the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author; for details, see [7]-[15] and references therein.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [16]-[25]. Moreover, in 2007,

2000 *Mathematics Subject Classification.* 47H04, 47H10, 49J27, 49J35, 54C60, 54H25, 91B50.
Key words and phrases. Abstract convex space; KKM space; Generalized (G-) convex space; Multimap (map); Map classes $\mathfrak{KC}, \mathfrak{KO}$; ϕ_A-space; GFC-space.

we found that most of variants of G-convex spaces can be subsumed in the concept of ϕ_A-spaces $(X,D;\{\phi_A\}_{A\in\langle D\rangle})$ and made into G-convex spaces; see [25]-[30]. We also found that most of the so-called generalized KKM maps are simply KKM maps on G-convex spaces or abstract convex spaces; see [28]. Such contents of the KKM theory have numerous applications on various fields, especially, on fixed point theory [14]-[24].

In the present paper, we review some known facts for abstract convex spaces and obtain a new KKM type theorem. This includes a large number of particular cases for the classes of G-convex spaces and ϕ_A-spaces. Finally, further comments on the contents of the recent work [31] are given.

2. **Abstract convex spaces**

For the abstract convex spaces and KKM spaces, the reader may consult with our previous work [16]-[25].

Definition 2.1. An *abstract convex space* $(E,D;\Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma:\langle D\rangle\to 2^E$ with nonempty values $\Gamma_A:=\Gamma(A)$ for $A\in\langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D.

For any $D'\subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A\in\langle D'\rangle\}\subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E,D;\Gamma)$ relative to D' if for any $N\in\langle D'\rangle$, we have $\Gamma_N\subset X$, that is, $\mathrm{co}_\Gamma D'\subset X$.

When $D\subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X\cap D)\subset X$; in other words, X is Γ-convex relative to $D':=X\cap D$. In case $E=D$, let $(E;\Gamma):=(E,E;\Gamma)$.

Definition 2.2. Let $(E,D;\Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F:E\to 2^Z$ with nonempty values, if a multimap $G:D\to 2^Z$ satisfies

$$F(\Gamma_A)\subset G(A):=\bigcup_{y\in A}G(y)\quad\text{for all } A\in\langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G:D\to 2^E$ is a KKM map with respect to the identity map 1_E.

A multimap $F:E\to 2^Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G:D\to 2^Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F\in\mathfrak{KC}(E,D,Z)$ [resp, $F\in\mathfrak{KO}(E,D,Z)$]. Some authors use KKM instead of $\mathfrak{KC}$.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E,D;\Gamma)$ is the statement $1_E\in\mathfrak{KC}(E,D,E)$; that is, for any closed-valued KKM map $G:D\to 2^E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E\in\mathfrak{KC}(E,D,E)\cap\mathfrak{KO}(E,D,E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

We have abstract convex subspaces as the following simple observation:

Proposition 2.4 ([19]). *For an abstract convex space $(E, D; \Gamma)$ and a nonempty subset D' of D, let X be a Γ-convex subset of E relative to D' and $\Gamma' : \langle D' \rangle \to 2^X$ a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \ for\ A \in \langle D' \rangle.$$

Then $(X, D'; \Gamma')$ itself is an abstract convex space called a subspace relative to D'.

We need the following:

Proposition 2.5. *Let $(E, D; \Gamma)$ be an abstract convex space, $(X, D'; \Gamma')$ a subspace, and Z a topological space. If $F \in \mathfrak{KC}(E, D, Z)$, then $F|_X \in \mathfrak{KC}(X, D', \overline{F(X)})$.*

Proof. Suppose that a closed-valued map $G' : D' \to 2^{\overline{F(X)}}$ satisfies

$$F|_X(\Gamma'_A) \subset G'(A) \ \text{ for all } A \in \langle D' \rangle.$$

Define a map $G : D \to 2^Z$ by

$$G(y) := \begin{cases} G'(y) & \text{for } y \in D' \\ \overline{F(X)} & \text{otherwise.} \end{cases}$$

Then

$$F(\Gamma_A) = F|_X(\Gamma'_A) \subset G'(A) = G(A) \ \text{ for } A \in \langle D' \rangle; \text{ and}$$
$$F(\Gamma_A) \subset \overline{F(X)} = G(A) \ \text{ for } A \in \langle D \rangle \setminus \langle D' \rangle.$$

Since $F \in \mathfrak{KC}(E, D, Z)$ and G has closed values, the family $\{G(y)\}_{y \in D}$ has the finite intersection property, and hence so does its subfamily $\{G'(y)\}_{y \in D'}$. Therefore, $F|_X \in \mathfrak{KC}(X, D', \overline{F(X)})$. □

Proposition 2.6. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \to 2^Z$ a multimap. Then $F \in \mathfrak{KC}(E, D, Z)$ if and only if for any closed-valued KKM map $G : D \to 2^Z$, we have $\overline{F(E)} \cap \bigcap \{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.*

Proof. For the necessity, since G is a KKM map, for any $N \in \langle D \rangle$, we have $F(\Gamma_N) \subset \overline{F(\Gamma_N)} \cap G(N) \subset \bigcup_{y \in N} \{\overline{F(E)} \cap G(y)\}$. Since F is a $\mathfrak{KC}$-map, the family $\{\overline{F(E)} \cap G(y)\}_{y \in D}$ has the finite intersection property. The sufficiency is clear. □

Under an additional requirement, we have the whole intersection property for the map-values of a KKM map:

Proposition 2.7. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp., $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \to 2^E$ a multimap satisfying*

(1) *G has closed [resp., open] values;*
(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map); and*
(3) *$\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$.*

Then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Proof. Since $1_E \in \mathfrak{KC}(E,D,E)$ [resp., $1_E \in \mathfrak{KO}(E,D,E)$], by (1) and (2), $\{G(y)\}_{y\in D}$ has the finite intersection property. Now the whole intersection property follows from the compactness in (3). □

Remark 2.8. (a) You may prefer to adopt "compactly" closed [resp., open] values in (1). This is impractical and superfluous. In fact, replacing the topology of E by its compactly generated extension, we can eliminate that kind of inadequate terminology; see [9].

(b) Some authors call G a transfer closed map when $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$. In this case, the conclusion of Proposition 2.7 becomes $\bigcap_{y\in D} G(y) \neq \emptyset$.

From the partial KKM principle we have the whole intersection property:

Theorem 2.9 ([22]). *Let $(X,D;\Gamma)$ be an abstract convex space satisfying the partial KKM principle (that is, $1_X \in \mathfrak{KC}(X,D,X)$), and $G : D \to 2^X$ a map such that*

(1) $\bigcap_{y\in D} G(y) = \bigcap_{y\in D} \overline{G(y)}$ *[that is, G is transfer closed-valued];*

(2) $\overline{G}$ *is a KKM map; and*

(3) *there exists a nonempty compact subset K of X such that either*
   (i) $K \supset \bigcap\{\overline{G(z)} \mid z \in M\}$ *for some $M \in \langle D\rangle$; or*
   (ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*
   $$K \supset L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\}.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

This subsumes a very large number of particular forms in the literature and has a number of equivalent formulations; see [22].

This can be extended to $F \in \mathfrak{KC}(X,D,Z)$ instead of $1_X \in \mathfrak{KC}(X,D,X)$ as the following main theorem in this paper shows:

Theorem 2.10. *Let $(X,D;\Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(X,D,Z)$, and $G : D \to 2^Z$ a map such that*

(1) $\bigcap_{y\in D} G(y) = \bigcap_{y\in D} \overline{G(y)}$ *[that is, G is transfer closed-valued];*

(2) $\overline{G}$ *is a KKM map with respect to F; and*

(3) *there exists a nonempty compact subset K of Z such that either*
   (i) $K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ *for some $M \in \langle D\rangle$; or*
   (ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$, $F(L_N)$ is compact, and*
   $$K \supset \overline{F(L_N)} \cap \bigcap\{\overline{G(z)} \mid z \in D'\}.$$

Then $\overline{F(X)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Proof. Case (i): Since $F(\Gamma_N) \subset \overline{G}(N)$ for each $N \in \langle D \rangle$ by (2), we have

$$F(\Gamma_N) \subset F(X) \cap \overline{G}(N) \subset \overline{F(X)} \cap \overline{G}(N) =: G'(N),$$

where $G'(y) := \overline{F(X)} \cap \overline{G(y)}$ is closed for each $y \in D$. Then, by Proposition 2.6, $\{G'(y) \mid y \in D\}$ has the finite intersection property. Since the requirement (i) implies

$$\overline{F(X)} \cap K \supset \overline{F(X)} \cap \bigcap_{y \in M} \overline{G(y)} = \bigcap_{y \in M} G'(y),$$

$\bigcap_{y \in M} G'(y)$ is compact. Therefore $\bigcap\{G'(y) \mid y \in D\} \neq \emptyset$ and hence

$$\overline{F(X)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

This is the conclusion in view of (1).

Case (ii): Suppose that

$$\overline{F(X)} \cap K \cap \bigcap_{y \in D} G(y) = \overline{F(X)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} = \emptyset.$$

Since $\overline{F(X)} \cap K$ is compact, $\overline{F(X)} \cap K \subset \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ for some $N \in \langle D \rangle$. Let L_N be the Γ-convex subset of X in (ii). Define $G' : D' \to 2^{\overline{F(L_N)}}$ by $G'(y) := \overline{G(y)} \cap \overline{F(L_N)}$ for $y \in D'$. For each $A \in \langle D' \rangle$, define $\Gamma'_A := \Gamma_A \cap L_N$. Then $(L_N, D'; \Gamma')$ is an abstract convex space. Moreover,

$$(F|_{L_N})(\Gamma'_A) \subset F(\Gamma_A) \cap F(L_N) \subset \overline{G}(A) \cap \overline{F(L_N)} = G'(A)$$

for each $A \in \langle D' \rangle$ by (6.2); and hence $G' : D' \to 2^{\overline{F(L_N)}}$ is a KKM map with respect to $F|_{L_N}$ on the abstract convex space $(L_N, D'; \Gamma')$ with closed values in $\overline{F(L_N)}$. Since $F \in \mathfrak{KC}(X, D, Z)$, by Proposition 2.5, we have $F|_{L_N} \in \mathfrak{KC}(L_N, D', \overline{F(L_N)})$ and hence, $\{G'(y) \mid y \in D'\} = \{\overline{G(y)} \cap \overline{F(L_N)} \mid y \in D'\}$ has the finite intersection property. Since we assumed that $\overline{F(L_N)}$ is compact, each $G'(y)$ is compact. Hence $\bigcap\{G'(y) \mid y \in D'\} \neq \emptyset$ and there exists a

$$z \in \bigcap_{y \in D'} G'(y) = \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K$$

by (ii). Since $z \in K$ and $z \in \overline{F(L_N)}$, we have $z \in \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ by our assumption. So $z \notin \overline{G(y)}$ for some $y \in N \subset D'$, and hence $z \notin \bigcap\{\overline{G(y)} \mid y \in D'\}$. This contradicts $z \in \bigcap\{G'(y) \mid y \in D'\}$. $\square$

Remark 2.11. (1) In requirement (ii), $\overline{F(L_N)}$ is not necessarily compact whenever we assume

$$K \supset \overline{F(L_N)} \cap \bigcap_{y \in M} \overline{G(y)} \quad \text{for some } M \in \langle D' \rangle.$$

(2) Note that Theorem 2.9 subsumes a very large number of particular forms in the literature and can be reformulated to the equivalent forms of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties as in [7, 15, 22].

3. *G*-convex spaces

All results in the preceding section work for G-convex spaces due to the author. More precisely, in this section, we show that two well-known general KKM type theorems for G-convex spaces are consequences of Theorem 2.9.

Definition 3.1. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with the set of vertices $V := \{e_i\}_{i=0}^n$, and Δ_J is the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For details, see references of [7]-[15].

Example 3.2. The following are typical examples of G-convex spaces:

(1) The triple $(\Delta_n, V; \mathrm{co})$ in the original KKM theorem [1], where $\mathrm{co} : \langle V \rangle \to 2^{\Delta}_n$ is the convex hull operation.

(2) A triple $(X, D; \Gamma)$, where X and D are subsets of a t.v.s. E such that $\mathrm{co}\, D \subset X$ and $\Gamma := \mathrm{co}$. Fan's celebrated KKM lemma [32] is for $(E, D; \mathrm{co})$, where D is a nonempty subset of E.

(3) A *convex space* $(X, D; \Gamma)$, where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$; see [4]. However he obtained several KKM type theorems with respect to $(X, D; \Gamma)$.

(4) An *H-space* $(X, D; \Gamma)$, where X is a topological space, D a nonempty subset of X, and $\Gamma = \{\Gamma_A\}$ a family of contractible (or, more generally, ω-connected) subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$. If $D = X$, $(X; \Gamma)$ is called a C-space by Horvath [5, 6].

The origin of the class $\mathfrak{KC}$ and Theorem 2.9 is the following [7, Theorem 3]:

Theorem 3.3. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, and $F \in \mathfrak{A}^{\kappa}_c(X, Y)$. Let $G : D \to 2^Y$ be a map such that*

(1) *for each $x \in D$, $G(x)$ is closed in Y;*

(2) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; and*

(3) *there exist a nonempty compact subset K of Y such that either*

 (i) *$\bigcap \{Gx \mid x \in M\} \subset K$ for some $M \in \langle D \rangle$; or*

(ii) *for each* $N \in \langle D \rangle$, *a compact G-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \cap \bigcap\{Gx \mid x \in L_N \cap D\} \subset K$.

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx \mid x \in D\} \neq \emptyset$.

Remark 3.4. (1) Theorem 3.3 was given originally under the assumption that $D \subset X$, which is redundant in view of condition (ii) of Theorem 2.9. In Theorem 3.3, the admissible class $\mathfrak{A}_c^\kappa$ is a subclass of $\mathfrak{KC}$; and note that $F(L_N)$ is compact since L_N is compact and F can be regarded as a composition of u.s.c. maps having compact values (by the definition of $\mathfrak{A}_c^\kappa$).

(2) In [7], we gave ten equivalent formulation of Theorem 3.3 in the form of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties. Similarly, we can make equivalent formulations of Theorem 2.9.

Example 3.5. For the references of the following, see [7].

(1) The origin of Theorem 3.3 goes back to Sperner and Knaster, Kuratowski, and Mazurkiewicz [1] for $X = Y = K = \Delta_n$ an n-simplex, D its set of vertices, and $F = 1_X$.

(2) For a convex space X, Theorem 3.3 was due to Park [33]. As Park noted in [2], a particular form [2, Theorem 3] for $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$ includes earlier works of Fan, Lassonde [4], Chang, and Park. Moreover, Park showed that [2, Theorem 3] also extends a number of KKM type theorems due to Sehgal, Singh, and Whitfield, Lassonde [4], Shioji, Liu, Chang and Zhang, and Guillerme.

(3) For an H-space X and $F = 1_X$, Theorem 3.3 generalizes Horvath [5, 6]; Bardaro and Ceppitelli; Ding and Tan; Ding, Kim, and Tan; Park; and Ding.

(4) After 1997 [7], there have appeared too many works on G-convex spaces and modifications of Theorem 3.3 to trace out all of them.

Park and Lee [10] defined generalized KKM maps on G-convex spaces as follows:

Definition 3.6. Let $(X, D; \Gamma)$ be a G-convex space and I a nonempty set. A map $G : I \to 2^X$ is called a *generalized KKM map* provided that for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset G(M)$ for each $M \in \langle N \rangle$.

In [15], H. Kim and the author showed that a generalized KKM map G is a KKM map on a new G-convex space $(X, I; \Gamma^G)$ and obtained the following KKM type theorem for generalized KKM maps with closed values:

Theorem 3.7 ([15]). *Let* I *be a set,* $(X, D; \Gamma)$ *a G-convex space, and* $G : I \to 2^X$ *a map. Suppose that*

(1) $\bigcap_{z\in I} G(z) = \bigcap_{z\in I} \overline{G(z)}$ [*that is,* G *is transfer closed-valued*];

(2) $\overline{G}$ *is a generalized KKM map; and*

(3) *there exists a nonempty compact subset* K *of* X *such that either*

(i) $\bigcap_{z\in M}\overline{F(z)}\subset K$ *for some* $M\in\langle I\rangle$; *or*

(ii) *if* $X\supset D$ *and, for each* $J\in\langle I\rangle$ *and each function* $\sigma : J\to D$, *there exists a* Γ*-convex subset* L_N *of* X *containing* $N=\sigma(J)$ *such that*

$$L_N\cap\bigcap_{z\in J}\overline{F(z)}\subset K.$$

Then $K\cap\bigcap_{z\in I}F(z)\neq\emptyset$.

Every G-convex space is an abstract convex space satisfying the partial KKM principle and, since $\overline{G}$ is a KKM map on the new G-convex space $(X,I;\Gamma^G)$, Theorem 3.7 follows from Theorem 2.8.

Remark 3.8. (1) Condition (ii) can be improved as in Theorem 2.8 without assuming $X\supset D$. If $X=K$ itself is compact, then the conclusion holds without assuming (i) or (ii).

(2) In (ii), L_N does not need to be compact in view of Remark 2.11(1).

(3) This kind of KKM theorems originates from Tian. If $(X;\Gamma)$ is an H-space, Theorem 3.7 improves results of Kim and Chang-Ma. Further, if $X=D$ is a convex subset of a topological vector space, Theorem 3.7 was due to Chang-Zhang and Kassay-Kolumb n. For the references, see [15].

4. ϕ_A-spaces

Motivated by many imitations of G-convex spaces, we introduced the following reformulation of the class of G-convex spaces in [25]-[30]:

Definition 4.1. A ϕ_A*-space*

$$(X,D;\{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A:\Delta_n\to X$ (that is, singular n-simplices) for $A\in\langle D\rangle$ with the cardinality $|A|=n+1$.

Note that, by letting $\Gamma_A:=\phi_A(\Delta_n)$, $(X,D;\Gamma)$ becomes an abstract convex space which is not necessarily G-convex.

Any G-convex space is a ϕ_A-space. The converse also holds in the following sense:

Proposition 4.2 ([25, 26]). *A* ϕ_A*-space* $(X,D;\{\phi_A\}_{A\in\langle D\rangle})$ *can be made into a* G*-convex space* $(X,D;\Gamma)$.

Now we have the following diagram for triples $(E,D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, a topological space Z, and a map $F : X \to 2^Z$, there are several ways to define a KKM map $G : D \to 2^Z$ with respect to F as follows:

4.1. *As an abstract convex space*

Every ϕ_A*-space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ is an abstract convex space by letting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$ with $|A| = n+1$. In this case, we have the following:

(1) A map $G : D \to 2^Z$ is a KKM map with respect to $F : X \to 2^Z$ whenever $F(\phi_A(\Delta_n)) \subset G(A)$ for all $A \in \langle D\rangle$.
(2) For every simplex, convex subsets of t.v.s., or Lassonde type convex spaces, we assumed $D \subset X$ and $\Gamma_A := \operatorname{co} A = \phi_A(\Delta_n)$. Therefore, the definition (1) works.
(3) All contents of Section 2 are applicable.

4.2. *As a G-convex space*

The following definition is given in [28]-[30]:

Definition 4.3. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, any map $G : D \to 2^X$ satisfying

$$\phi_A(\Delta_J) \subset G(J) \text{ for each } A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

Note that if $\phi_A(\Delta_J) = \phi_J(\Delta_J)$ for each A and J (as for the case Lassonde type convex spaces), then this definition becomes the one in 4.1.

Moreover, recall that

Proposition 4.4 ([28]-[30]). (1) *A KKM map $G : D \to 2^X$ on a G-convex space $(X, D; \Gamma)$ is a KKM map on the corresponding ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$.*
(2) *A KKM map $T : D \to 2^X$ on a ϕ_A-space $(X, D; \{\phi_A\})$ is a KKM map on a new G-convex space $(X, D; \Gamma)$.*

The following is a KKM theorem for ϕ_A-spaces given in [28]-[30]:

Theorem 4.5. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \to 2^X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n+1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

(3) *$\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D\rangle$,*

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

Remark 4.6. (1) We may assume that, for each $z \in D$ and $N \in \langle D\rangle$, $G(a)\cap\phi_N(\Delta_n)$ is closed [resp., open] in $\phi_N(\Delta_n)$. This is said by some authors that G has finitely closed [resp., open] values. However, by replacing the topology of X by its finitely generated extension, we can eliminate “finitely”; see [9].

(2) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \mathrm{co}$, the convex hull, Theorem 4.5 reduces to the original KKM theorem and its open version; see [1, 3].

(3) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 3 extends Fan's KKM lemma; see [3, 32].

The preceding definition of a KKM map on a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ can be extended to a KKM map with respect to a map $F : X \to 2^Z$ for a topological space Z as follows:

Definition 4.7. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and a topological vector space Z, let $F : X \to 2^Z$ be a map. Then a map $G : D \to 2^X$ satisfying

$$F(\phi_A(\Delta_J)) \subset G(J) \text{ for each } A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map with respect to F*.

Here, for the simplex Δ_n corresponding to $|A| = n + 1$, Δ_J is its face corresponding to $J \subset A$, as usual.

Recall that $F \in \mathfrak{KC}(X, D, Z)$ whenever for any closed-valued KKM map $G : D \to 2^Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property.

Note that, when $F \neq 1_X$, the above concept of a KKM map with respect to $F : X \to 2^Z$ for a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ is

(1) different from the one for the abstract convex space case [see Subsection 4.1(1)], and

(2) may not a KKM map with respect to F for the new G-convex space corresponding to the ϕ_A-space in Proposition 4.4(2).

Therefore we have the following problem:

Problem. *Let* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ *be a* ϕ_A*-space,* Z *a topological space,* $F \in \mathfrak{KC}(X, D, Z)$*, and* $G : D \to 2^Z$*. Under* Definition 4.7*, does* Theorem 2.9 *hold?*

There have also appeared a large number of the so-called generalized KKM maps in the literature. In fact, a number of authors tried to generalize the concept of KKM maps on particular cases of ϕ_A-spaces. All such KKM maps are known to be the ones for certain G-convex spaces; see [18].

5. GFC-spaces

Recently, in 2008, Khanh *et al.* [31] introduced generalized finitely continuous topological spaces (simply, GFC-spaces) and certain generalized KKM mappings. They noted incorrectly that G-convex spaces are special case of GFC-spaces and that the G-convex space and the FC-space are incomparable. They missed to give any proper examples to support their claims. Actually, their GFC-spaces are simply our ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and, hence, can be made into G-convex spaces.

In [31], the authors used $(X, D; \Phi)$ to denote their spaces.

Definition 5.1 ([31]). (ii) Let $P, Q \subset D$ and $S : D \to 2^X$ be given. P is called an S-subset of D (an S-subset of D with respect to Q) if $\forall\ N = \{y_0, y_1, \dots, y_n\} \in \langle D \rangle$, $\forall\ \{y_{i_0}, y_{i_1}, \dots, y_{i_k}\} \subset N \cap P$ ($N \cap Q$, resp.), $\phi_N(\Delta_k) \subset S(P)$, where Δ_k is the face of Δ_n corresponding to $\{y_{i_0}, y_{i_1}, \dots, y_{i_k}\}$.

In this definition, they followed the definition of FC-subspace due to Ding, which was already improved in [25, 26]. Similarly, the above definition can be improved.

Any space $(X, D; \Phi)$ becomes an abstract convex space $(X, D; \Gamma)$ by putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D \rangle$ with $|A| = n + 1$; see [25].

Definition 5.2 ([31]). (i) Let $(X, D; \Phi)$ be a GFC-space and Z be a topological space. Let $G : D \to 2^Z$ and $F : X \to 2^Z$ be set-valued mappings. G is called a generalized KKM mapping with respect to F (F-KKM mapping in short) if, for each $N = \{y_0, y_1, \dots, y_n\} \in \langle D \rangle$ and each $\{y_{i_0}, y_{i_1}, \dots, y_{i_k}\} \subset N$, one has $F(\phi_N(\Delta_k)) \subset \bigcup_{j=0}^{k} G(y_{i_j})$, where Δ_k is as above.

(ii) We say that a set-valued mapping $F : X \to 2^Z$ has the generalized KKM property if for each F-KKM mapping $G : D \to 2^Z$, the family $\{\overline{F(y)} : y \in D$ has the finite intersection property. By KKM(X, D, Z) we denote the class of all the mappings $F : X \to 2^Z$ which enjoy the generalized KKM property.

The following is their KKM theorem comparable to our Theorem 2.9:

Theorem 5.3 ([31]). *Let (X, Y, Φ) be a GFC-space, Z be a topological space, $S : Y \to 2^X$, $T : X \to 2^Z$ and $F : Y \to 2^Z$ be multifunctions, where $T \in$* KKM(X, Y, Z)*. assume that*

(i_1) *for each compact subset $D \subset X$, $\overline{T(D)}$ is compact;*

(i_2) *there is a compact subset K of Z such that for each $N \in \langle Y \rangle$, there is an S-subset L_N of Y, containing N with $S(L_N)$ or $\overline{S(L_N)}$ being compact and*

$$\overline{T(S(L_N))} \cap \bigcap_{y \in L_N} \mathrm{ccl} F(y) \subset K;$$

(ii) *F is T-KKM and transfer compactly closed-valued.*

Then

$$\overline{T(S(Y))} \cap \bigcap_{y \in Y} F(y) \neq \emptyset.$$

From the seminal works of Ky Fan, this kind of whole intersection property can be reformulated to many equivalent forms like coincidence theorems or geometric forms. This was done in [31].

Interested readers can compare their results with our previous works [7, 8, 19, 21, 22].

References

[1] B. Knaster, K. Kuratowski and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes f r n-Dimensionale Simplexe, *Fund. Math.* **14** (1929), 132–137.

[2] S. Park, Some coincidence theorems on acyclic multifunctions and applications to KKM theory, *Fixed Point Theory and Applications* (K.-K. Tan, ed.), 248–277, World Sci. Publ., River Edge, NJ, 1992.

[3] S. Park, Ninety years of the Brouwer fixed point theorem, *Vietnam J. Math.* **27** (1999), 193–232.

[4] M. Lassonde, On the use of KKM multifunctions in fixed point theory and related topics, *J. Math. Anal. Appl.* **97** (1983), 151–201.

[5] C.D. Horvath, Contractibility and generalized convexity, *J. Math. Anal. Appl.* **156** (1991), 341–357.

[6] C.D. Horvath, Extension and selection theorems in topological spaces with a generalized convexity structure, *Ann. Fac. Sci. Toulouse* **2** (1993), 253–269.

[7] S. Park and H. Kim, Foundations of the KKM theory on generalized convex spaces, *J. Math. Anal. Appl.* **209** (1997), 551–571.

[8] S. Park, Elements of the KKM theory for generalized convex spaces, *Korean J. Comput. Appl. Math.* **7** (2000), 1–28.

[9] S. Park, Remarks on topologies of generalized convex spaces, *Nonlinear Func. Anal. Appl.* **5** (2000), 67–79.

[10] S. Park and W. Lee, A unified approach to generalized KKM maps in generalized convex spaces, *J. Nonlinear Convex Anal.* **2** (2001), 157–166.

[11] S. Park, New topological versions of the Fan-Browder fixed point theorem, *Nonlinear Anal.* **47** (2001), 595–606.

[12] S. Park and H. Kim, Remarks on the KKM property for open-valued multimaps on generalized convex spaces, *J. Korean Math. Soc.* **42** (2005), 101–110.

[13] S. Park, The KKM, matching, and fixed point theorems in generalized convex spaces, *Nonlinear Funct. Anal. Appl.* **11** (2006), 139–154.

[14] S. Park, A unified fixed point theory in generalized convex spaces, *Acta Math. Sinica*, English Series **23**(8) (2007), 1509–1536.

[15] H. Kim and S. Park, Generalized KKM maps, maximal elements, and almost fixed points, *J. Korean Math. Soc.* **44**(2) (2007), 393–406.

[16] S. Park, On generalizations of the KKM principle on abstract convex spaces, *Nonlinear Anal. Forum* **11** (2006), 67–77.

[17] S. Park, Remarks on $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps in abstract convex spaces, *Nonlinear Anal. Forum* **12**(1) (2007), 29–40.

[18] S. Park, Comments on some abstract convex spaces and the KKM maps, *Nonlinear Anal. Forum* **12**(2) (2007), 125–139.

[19] S. Park, Elements of the KKM theory on abstract convex spaces, *J. Korean Math. Soc.* **45**(1) (2008), 1–27.

[20] S. Park, Remarks on weakly KKM maps in abstract convex spaces, *Inter. J. Math. Math. Sci.* **2008**, Article ID 423596, 10pp. doi:10.1155/2008 /423596.

[21] S. Park, Equilibrium existence theorems in KKM spaces, *Nonlinear Anal.* **69** (2008), 4352–4364.

[22] S. Park, New foundations of the KKM theory, *J. Nonlinear Convex Anal.* **9**(3) (2008), 331–350.

[23] S. Park, Remarks on the partial KKM principle, *Nonlinear Anal. Forum* **14**(1) (2009), 1–12.

[24] S. Park, Fixed point theory of multimaps in abstract convex uniform spaces, *Nonlinear Anal. TMA* **71**(2009), 2468–2480.

[25] S. Park, Various subclasses of abstract convex spaces for the KKM theory, *Proc. Nat. Inst. Math. Sci.* **2**(4) (2007), 35–47.

[26] S. Park, Generalized convex spaces, L-spaces, and FC-spaces, *J. Global Optim.* (2008), DOI:10. 1007/s10898-008-9363-1.

[27] S. Park, Comments on the KKM theory on ϕ_A-spaces, *PanAmerican Math. J.* **18**(2) (2008), 61–71.

[28] S. Park, Comments on recent studies on abstract convex spaces, N*onlinear Anal. Forum* **13**(1) (2008), 1–17.

[29] S. Park, Remarks on fixed points, maximal elements, and equilibria of economies in abstract convex spaces, *Taiwan. J. Math.* **12**(6) (2008), 1365–1383.

[30] S. Park, Remarks on KKM maps and fixed point theorems in generalized convex spaces, *CUBO, Math. J.* **10**(4) (2008), 1–13.

[31] P.Q. Khanh, N.H. Quan, J.C. Yao, Generalized KKM type theorems in GFC-spaces and applications, *Nonlinear Anal.* (2008), doi:10.1016/j.na. 2008.11.055.

[32] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* **142** (1961), 305–310.

[33] S. Park, Foundations of the KKM theory via coincidences of composites of admissible u.s.c. maps, *J. Korean Math. Soc.* **31** (1994), 493–519.

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Received May 4, 2009
Accepted June 30, 2009