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Projectively Flat Finsler Space with A r-th Series (α,β)-Metric

Research Article

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Abstract. The (α,β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β it has been sometimes treated in theoretical physics [8]. The condition for a Finsler space with an (α,β)-metric $L(\alpha,\beta)$ to be projectively flat was given by Matsumoto. In this paper, we discuss the r-th series (α,β)-metric to be projectively flat on the basis of Matsumoto's results.

Keywords. Finsler Space; r-th series (α,β)-metric; Projectively flat

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1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, that is, an n-dimensional differential manifold M^n equipped with a fundamental function $L(x,y)$. The concept of an (α,β)-metric $L(\alpha,\beta)$ was introduced by Matsumoto [5] and was investigated and study in detail by Hashiguchi and Ichijyo [3] have studied in detail on some special (α,β)-metric. A Finsler metric $L(x,y)$ is called an (α,β)-metric $L(\alpha,\beta)$ if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^n. Lee and Park [6] have studied Finsler spaces with infinite series (α,β)-metric. In this paper by using

r-th series (α,β)-metric and proved some results that a r-th series (α,β) to be a projectively flat.

A Finsler space is called Projectively flat, or with rectilinear geodesic, if the space is covered by coordinate neighborhoods in which the geodesics can be represented by $(n-1)$ linear equations of the coordinates. Such a coordinate system is called rectilinear. The coordinate for a Finsler space to be projectively flat was studied by L. Berwald [2].

The purpose of the present paper is to consider the projective flatness of Finsler space with an r-th series (α,β)-metric.

2. Preliminaries

The study of some well known (α,β)-metrics are Randers metric $\alpha+\beta$, Kropina metric α^2/β and generalized Kropina metric α^{m+1}/β^m have greatly contributed to the growth of Finsler geometry and its applications to theory of relativity.

The derivative of the (α,β)-metric with respect to α and β are given by,

$$L_\alpha = \partial L/\partial\alpha, \quad L_\beta = \partial L/\partial\beta, \quad L_{\alpha\alpha} = \partial L_\alpha/\partial\alpha, \quad L_{\beta\beta} = \partial L_\beta/\partial\beta, \quad L_{\alpha\beta} = \partial L_\alpha/\partial\beta.$$

The r-th series (α,β)-metric [4] is expressed as the form

$$L(\alpha,\beta) = \beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k, \tag{2.1}$$

where we assume $\alpha < \beta$.

If $r=0$, then $L=\beta$ is a one form metric. If $r=1$, then $L=\alpha+\beta$ is a Randers metric. We shall deal with arbitrary integer r greater than 3 in the paper. We shall call the (α,β)-metric (2.1) is the r-th series (α,β)-metric.

The geodesics of a Finsler space $F^n=(M^n,L)$ are given by the system of differential equations including the function

$$4G^i(x,y) = g^{ij}(y^r\dot{\partial}_j\partial_r L^2 - \partial_j L^2).$$

For an (α,β)-metric $L(\alpha,\beta)$ the space $R^n=(M^n,\alpha)$ is called the associated Riemannian space with $F^n=(M^n,L(\alpha,\beta))$ ([1], [6]). The covariant differentiation with respect to the Levi-Civita connection $\gamma_j{}^i{}_k(x)$ of R^n is denoted by $(;)$. We put $a^{ij}=(a_{ij})^{-1}$, and use the symbols as follows:

$$r_{ij} = \tfrac{1}{2}(b_{i;j}+b_{j;i}),\ s_{ij} = \tfrac{1}{2}(b_{i;j}-b_{j;i}),\ r^i{}_j = a^{ir}r_{rj},\ s^i{}_j = a^{ir}s_{rj},$$

$$r_j = b_r r^r{}_j,\ s_j = b_r s^r{}_j,\ b^i = a_{ir}b_r,\ b^2 = a^{rs}b_r b_s.$$

Now the following Matsumoto's theorem [7] is well known.

Theorem 1. *A Finsler space (M,L) with an (α,β)-metric $L(\alpha,\beta)$ is projectively flat if and only if for any point of space M there exist local coordinate neighborhoods containing the point such that γ^i_{jk} satisfies:*

$$(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C+\alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0, \tag{2.2}$$

where C is given by

$$C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha/\beta^2 L_\alpha})(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0. \tag{2.3}$$

The equation is written in the form

$$(C + \alpha r_{00}/2\beta)\{1 + (\alpha L_{\alpha\alpha/\beta^2 L_\alpha})(\alpha^2 b^2 - \beta^2)\} - (\alpha/2\beta)\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\} = 0, \tag{2.4}$$

that is,

$$(C + \alpha r_{00}/2\beta) = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha L_\beta s_0)}{2\{\beta^2 L_\alpha + \alpha L_{\alpha\alpha}(\alpha^2 b^2 - \beta^2)\}}.$$

Therefore (2.2) *leads us to*

$$\begin{aligned}&\{L_\alpha(\alpha^2\gamma^i_{0\,0} - \gamma_{000}y^i) + 2\alpha^3 L_\beta s^i_0\}\{\beta^2 L_\alpha + \alpha L_{\alpha\alpha}(\alpha^2 b^2 - \beta^2)\}\\ &\quad + \alpha^3 L_{\alpha\alpha}(r_{00}L_\alpha - 2\alpha L_\beta s_0)(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{2.5}$$

3. Projectively Flat Finsler Space

In an n-dimensional Finsler space F^n with the r-th ($r \geq 3$) series (α,β)-metric (2.1), we have

$$L_\alpha = \sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1}, \quad L_\beta = -\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}, \quad L_{\alpha\alpha} = \frac{1}{\beta}\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}. \tag{3.1}$$

Substituting (3.1) into (2.5), we have

$$\begin{aligned}&\left\{\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1}(\alpha^2\gamma^i_{0\,0} - \gamma_{000}y^i) - 2\alpha^3\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k} s^i_0\right\}\\ &\times\left\{\beta^2\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1} + \frac{\alpha}{\beta}\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}(\alpha^2 b^2 - \beta^2)\right\}\\ &+\frac{\alpha^3}{\beta}\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}\left\{r_{00}\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1} + 2\alpha\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k} s_0\right\}(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{3.2}$$

We shall divide our consideration in two cases of which r is even or odd.

Case (i). $r = 2h$ (h is a positive integer).

When $r = 2h$, we have

$$\begin{aligned}&\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k} = \alpha^{2h}\sum_{k=0}^{2h}(2h-k)\alpha^{-k}\beta^{-2h+k},\\ &\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k+1} = \alpha\left\{\alpha^{2h}\sum_{k=0}^{2h}(2h-k-1)\alpha^{-k}\beta^{-2h+k-1}\right\},\end{aligned}$$

$$\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^k = \alpha^{2h}\sum_{k=0}^{2h}(2h-k)(2h-k-1)\alpha^{-k}\beta^{-2h+k}. \tag{3.3}$$

Put $-k = j$ and separating the rational and irrational parts in y^i, we have

$$\begin{aligned}
&\sum_{j=0}^{2h}(2h+j)\alpha^j\beta^{-2h-j} = \sum_{j=0}^{h}(2h+2j)\alpha^{2j}\beta^{-2h-2j} + \alpha\sum_{j=0}^{h-1}(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1},\\
&\sum_{j=0}^{2h}(2h+j-1)\alpha^j\beta^{-2h-j-1} = \sum_{j=0}^{h}(2h+2j-1)\alpha^{2j}\beta^{-2h-2j-1} + \alpha\sum_{j=0}^{h-1}(2h+2j)\alpha^{2j}\beta^{-2h-2j-2},\\
&\sum_{j=0}^{2h}(2h+j)(2h+j-1)\alpha^j\beta^{-2h-j} = \sum_{j=0}^{h}(2h+2j)(2h+2j-1)\alpha^{2j}\beta^{-2h-2j}\\
&\qquad\qquad + \alpha\sum_{j=0}^{h-1}(2h+2j)(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1}.
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
&A = \sum_{j=0}^{h}(2h+2j)\alpha^{2j}\beta^{-2h-2j}, && B = \sum_{j=0}^{h-1}(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1},\\
&D = \sum_{j=0}^{h}(2h+2j-1)\alpha^{2j}\beta^{-2h-2j-1}, && E = \sum_{j=0}^{h-1}(2h+2j)\alpha^{2j}\beta^{-2h-2j-2},\\
&F = \sum_{j=0}^{h}(2h+2j)(2h+2j-1)\alpha^{2j}\beta^{-2h-2j}, && G = \sum_{j=0}^{h-1}(2h+2j)(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1}.
\end{aligned} \tag{3.5}$$

Substituting (3.3) and (3.4) into (3.2), we have

$$\begin{aligned}
&\Big[(\alpha^2\gamma^i_{0\,0}-\gamma_{000}y^i)\{\beta^2(A^2+\alpha^2B^2+2\alpha AB)+(AF+\alpha^2BG+\alpha(AG+BF))(\alpha^2b^2-\beta^2)\}\\
&\quad -2\alpha^4 s^i_0\{(AD+\alpha^2BE+\alpha(BD+AE))\beta^2+(DF+\alpha^2GE+\alpha(DG+EF))(\alpha^2b^2-\beta^2)\}\\
&\quad +\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AF+\alpha^2BG+\alpha(BF+AG))+2\alpha^2 s_0(DF+\alpha^2GE+\alpha(FE+DG))\}\Big]=0.
\end{aligned} \tag{3.6}$$

That is,

$$P + \alpha Q = 0,$$

where

$$\begin{aligned}
P &= \Big[(\alpha^2\gamma^i_{0\,0}-\gamma_{000}y^i)\{\beta^2(A^2+\alpha^2B^2)+(AF+\alpha^2BG)(\alpha^2b^2-\beta^2)\}\\
&\quad -2\alpha^4 s^i_0\{(AD+\alpha^2BE)\beta^2+(DF+\alpha^2GE)(\alpha^2b^2-\beta^2)\}\\
&\quad +\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AF+\alpha^2BG)+2\alpha^2 s_0(DF+\alpha^2GE)\}\Big],\\
Q &= \Big[(\alpha^2\gamma^i_{0\,0}-\gamma_{000}y^i)\{\beta^2(2AB)+(AG+BF)(\alpha^2b^2-\beta^2)\}\\
&\quad -2\alpha^4 s^i_0\{(BD+AE)\beta^2+(DG+EF))(\alpha^2b^2-\beta^2)\}\\
&\quad +\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(BF+AG)+2\alpha^2 s_0(FE+DG)\}\Big].
\end{aligned} \tag{3.7}$$

Since P, Q are rational parts and α is an irrational part in y^i, $P=0$ and $Q=0$, that is,

$$\Big[(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)\{\beta^2(A^2+\alpha^2B^2)+(AF+\alpha^2BG)(\alpha^2b^2-\beta^2)\} \\ -2\alpha^4s_0^i\{(AD+\alpha^2BE)\beta^2+(DF+\alpha^2GE)(\alpha^2b^2-\beta^2)\} \\ +\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(AF+\alpha^2BG)+2\alpha^2s_0(DF+\alpha^2GE)\}\Big]=0, \tag{3.8}$$

$$\Big[(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)\{\beta^2(2AB)+(AG+BF)(\alpha^2b^2-\beta^2)\} \\ -2\alpha^4s_0^i\{(BD+AE)\beta^2+(DG+EF))(\alpha^2b^2-\beta^2)\} \\ +\alpha^2(\alpha^2b^i-\beta y^i)\{r_{00}(BF+AG)+2\alpha^2s_0(FE+DG)\}\Big]=0. \tag{3.9}$$

Eliminating $(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)$ from (3.8) and (3.9), we have

$$2\alpha^2s_0^i[-\{\beta^2(AD+\alpha^2BE)+(DF+\alpha^2GE)(\alpha^2b^2-\beta^2)\}\{2AB\beta^2+(AG+BF)(\alpha^2b^2-\beta^2)\} \\ +\{(BD+AE)\beta^2+(DG+EF)(\alpha^2b^2-\beta^2)\}\{(A^2+\alpha^2B^2)\beta^2+(AF+\alpha^2BG)(\alpha^2b^2-\beta^2)\}] \\ +(\alpha^2b^i-\beta y^i)[\{r_{00}(AF+\alpha^2BG)+2\alpha^2s_0(DF+\alpha^2GE)\}\{2AB\beta^2+(AG+BF)(\alpha^2b^2-\beta^2)\} \\ -\{r_{00}(BF+AG)+2\alpha^2s_0(EF+DG)\}\{(A^2+\alpha^2B^2)\beta^2+(AF+\alpha^2BG)(\alpha^2b^2-\beta^2)\}]=0. \tag{3.10}$$

Transvecting (3.10) by b_i, we have

$$2\alpha^2s_0[-(AD+\alpha^2BE)\{2AB\beta^2+(AG+BF)(\alpha^2b^2-\beta^2)\} \\ +(BD+AE)\{(A^2+\alpha^2B^2)\beta^2+(AF+\alpha^2BG)(\alpha^2B^2-\beta^2)\}] \\ +r_{00}(\alpha^2b^2-\beta^2)\{2(AF+\alpha^2BG)AB-(BF-AG)(A^2+\alpha^2B^2)\}=0. \tag{3.11}$$

The term of (3.11) which does not contain α^2 is

$$r_{00}\beta^2[A^2(AG-BF)]=0. \tag{3.12}$$

That is $r_{00}8h^3(2h+1)\beta^{1-8h}=0$.

Therefore there exist $hp(1-8h):V_{(1-8h)}$ such that

$$r_{00}8h^3(2h+1)\beta^{1-8h}=\alpha^2V_{(1-8h)}. \tag{3.13}$$

We suppose that $\alpha^2\not\equiv 0(\mathrm{mod}\,\beta)$. In this case, there exist form (3.13) a function $k=k(x)$ satisfying $V_{(1-8h)}=k\beta^{1-8h}$, and hence

$$r_{00}=\lambda\alpha^2, \tag{3.14}$$

where $\lambda=k/8h^3(2h+1)$. Substituting (3.14) into (3.11), we have

$$2s_0[-(AD+\alpha^2BE)\{2AB\beta^2+(AG+BF)(\alpha^2b^2-\beta^2)\} \\ +(BD+AE)\{(A^2+\alpha^2B^2)\beta^2+(AF+\alpha^2BG)(\alpha^2b^2-\beta^2)\}] \\ +\lambda(\alpha^2b^2-\beta^2)\{2(AF+\alpha^2BG)AB-(BF+AG)(A^2+\alpha^2B^2)\}=0. \tag{3.15}$$

It is observed from (3.15) that must have a factor is $[2s_0\{A^2(GD-BD+AE-EF)\}-\lambda\{A^2(FB-AG)\}]=0$, that is

$$(c_1s_0+c_2\lambda\beta)\beta^{-8h}=\alpha^2W_{-(1+8h)},$$

where $c_1=16h^3(2h-1)$, $c_2=8h^3(2h+1)$. Since $\alpha^2\not\equiv 0 \pmod{\beta}$, $c_1s_0+c_2\lambda\beta=0$, that is $c_1s_i+c_2\lambda b_i=0$. Transvecting this by b^i, we have $c_2\lambda b^2=0$.

(a) If $c_2=0$, that is, $h=0$, then

$$A=0,\quad B=\frac{\alpha^2-\beta}{\alpha^2\beta},\quad D=\frac{1}{\beta},\quad E=-\frac{2}{\alpha^2},\quad F=0,\quad G=\frac{2\beta}{\alpha^2}\ .$$

Hence (3.8) and (3.9) is written as

$$(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)\{(\alpha^2-\beta)^2+2(\alpha^2-\beta)(\alpha^2b^2-\beta^2)\}+2\alpha^4s_0^i\{2(\alpha^2-\beta)\beta+4\beta(\alpha^2b^2-\beta^2)\}$$
$$+\alpha^2(\alpha^2b^i-\beta y^i)\{2r_{00}(\alpha^2-\beta)-8s_0\alpha^2\beta\}=0. \tag{3.16}$$
$$s_0^i\{(\alpha^2-\beta)+(\alpha^2b^2-\beta^2)\}-2s_0(\alpha^2b^i-\beta y^i)=0. \tag{3.17}$$

Transvecting (3.17) by b_i, we have $s_0(\alpha^2-\beta)=0$. Since $(\alpha^2-\beta)\neq 0$, we get $s_0=0$. Substituting this into (3.17), we have

$$s_0^i\{(\alpha^2-\beta)+2(\alpha^2b^2-\beta^2)\}=0,$$

from which $s_0^i=0$, that is $s_{ij}=0$. The term which does not contain α^2 in (3.16), is $-\gamma_{000}y^i\beta^2$. Therefore there exists $hp(1):\mu_0=\mu_i(x)y^i$ such that

$$\gamma_{000}=\mu_0\alpha^2. \tag{3.18}$$

Substituting $s_0^i=0$, $s_0=0$ and (3.18) into (3.16), we have

$$\{(\alpha^2-\beta)+2(\alpha^2b^2-\beta^2)\}(\gamma_{0\,0}^{\,i}-\mu_0y^i)+2r_{00}(\alpha^2b^i-\beta y^i)=0. \tag{3.19}$$

The term of $\beta(1+2\beta)(\gamma_{0\,0}^{\,i}-\mu_0y^i)+2r_{00}\beta y^i$ of (3.19) must contain the factor α^2. Hence there exists 1-form $v_0^i=v_j^i(x)y^j$ such that

$$(1+2\beta)(\gamma_{0\,0}^{\,i}-\mu_0y^i)+2r_{00}\beta y^i=v_0^i\alpha^2. \tag{3.20}$$

Transvecting (3.20) by y_i, we have

$$2r_{00}=v_0^iy_i. \tag{3.21}$$

On the other hand, (3.19) is rewritten as the form

$$\alpha^2\{(1+2b^2)(\gamma_{0\,0}^{\,i}-\mu_0y^i)+2r_{00}b^i\}=\beta\{(1+2\beta)(\gamma_{0\,0}^{\,i}-\mu_0y^i)+2r_{00}y^i\}, \tag{3.22}$$

from which it is reduces to

$$(1+2b^2)(\gamma^i_{0\,0}-\mu_0 y^i)+2r_{00}b^i=\beta v^i_0, \tag{3.23}$$

by virtue of (3.19). Substituting (3.20) into (3.23), we get

$$(1+2b^2)(\gamma^i_{0\,0}-\mu_0 y^i)=\beta v^i_0-v_{00}b^i, \tag{3.24}$$

where $v_{ij}=a_{ir}v^r_j$. From (3.19) and (3.24) we have

$$\begin{aligned} &v^i_0\beta\{\alpha^2(1+2b^2)-\beta(1+2\beta)\} \\ &\quad=\mu_{00}[\beta y_i(1+2b^2)+b^i\{2b^2(1+2b^2-\beta(1+2\beta))-\beta(1+2\beta)\}], \end{aligned} \tag{3.25}$$

from which

$$v^{ij}\{\beta(1+2b^2)a_{kh}-(1+2\beta)b_k b_h\}+(jkh),\quad v^{jk}\{\beta(1+2b^2)a_{ih}-(1+2\beta)b_i b_h\}+(jkh), \tag{3.26}$$

where (jkh) denote the cyclic permutation of indices j, k, h. It is easy to show that the tensor $\beta(1+2b^2)a_{kh}-(1+2\beta)b_k b_h$ has reciprocal

$$M^{ij}=[\beta a^{ij}+(1+2\beta)b^i b^j/(1-b^2)]/(1+2b^2).$$

Transvecting (3.26) by M^{hk}, we get

$$v_{ij}=M[\beta(1+2b^2)a_{ij}-(1+2\beta)b_i b_j], \tag{3.27}$$

where $M=M^{hk}v_{hk}/n$. Therefore, from (3.20) we have

$$r_{ij}=\frac{1}{2}M[\beta(1+2b^2)a_{ij}-(1+2\beta)b_i b_j]. \tag{3.28}$$

Hence we have

$$b_{i;j}=\frac{1}{2}M[\beta(1+2b^2)a_{ij}-(1+2\beta)b_i b_j]. \tag{3.29}$$

Next, from (3.27) the equation (3.24) is reduced in the form

$$(\gamma^i_{0\,0}-\mu_0 y^i)=M\beta[\beta y^i-\alpha^2 b^i], \tag{3.30}$$

that is,

$$\gamma^i_{j\,k}=\frac{1}{2}\{(\mu_j\delta^i_k+Mb_j b_k y^i)+\frac{1}{2}(\mu_k\delta^i_j+Mb_k b_j y^i)\}-Ma_{jk}b^i. \tag{3.31}$$

(b) For $h>0$, $\lambda=0$ or $b^2=0$.

First, if $\lambda=0$, then $s_i=0$ and $r_{00}=0$ from 3.14. Therefore, from (3.10) we have

$$\begin{aligned} &2\alpha^2 s^i_0[-\{\beta^2(AD+\alpha^2BE)+(DF+\alpha^2GE)(\alpha^2b^2-\beta^2)\}\{2AB\beta^2+(AG+BF)(\alpha^2b^2-\beta^2)\} \\ &\quad+\{(BD+AE)\beta^2+(DG+EF)(\alpha^2b^2-\beta^2)\}\{(A^2+\alpha^2B^2)\beta^2+(AF+\alpha^2BG)(\alpha^2b^2-\beta^2)\}]=0. \end{aligned} \tag{3.32}$$

The term which does not contain α^2 is

$$2s^i_0[A^2(AE-BD-2EF)+F^2(AE-BD)+2ADBF]=0,$$

that is $32h^2(1-h)s^i_0\beta^{-(2+8h)}=0$. Therefore there exists $hp-(3+8h):U_{-(3+8h)}$ such that

$$32h^2(1-h)s^i_0\beta^{-(2+8h)}=\alpha^2U_{-(3+8h)}.$$

Hence $s^i_0=0$, that is, $s_{ij}=0$. From this $r_{ij}=0$, we have

$$b_{i;j}=0. \tag{3.33}$$

Substituting $s^i_0=0$, $r_{ij=0}$ and $s_0=0$ into (3.8), we must have $hp(1):\sigma_0=\sigma_i(x)y^i$ satisfying $\gamma_{000}=\sigma_0\alpha^2$. Therefore $\gamma^{\ i}_{0\ 0}=\sigma_0y^i$, that is,

$$2\gamma^{\ i}_{j\ k}=\sigma_j\delta^i_k+\sigma_k\delta^i_j, \tag{3.34}$$

Which shows that the associated Riemannian space is projectively flat.

Secondly, if $b^2=0$, then (3.15) is reduces to

$$\begin{aligned}&2s_0[-(AD+\alpha^2BE)\{2AB\beta^2-(AG+BF)\beta^2\}+(BD+AE)\{(A^2+\alpha^2B^2)\beta^2-(AF+\alpha^2BG)\beta^2\}]\\&\quad+\lambda(\alpha^2b^i-\beta^2)\{2(AF+\alpha^2BG)AB-(BF+AG)(A^2+\alpha^2B^2)\}=0.\end{aligned} \tag{3.35}$$

The term of (3.35) which does not contain α^2 is

$$2s_0[\alpha^2(DG-BD+AE-EF)]=0,$$

that is $2s_0\beta^{-(8h+2)}=\alpha^3U_{-(4+8h)}$, where $U_{-(4+8h)}$ is $hp-(4+8h)$. Therefore $s_0=0$, and hence $\lambda=0$. Thus we obtain (3.33) and (3.34).

(i) Case of $r=2h+1$ (h is a positive integer)

In this case, we have

$$\begin{aligned}\sum_{j=0}^{2h+1}(2h+j+1)\alpha^j\beta^{-2h-j-1}&=\sum_{j=0}^{h}(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1}+\alpha\sum_{j=0}^{h}(2h+2j+2)\alpha^{2j}\beta^{-2h-2j-2},\\ \sum_{j=0}^{2h+1}(2h+j)\alpha^j\beta^{-2h-j-2}&=\sum_{j=0}^{h}(2h+2j)\alpha^{2j}\beta^{-2h-2j-2}+\alpha\sum_{j=0}^{h}(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-3},\\ \sum_{j=0}^{2h+1}(2h+j)(2h+j+1)\alpha^j\beta^{-2h-j-1}&=\sum_{j=0}^{h}(2h+2j)(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1}\\&\quad+\alpha\sum_{j=0}^{h}(2h+2j+1)(2h+2j+2)\alpha^{2j}\beta^{-2h-2j-2}.\end{aligned} \tag{3.36}$$

where

$$H=\sum_{j=0}^{h}(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1},\qquad I=\sum_{j=0}^{h}(2h+2j+2)\alpha^{2j}\beta^{-2h-2j-2},$$
$$J=\sum_{j=0}^{h}(2h+2j)\alpha^{2j}\beta^{-2h-2j-2},\qquad K=\sum_{j=0}^{h}(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-3},\tag{3.37}$$
$$L=\sum_{j=0}^{h}(2h+2j)(2h+2j+1)\alpha^{2j}\beta^{-2h-2j-1},\quad N=\sum_{j=0}^{h}(2h+2j+1)(2h+2j+2)\alpha^{2j}\beta^{-2h-2j-2}.$$

$$\begin{aligned}&(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)\{\beta^2(H^2+\alpha^2I^2+2\alpha HI)+(LH+\alpha^2NI+\alpha(HN+LI))(\alpha^2b^2-\beta^2)\}\\&-2\alpha^4s_0^i\{\beta^2(HJ+\alpha^2IK+\alpha(HK+IJ))+(JL+\alpha^2NK+\alpha(JN+LK))(\alpha^2b^2-\beta^2)\}\\&+\alpha^2\{r_{00}(HL+\alpha^2NI+\alpha(LI+NH))+2\alpha^2S_0(JL+\alpha^2NK+\alpha(NJ+LK))\}(\alpha^2b^i-\beta y^i)=0.\end{aligned}\tag{3.38}$$

Separating the rational and irrational parts in y^i, we have

$$P'+\alpha Q'=0,\tag{3.39}$$

where

$$\begin{aligned}P'&=(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)\{\beta^2(H^2+\alpha^2I^2)+(LH+\alpha^2NI)(\alpha^2b^2-\beta^2)\}\\&\quad-2\alpha^4s_0^i\{\beta^2(HJ+\alpha^2IK)+(JL+\alpha^2NK)(\alpha^2b^2-\beta^2)\}\\&\quad+\alpha^2\{r_{00}(HL+\alpha^2NI)+2\alpha^2S_0(JL+\alpha^2NK)\}(\alpha^2b^i-\beta y^i)=0.\end{aligned}\tag{3.40}$$

$$\begin{aligned}Q'&=(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)\{\beta^2 2HI+(HN+LI)(\alpha^2b^2-\beta^2)\}\\&\quad-2\alpha^4s_0^i\{\beta^2(HK+IJ)+(JN+LK)(\alpha^2b^2-\beta^2)\}\\&\quad+\alpha^2\{r_{00}(LI+NH)+2\alpha^2S_0(NJ+LK)\}(\alpha^2b^i-\beta y^i)=0.\end{aligned}\tag{3.41}$$

From (3.38) we have

$$\gamma_{000}y^iH\{H-L\}=0,$$

that is $(2h+1)(1-4h^2)\gamma_{000}y^i\beta^{-(2+4h)}=\alpha^2\beta^{-4h}$, where $V_{-(2+4h)}$ is a $hp-(2+4h)$. Therefore there exists $hp(1):v_0$ satisfying

$$\gamma_{000}=v_0\alpha^2.\tag{3.42}$$

Next, eliminating $(\alpha^2\gamma_{0\,0}^{\,i}-\gamma_{000}y^i)$ from (3.40) and (3.41), we have

$$\begin{aligned}&-2\alpha^2s_0^i[\{\beta^2(HJ+\alpha^2IK)+(JL+\alpha^2NK)(\alpha^2b^2-\beta^2)\}\{\beta^2 2HI+(HN+LI)(\alpha^2b^2-\beta^2)\}\\&\quad+\{\beta^2(HK+IJ)+(JN+LK)(\alpha^2b^2-\beta^2)\}\{\beta^2(H^2+\alpha^2I^2)+(LH+\alpha^2NI)(\alpha^2b^2-\beta^2)\}\\&\quad+\{r_{00}(HL+\alpha^2NI)+2\alpha^2S_0(JL+\alpha^2NK)\}(\alpha^2b^i-\beta y^i)\{\beta^2 2HI+(HN+LI)(\alpha^2b^2-\beta^2)\}\\&\quad-\{r_{00}(LI+NH)+2\alpha^2S_0(NJ+LK)\}(\alpha^2b^i-\beta y^i)\\&\quad\times\{\beta^2(H^2+\alpha^2I^2)+(LH+\alpha^2NI)(\alpha^2b^2-\beta^2)\}]=0.\end{aligned}\tag{3.43}$$

The term of (3.43) which does not contain α^2 is

$$r_{00}y^i\{H^2(NH-LI)\}=0,$$

that is $2(h+1)(2h+1)^3\beta^{-(5+8h)}r_{00}y^i=0$. Therefore there exists a function

$$r_{00}=\rho\alpha^2. \tag{3.44}$$

Substituting (3.44) into (3.43) which does not contain α^2 is and transvecting it by b_i, we have

$$\begin{aligned}&-2s_0[(HJ+\alpha^2IK)\{\beta^2 2HI+(HN+LI)(\alpha^2b^2-\beta^2)\}\\&\quad+(HK+IJ)\{\beta^2(H^2+\alpha^2I^2)+(LH+\alpha^2NI)(\alpha^2b^2-\beta^2)\}]\\&\quad+\rho\{(HL+\alpha^2NI)2HI-(LI+NH)(H^2+\alpha^2I^2)\}(\alpha^2b^2-\beta^2)=0.\end{aligned} \tag{3.45}$$

The term of (3.45) which does not contain α^2 is

$$2s_0[H^2(3JI-JN+HK-LK)-2LIHJ]+\rho H^2(LI-NH)=0,$$

that is $(2h+1)^2\beta{-}(8h+6)[2s_0(64h^5+128h^4+96h^3+24h^2-10h-1)+\rho\beta(4h^2+6h+2)]=0$.

The above equation can be written as $(c'_1s_0+c'_2\rho\beta)\beta{-}(8h+6)$, where $c'_1=2(2h+1)^2(64h^5+128h^4+96h^3+24h^2-10h-1)$ and $c'_2=(4h^2+6h+2)$.

Therefore $c'_1s_0+c'_2\rho\beta=0$, that is, $c'_1s_i+c'_2\rho b_i=0$. Transvecting this equation by b_i, we have $c'_2\rho b^2=0$. Since $c'_2\neq 0$ for a positive integer, $\rho=0$ or $b^2=0$.

First, if $\rho=0$, then $s_0=0$, that is, $s_i=0$ and $r_{00}=0$ from (3.44). Therefore we have from $s^i_0=0$, that is $s_{ij}=0$. Hence $b_{i;j}=0$. Substituting $b_{i;j}=0$ and (3.42) into (3.40), we have $\gamma^i_{0\,0}=v_0y^i$, that is, the associated Riemannian space is projectively flat. Secondly, if $b^2=0$, we have easily the above result by the same method of the case of $r=2h$. Thus we have the following

Theorem 2. *A Finsler space F^n with the r-th series (α,β)-metric* (2.1) *provided* $\alpha^2\not\equiv 0(\text{mod }\beta)$ *is projectively flat if and only if*

(i) *when $r=2$, $b_{i;j}$ satisfies* (3.26) *and the Chrisroffel symbols of the associated Riemannian space are written in the form* (3.31).

(ii) *when $r>2$, $b_{i;j}=0$ and the associated Riemannian space is projectively flat.*

4. Conclusion

The knowledge of Finsler geometry already in the consideration of Riemann, to have a norm function depends homogeneously on a line element with its position. The present paper, We discussed the r-th series (α,β)-metric to be projectively flat on the basis of Matsumoto's results.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] P.L. Antonelli, R.S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Acad., Dordrecht (1993).

[2] L. Berwald, Über Fienslwiche und cartansche geometric IV, PRojective Krummung allegemeiner affiner Raume und Finslercher Raume sckalerrer krummung, *Ann. of Math.* **48** (1947), 755–781.

[3] M. Hashiguchi and Y. Ichijyo, On some special (α, β)-metric, *Rep. Fac. Sci. Kagasima Univ.* (*Math., Phys., Chem.*) **8** (1975), 39–46.

[4] I.-Y. Lee and H.-S. Park, Finsler spaces with infinite Series (α, β)-metric, *J. Korean Math. Soc.* **41** (2004), 567–589.

[5] M. Matsumoto, On Finsler space with Randers metric and special forms of importent tensors, *J. Math. Kyoto Univ.* **14** (1974), 477–498.

[6] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha press, Saikawa, Ōtsu, Japan (1986).

[7] M. Matsumoto, Projectively flat Finsler spaces with (α, β)-metric, *Rep. Math. Phys.* **30** (1991), 15–20.

[8] C. Shibata, On Finsler space with an (α, β)-metric, *J. Hokkaido Univ. of Education, IIA* **35** (1984), 1–16.