A Study on Linear Jaco Graphs

Research Article

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Abstract. We introduce the concept of a family of finite directed graphs which are derived from an infinite directed graph called the $f(x)$-root digraph. The $f(x)$-root digraph has four fundamental properties which are; $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$ and, if v_j is the head of an arc then the tail is always a vertex v_i, $i < j$ and, if v_k for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices v_ℓ, $k < \ell < j$ are tails of arcs to v_j and finally, the degree of vertex v_k is $d(v_k) = mk + c$. The family of finite directed graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and corresponding arcs) v_t, $t > n$. Hence, trivially we have $d(v_i) \leq mi + c$ for $i \in \mathbb{N}$.

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1. Introduction

For general notation and concepts in graph theory, we refer to [2, 3, 4, 8]. All graphs mentioned in this paper are simple, connected finite and directed graphs unless mentioned otherwise. We introduce the concept of a family of finite Jaco graphs which are directed graphs derived from the infinite Jaco graph (see Definition 1.1), called the $f(x)$-root Jaco graph. In this study we generalise a number of results found in [6, 7] because the aforesaid studies relate to special cases of linear Jaco graphs.

Definition 1.1. Let $f(x) = mx + c$; $x, m \in \mathbb{N}$, $c \in \mathbb{N}_0$. The family of infinite linear Jaco graphs denoted by $\{J_\infty(f(x)) : f(x) = mx + c; x, m \in \mathbb{N} \text{ and } c \in \mathbb{N}_0\}$ is defined by $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$,

$A(J_\infty(f(x))) \subseteq \{(v_i, v_j) : i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in A(J_\infty(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

From Definition 1.1 we note the $f(x)$-root Jaco graph has four fundamental properties which are:

(a) $V(J_\infty(f(x))) = \{v_i : i \in \mathbb{N}\}$ and,

(b) if v_j is the head of an arc then the tail is always a vertex v_i, $i < j$ and,

(c) if v_k for smallest $k \in \mathbb{N}$ is a tail vertex then all vertices v_ℓ, $k < \ell < j$ are tails of arcs to v_j and finally,

(d) the degree of vertex k is $d(v_k) = mk + c$.

Definition 1.2. For $f(x) = mx + c; x, m \in \mathbb{N}$ and $c \in \mathbb{N}_0$, we define the series $(c_{f(x),n})_{n \in \mathbb{N}_0}$ by

$$c_{f(x),0} = 0, \quad c_{f(x),1} = 1, \quad c_{f(x),n \geq 2} = \min\{k < n : mk + c_{f(x),k} \geq n\}.$$

The connection between the $f(x)$-root digraph $J_\infty(f(x))$ and the series $(c_{f(x),n})_{n \in \mathbb{N}_0}$ is explained by the following lemma.

Lemma 1.1. *Consider the Jaco graph $J_\infty(f(x))$ and let $n \in \mathbb{N}$ then the following hold:*

(a) $d^+(v_n) + d^-(v_n) = f(n)$.

(b) $d^-(v_{n+1}) \in \{d^-(v_n), d^-(v_n) + 1\}$.

(c) *If $(v_i, v_k) \in A(J_\infty(f(x)))$ and $i < j < k$, then $(v_j, v_k) \in A(J_\infty(f(x)))$.*

(d) $d^+(v_n) = ((m-1)n + c) + c_{f(x),n}$, $n \geq 2$.

Proof. (a) Invoking Definition 1.1 and since $d^+(v_n) = (f(n) + n) - n - d^-(v_n)$, the result is obvious.

(b) and (c): We prove (b) and (c) simultaneously through induction on n. First of all, $d^-(v_1) = 0$ implying $d^-(v_2) = 1 = d^-(v_1) + 1$.

Let $n \geq 2$ and assume result (b) holds for $m \leq n$ and (c) holds for $m \leq n - 1$. In particular, $d^-(v_n) > 0$. Let $\ell < n$ be minimal with $(v_\ell, v_n) \in A(J_\infty(f(x)))$, so $(f(\ell) + \ell) - d^-(v_\ell) \geq n$. Let $\ell < j < n$. By induction, we have $d^-(v_\ell) \leq d^-(v_j) \leq d^-(v_\ell) + j - \ell$ and $(f(j) + j) - d^-(v_j) \geq n$. Hence, and by choice of ℓ, we have $(v_k, v_n) \in A(J_\infty(f(x)))$ if and only if $\ell \leq k < n$, hence result (c) is valid for n, while $d^-(v_n) = n - \ell$ and $d^+(v_n) = (f(n) + n) - (n - \ell) = f(n) + \ell$.

If $(f(\ell) + \ell) - d^-(v_\ell) \geq n + 1$, then ℓ is minimal with $(f(\ell) + \ell) - d^-(v_\ell) \geq n + 1$. If $(f(\ell) + \ell) - d^-(v_\ell) = n$, we still, as $\ell + 1 \leq n$, have $d^-(v_{\ell+1}) \in \{d^-(v_\ell), d^-(v_{\ell)+1}\}$ and $(f(\ell+1) + \ell + 1) - d^-(v_{\ell+1}) \geq n + 1$. Either way, $(v_{\ell+1}, v_{n+1}) \in A(J_\infty(f(x)))$. If $\ell + 1 < j < n$, then induction yields $d^-(v_{\ell+1}) \leq d^-(v_j) \leq d^-(v_{\ell+1}) + (j - \ell - 1)$ and $(f(j) + j) - d^-(v_j) \geq n + 1$. As $d^-(v_n) \leq n - 1, (v_n, v_{n+1}) \in A(J_\infty(f(x)))$, so $(v_k, v_{n+1}) \in A(J_\infty(f(x)))$ whenever $\ell + 1 \leq k \leq n$. Depending on whether $(v_\ell, v_{n+1}) \in A(J_\infty(f(x)))$ or not, we obtain $d^-(v_{n+1}) = n + 1 - \ell = (n - \ell) + 1 = d^-(v_\ell) + 1$ or $d^-(v_{n+1}) = n + 1 - (\ell + 1) = d^-(v_n)$.

(d) Let $n \geq 3$, and, as before, choose ℓ minimal with $(v_\ell, v_n) \in A(J_\infty(a))$. We prove the result by induction on n and apply arguments very similar to the ones already used. First of all, $d^-(v_1) = 0$, and $d^+(v_1) = f(1) = ((m-1)1 + c) + c_{f(x),1}$. Now let $n > 1$, and, as before, choose ℓ minimal such that $(v_\ell, v_n) \in A(J_\infty(a))$. By (a) and (c), $d^-(v_n) = n - \ell$, and $d^+(v_n) = (mn + c) - n + \ell = ((m-1)n + c) + \ell$.

Induction yields that $d^+(v_k) = d^+(v_k) = ((m-1)k + c) + c_{f(x),k}$, whenever $k < n$. The definition of ℓ says that ℓ is minimal with $\ell + d^+(v_\ell) = ((m-1)\ell + c) + c_{f(x),\ell} \geq n$, which means that $\ell = c_{f(x),n}$. ☐

Corollary 1.2. *Note that* (a) *and* (b) *of Lemma* 1.1 *entail that* $d^-(v_{n+1}) = (n+1) - c_{f(x),n+1} \in \{n - c_{f(x),n}, n - c_{f(x),n} + 1\}$ *and that* (d) *then implies that the series* $(c_{f(x),n})$ *are well-defined and ascending, more specifically,* $c_{f(x),n+1} \in \{c_{f(x),n}, c_{f(x),n} + 1\}$, $(n \in \mathbb{N}_0)$.

Consider the function $g(x) = m_1 x + c_1$; $x, m_1 \in \mathbb{N}$, $c_1 \in \mathbb{N}_0$. In respect of $f(x)$ and $g(x)$ we have the following proposition.

Proposition 1.3. *Let* $k \in \mathbb{N}$, *and* $0 \leq g(x) < f(x)$. *Then* $c_{f(x),mk+c_{f(x),k}-g(x)} = k$.

Proof. Let $mk + c_{f(x),k} - g(x) = \ell$. Certainly, $mk + c_{f(x),k} \geq \ell$ hence, $c_\ell \leq k$. On the other hand, $((m-1)k+c) + c_{f(x),k} = (mk+c) + c_{f(x),k} - k < \ell$, so Corollary 1.2 says $((m-1)k+c) + c_{f(x),k-1} < \ell$ and $c_\ell = k$. ☐

2. Finite Linear Jaco Graphs

The family of finite linear Jaco graphs are those limited to $n \in \mathbb{N}$ vertices by lobbing off all vertices (and corresponding arcs) v_t, $t > n$. Hence, trivially we have $d(v_i) \leq f(i)$ for $i \in \mathbb{N}$.

Definition 2.1. The family of finite linear Jaco graphs denoted by $\{J_n(f(x)) : f(x) = mx + c;$ $x, m \in \mathbb{N}$ and $c \in \mathbb{N}_0\}$ is the defined by $V(J_n(f(x))) = \{v_i : i \in \mathbb{N}, i \leq n\}$, $A(J_n(f(x))) \subseteq \{(v_i, v_j) \mid i, j \in \mathbb{N}, i < j \leq n\}$ and $(v_i, v_j) \in A(J_n(f(x)))$ if and only if $(f(i) + i) - d^-(v_i) \geq j$.

Definition 2.2. Vertices with degree $\Delta(J_n(f(x)))$ is called Jaconian vertices and the set of vertices with maximum degree is called the Jaconian set of the linear Jaco graph $J_n(f(x))$, and denoted, $\mathbb{J}(J_n(f(x)))$ or, $\mathbb{J}_n(f(x))$ for brevity.

Definition 2.3. The lowest numbered (indexed) Jaconian vertex is called the prime Jaconian vertex of a linear Jaco graph.

Definition 2.4. If v_i is the prime Jaconian vertex, the complete subgraph on vertices $v_{i+1}, v_{i+2}, \cdots, v_n$ is called the Hope subgraph of a linear Jaco graph and denoted, $\mathbb{H}(J_n(f(x)))$ or, $\mathbb{H}_n(f(x))$ for brevity.

2.1 Basic Properties of Linear Jaco Graphs

Property 1. From the definition of a linear Jaco graph $J_n(f(x))$, it follows that, if for the prime Jaconian vertex v_i, we have $d(v_i) = f(i)$, then in the underlying Jaco graph denoted $J_n^*(f(x))$ we have $d(v_m) = f(m)$ for all $m \in \{1, 2, 3, \cdots, i\}$.

Property 2. From the definition of a linear Jaco graph $J_n(f(x))$, it follows that $\Delta(J_k(f(x))) \leq \Delta(J_n(f(x)))$ for all $k \leq n$.

Property 3. From the definition of a linear Jaco graph $J_n(f(x))$, it follows that the lowest degree attained by all Jaco graphs is $0 \leq \delta(J_n(f(x))) \leq f(1)$.

Property 4. The $d^-(v_k)$ for any vertex v_k of a linear Jaco graph $J_n(f(x)), n \geq k$ is equal to $d(v_k)$ in the underlying Jaco graph $J_k^*(f(x))$.

Note that henceforth, when the context is clear we interchangeably refer to a linear Jaco graph $J_n(f(x))$, (directed) and the underlying Jaco graph $J_n^*(f(x))$, (undirected) as a *Jaco graph*. Similar, when the context is clear we refer to either *arcs* or *edges*.
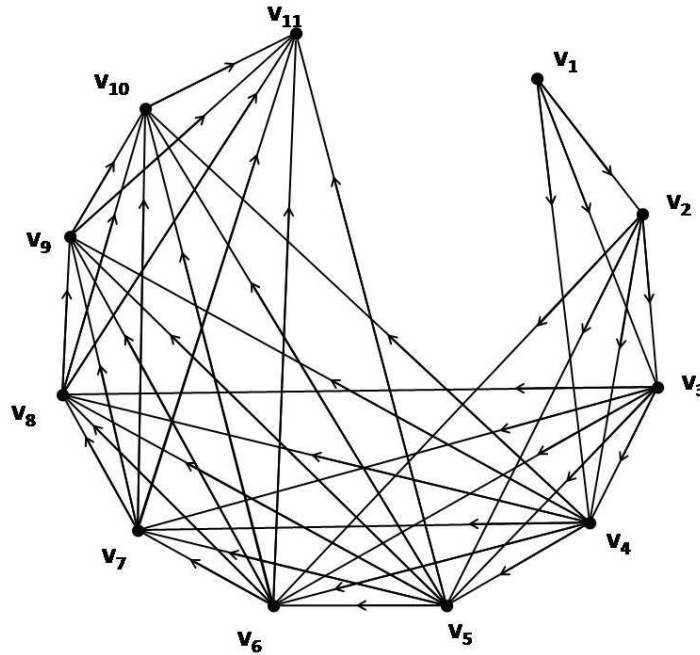

Figure 1

Illustration 1. *Figure 1 depicts the Jaco graph $J_{11}(f(x))$, $f(x) = 2x + 1$. We note that vertex v_4 is the prime Jaconian vertex and $\mathbb{J}_{11}(f(x)) = \{v_4, v_5, v_6\}$.*

2.2 Results on Linear Jaco Graphs

Lemma 2.1. *For the Jaco graphs $J_i(f(x))$, $i \in \{1, 2, 3, \ldots, f(1) + 1\}$ we have $\Delta(J_i(f(x))) = i - 1$ and $\mathbb{J}(J_i(f(x))) = \{v_k : 1 \leq k \leq i\} = V(J_i(f(x)))$.*

Proof. If $t = f(1) + 1$ then, $((f(1) + 1) + 1) - d^-(v_1) > (f(1) + 1)$ so from Definition 2.1 it follows that the arcs (v_1, v_i), $i = 2, 3, \ldots, (f(1) + 1)$ exist. It then follows that all arcs (v_i, v_j), $i < j$ exist. So the underlying graph $J_{f(1)+1}^*(f(x))$ is the complete graph $K_{f(1)+1}$. Since $\Delta(J_{f(1)+1}(f(x))) = (f(1) + 1) - 1 = f(1)$ and we have $d(v_i) = f(1)$ for all vertices in $K_{f(1)+1}$, it follows that $\Delta(J_{f(1)+1}(f(x))) = f(1)$ and $\mathbb{J}(J_{f(1)+1}(f(x))) = \{v_k : 1 \leq k \leq f(1)\} = V(J_{f(1)+1}(f(x)))$. The result follows similarly for $t < f(1) + 1$. $\square$

Proposition 2.2. *For the Jaco graphs $J_{f(i)}(f(x))$, $1 \leq i \leq f(1)$ the prime Jaconian vertex is the vertex v_i and $\mathbb{J}_{f(i)}(f(x)) = \{v_\ell : i \leq \ell \leq f(1) + 1\}$.*

Proof. Consider the Jaco graph $J_{f(1)}(f(x))$. It follows that the induced subgraph $J' = \langle v_i; 1 \leq i \leq v_{f(1)+1}\rangle$ is the complete graph $K_{f(1)+1}$, hence $d_{J'}(v_i) = f(1)$, $1 \leq i \leq f(1) + 1$. It follows immediately that v_1 is the prime Jaconian vertex of J' and $\mathbb{J}(J') = \{v_\ell : 1 \leq \ell \leq f(1) + 1\}$.

Also $f(i) > f(1)$, $2 \leq i \leq f(1)$ so besides the arcs in J', v_i has additional arcs (v_i, v_s); $f(1) + 2 \leq s \leq f(i) + 1$ in $J_{f(i)+1}(f(x))$. This amounts to $(f(i) - f(1)) - 1$ additional arcs $\forall\ v_\ell$, $i \leq \ell \leq f(1)$ in $J_{f(i)+1}(f(x))$. Clearly i is minimal such that arc $(v_i, v_{f(i)+1})$ exists so v_i is the prime Jaconian vertex of the Jaco graph $J_{f(i)+1}(f(x))$.

Because $d_{J'}((v_\ell) = f(1)$, $i \leq \ell \leq f(1) + 1$ it follows immediately that $d_{J'}(v_\ell) + (f(i) - f(1)) - 1 = d_{J'}(v_i) + (f(i) - f(1)) - 1$ in $J_{f(i)+1}(f(x))$. Hence, $\mathbb{J}_{f(i)}(f(x)) = \{v_\ell : i \leq \ell \leq f(1) + 1\}$. $\qquad\square$

Lemma 2.3. *If in a Jaco graph $J_n(f(x))$, and for smallest i, we have $d(v_i) = f(i)$ and the arc (v_i, v_n) is defined, then v_i is the prime Jaconian vertex of $J_n(f(x))$.*

Proof. If in the construction of a Jaco graph $J_n(f(x))$, and for smallest i, the arc (v_i, v_n) is defined, we have that in the underlying graph $J_n^*(f(x))$, $d(v_i) \leq f(i)$ and $d(v_j) \leq d(v_i)$ for all $j > i$. So it follows that $d(v_i) = \Delta(J_n(f(i)))$ so v_i is the prime Jaconian vertex of $J_n(f(x))$. $\qquad\square$

Lemma 2.4. *For all Jaco graphs $J_n(f(x))$, $n \geq 2$ and, $v_i, v_{i-1} \in V(J_n(f(x)))$ we have in the Jaco graph $J_n^*(f(x))$, that $|(d(v_i) - d(v_{i-1})| \leq m$.*

Proof. Clearly, $\max |(d(v_i) - d(v_{i-1})| = f(i) - f(i-1) = mi + c - (m(i-1) + c) = m$ hence, $|(d(v_i) - d(v_{i-1})| \leq m$. $\qquad\square$

Proposition 2.5. *For the Jaco graph $J_{f(i)}(f(x))$, $i \geq f(1) + 1$ the prime Jaconian vertex is a vertex v_j, $j \leq i$.*

Proof. We have that $f(i) - f(i-1) \geq 1$. From Lemma 1.1(b) it follows that $d^-(v_i) = d^-(v_{i-1})$ or $d^-(v_{i-1}) + 1$.

Case 1: Let $f(i) - f(i-1) = 1$.

    1.1: Let $d^-(v_i) = d^-(v_{i-1})$. It follows that $d_{J_{f(i)}}(v_{i-1}) = d_{J_{f(i)}}(v_i) + 1$ hence v_{i-1} is the prime Jaconian vertex. Let $j = i - 1$, so $j < i$.

    1.2: Let $d^-(v_i) = d^-(v_{i-1}) + 1$. It follows that $d_{J_{f(i)}}(v_{i-1}) = d_{J_{f(i)}}(v_i)$ hence v_{i-1} is the prime Jaconian vertex. Let $j = i - 1$, so $j < i$.

Case 2: Let $f(i) - f(i-1) = 2$.

    2.1: Let $d^-(v_i) = d^-(v_{i-1})$. It follows that $d_{J_{f(i)}}(v_{i-1}) = d_{J_{f(i)}}(v_i)$ hence v_{i-1} is the prime Jaconian vertex. Let $j = i - 1$, so $j < i$.

    2.1: Let $d^-(v_i) = d^-(v_{i-1}) + 1$. It follows that $d_{J_{f(i)}}(v_i) = d_{J_{f(i)}}(v_{i-1}) + 1$ hence v_i is the prime Jaconian vertex. So equality holds.

By similar reasoning it follows that for all $f(i) - f(i-1) \geq 3$ vetex v_i is the prime Jaconian vertex hence, the result is settled. $\qquad\square$

Note that the value of $\Delta(J_n(f(x)))$ might repeat itself as n increases to $n + 1$ and on an increase we always obtain, $\Delta(J_{n+1}(f(x))) = \Delta(J_n(f(x))) + m$.

Theorem 2.6. *The Jaco graph $J_k(f(x))$, $k = f((m+c)+1)+1$ is the smallest Jaco graph in $\{J_n(f(x)): n \in \mathbb{N}\}$ which has $\Delta(J_k(f(x))) = f((m+c)+1)$ with unique Jaconian vertex $v_{(m+c)+1}$.*

Proof. We will prove the result through *imbedded* induction.

Let $c = 0$ so consider the linear function $f(x) = mx$. For $m = 1$ let $f'(x) = x$. The graph $J_3(f'(x))$ is clearly the smallest Jaco graph in $\{J_n(f'(x)): n \in \mathbb{N}\}$ for which $\Delta(J_3(f'(x))) = (1+0)+1 = 2$ and $\mathbb{J}(J_3(f'(x))) = \{v_{((1+0)+1)}\} = \{v_2\}$. So the result holds for $m = 1$.

Assume the result holds for $m = \ell$, and let $f''(x) = \ell x$. So for the smallest Jaco graph $J_l(f''(x))$, $l = \ell(\ell+1)+1$ we have that $\Delta(J_l(f''(x))) = \ell(\ell+1)$ and $\mathbb{J}(J_l(f''(x))) = \{v_{(\ell+1)}\}$.

Now consider the linear function $f'''(x) = (\ell+1)x$. In the Jaco graph $J_l(f''(x))$ the vertex $v_{(\ell+1)+1} = v_{(\ell+2)}$ has $d(v_{(\ell+2)}) = d(v_{(\ell+1)}) - 1$. So in constructing the Jaco graph $J_l(f'''(x))$, amongst others the arc $(v_1, v_{(m+2)})$ is linked. So at least $v_{(\ell+1)}, v_{(\ell+2)} \in \mathbb{J}(J_l(f'''(x)))$. So $d(v_{(\ell+2)}) = \ell(\ell+1)$. If follows that the minimum number of additional vertices (smallest Jaco graph) say t, to be added to $J_l(f'''(x))$ to obtain $d(v_{(\ell+2)}) = (\ell+1)(\ell+2)$ and $\mathbb{J}(J_{(l+t)}(f'''(x))) = \{v_{(\ell+2)}\}$ in $J_{(l+t)}(f'''(x))$ is given by $t = (\ell+1)(\ell+2) - \ell(\ell+1) = 2(\ell+1)$. The number of vertices of $J_l(f''(x))$ is given by $\ell(\ell+1)+1$. Now $l + t = (\ell(\ell+1)+1) + 2(\ell+1) = (\ell+1)(\ell+2)+1$.

Clearly at least $v_{(\ell+2)} \in \mathbb{J}(J_k(f'''(x)))$, $k = l+t$, and $v_{(\ell+1)} \notin \mathbb{J}(J_k(f'''(x)))$ since $\ell(\ell+1) < (\ell+1)(\ell+2)$. In the construction of the Jaco graph $J_l(f'''(x))$, the arc $(v_1, v_{(\ell+3)})$ was not linked so $d(v_{(\ell+3)}) < d(v_{(\ell+2)})$ in $J_l(f'''(x))$. So it follows that $d(v_{(\ell+3)}) < d(v_{(\ell+2)})$ in $J_k(f'''(x))$, $k = l+t$. The latter implies that $J(J_k(f'''(x)))$ is the smallest such Jaco graph and $\mathbb{J}(J_k(f'''(x))) = \{v_{(\ell+2)}\}$.

Hence the result holds for $m = \ell+1$ implying it holds in general for $c = 0$.

Now let $c = 1$, and $f(x) = mx+1$. For $m = 1$ let $f'(x) = x+1$. Clearly $J_5(f'(x))$ is the smallest Jaco gaph in $\{J_n(f'(x)): n \in \mathbb{N}\}$ for which $\Delta(J_5(f'(x))) = f'(2)+1 = 4$ and $\mathbb{J}(J_5(f'(x))) = \{v_{(2+1)}\} = \{v_3\}$. So the result holds for $m = 1$.

Assume it holds for $m = \ell$, and let $f''(x) = \ell x+1$. So for the smallest Jaco graph $J_l(f''(x))$, $l = \ell^2 + 2\ell + 1$ we have that $\Delta(J_l(f''(x))) = \ell(\ell+2)$ and $\mathbb{J}(J_l(f''(x))) = \{v_{(\ell+2)}\}$.

Now consider the function $f'''(x) = (\ell+1)x+1$. In the Jaco graph $J_l(f''(x))$ the vertex $v_{(\ell+2)+1} = v_{\ell+3}$ has $d(v_{\ell+3}) = d(v_{\ell+2}) - 1$. So in the construction of the Jaco graph $J_l(f'''(x))$, amogst others the arc $(v_1, v_{\ell+3})$ is linked. So at least $v_{(\ell+2)}, v_{(\ell+3)} \in \mathbb{J}(J_l(f'''(x)))$. So $d(v_{(\ell+3)}) = \ell(\ell+2)$. If follows that the minimum number of additional vertices (smallest Jaco graph) say t, to be added to $J_l(f'''(x))$ to obtain $d(v_{(\ell+3)}) = (\ell+2)(\ell+3)$ and $\mathbb{J}(J_{(l+t)}(f'''(x))) = \{v_{(\ell+3)}\}$ in $J_{(l+t)}(f'''(x))$ is given by $t = (\ell+2)(\ell+3) - \ell(\ell+2) = 2(\ell+2)$. The number of vertices of $J_l(f''(x))$ is given by $(\ell+1)(\ell+2)+1$. Now $l + t = ((\ell+1)(\ell+2)+1) + 2((\ell+1)+1) = (\ell+2)(\ell+3)+1$.

Clearly at least $v_{(\ell+3)} \in \mathbb{J}(J_k(f'''(x)))$, $k = l+t$, and $v_{(\ell+2)} \notin \mathbb{J}(J_k(f'''(x)))$ since $(\ell+1)(\ell+2) < (\ell+2)(\ell+3)$. In the construction of the Jaco graph $J_l(f'''(x))$, the arc $(v_1, v_{(\ell+4)})$ was not linked so $d(v_{(\ell+4)}) < d(v_{(\ell+3)})$ in $J_l(f'''(x))$. So it follows that $d(v_{(\ell+4)}) < d(v_{(\ell+3)})$ in $J_k(f'''(x))$, $k = l+t$. The latter implies that $J(J_k(f'''(x)))$ is the smallest such Jaco graph and $\mathbb{J}(J_k(f'''(x))) = \{v_{(\ell+3)}\}$.

Hence the result holds for $m = \ell+1$ implying it holds in general for $c = 1$.

Now assume the result holds for $1 \le m \le \ell_1$ and $0 \le c \le c_\ell$. This means that for $f'(x) = \ell_1 x + c_\ell$ the smallest Jaco graph $J_l(f'(x))$ has $\Delta(J_l(f'(x))) = f'((\ell_1 + c_\ell) + 1) = \ell_1(\ell_1 + c_\ell) + 1) + c_\ell$. and $\mathbb{J})J_l(f'(x))) = \{v_{(\ell_1 + c_\ell)+1}\}$. Now consider the linear function $f''(x) = (\ell_1 + 1)x + (c_\ell + 1)$.

In the Jaco graph $J_l(f'(x))$ the vertex $v_{(\ell+c_\ell+2)+1} = v_{\ell+c_\ell+3}$ has $d(v_{\ell+c_\ell+3}) = d(v_{\ell+c_\ell+2}) - 1$. So in the construction of the Jaco graph $J_l(f''(x))$, amongst others the arc $(v_1, v_{\ell+c_\ell+3})$ is linked. So at least $v_{(\ell+c_\ell+2)}, v_{(\ell+c_\ell+3)} \in \mathbb{J}(J_l(f''(x)))$. So $d(v_{(\ell+c_\ell+3)}) = (\ell+c_\ell)(\ell+c_\ell+2)$. If follows that the minimum number of additional vertices (smallest Jaco graph) say t, to be added to $J_l(f''(x))$ to obtain $d(v_{(\ell+c_\ell+3)}) = (\ell+c_\ell+2)(\ell+c_\ell+3)$ and $\mathbb{J}(J_{(l+t)}(f''(x))) = \{v_{(\ell+c_\ell+3)}\}$ in $J_{(l+t)}(f''(x))$ is given by $t = (\ell+c_\ell+2)(\ell+c_\ell+3) - (\ell+c_\ell)(\ell+c_\ell+2) = 2(\ell+c_\ell+2)$. The number of vertices of $J_l(f'(x))$ is given by $(\ell+c_\ell+1)(\ell+c_\ell+2) + 1$. Now $l+t = ((\ell+c_\ell+1)(\ell+c_\ell+2)+1) + 2((\ell+c_\ell+1)+1) = (\ell+c_\ell+2)(\ell+c_\ell+3) + 1$.

Clearly at least $v_{(\ell+c_\ell+3)} \in \mathbb{J}(J_k(f''(x)))$, $k = l+t$, and $v_{(\ell+c_\ell+2)} \notin \mathbb{J}(J_k(f''(x)))$ since $(\ell+c_\ell+1)$ $\cdot(\ell+c_\ell+2) < (\ell+c_\ell+2)(\ell+c_\ell+3)$. In the construction of the Jaco graph $J_l(f''(x))$, the arc $(v_1, v_{(\ell=c_\ell+4)})$ was not linked so $d(v_{(\ell+c_\ell+4)}) < d(v_{(\ell+c_\ell+3)})$ in $J_l(f''(x))$. So it follows that $d(v_{(\ell+c_\ell+4)}) < d(v_{(\ell+c_\ell+3)})$ in $J_k(f''(x))$, $k = l+t$. The latter implies that $J(J_k(f''(x)))$ is the smallest such Jaco graph and $\mathbb{J}(J_k(f''(x))) = \{v_{(\ell+c_\ell+3)}\}$.

Hence the result holds for $m = \ell_1 + 1$ and $c_\ell + 1$ implying it holds in general for $f(x) = mx + c$. □

3. Number of Arcs of the Finite Jaco Graphs
$\{J_n(f(x)) : f(x) = mx + c; x, m \in \mathbb{N}$ **and** $c \in \mathbb{N}_0\}$

Note, in [6, 7] it was suggested that Theorem 3.7 combined with Binet's formula perhaps amounts to a closed formula for $d^+(v_n)$. The algorithms discussed in Ahlbach et al. [1] suggest that a general closed formula might not be found despite Binet's formula being closed. Theorem 3.8 provides a result for a subset of the family of Jaco graphs $J_n(f(x))$. First we present results for $c = 0$ hence for $f(x) = mx$.

Proposition 3.1. *The number of arcs of a Jaco graph $J_\ell(mx) = \frac{1}{2}\ell(\ell-1)$ if $\ell \le m + 1$.*

Proof. If $\ell = m + 1$ then, $((m+1)+1).1 - d^-(v_1) > (m+1)$ so from Definition 2.1 it follows that the arcs (v_1, v_i), $i = 2, 3, \ldots, (m+1)$ exist. It then follows that all arcs (v_i, v_j), $i < j$ exist. So the underlying graph $J^*_{m+1}(m)$ is the complete graph K_{m+1} hence, $\epsilon(J_{m+1}(m)) = \frac{1}{2}m(m+1) = \frac{1}{2}\ell(\ell-1)$. The result follows similarly for $\ell < m + 1$. □

Theorem 3.2. *If for the Jaco graph $J_n(mx)$, we have $\Delta(J_n(mx)) = k$, then $\epsilon(J_n(mx)) = \epsilon(\mathbb{H}(J_n(mx))) + \sum\limits_{i=1}^{k} d^+(v_i)$.*

Proof. For $n = 1$, $d^+(v_1) = 0$ and $J_1(mx)$ is the arcless graph on vertex v_1 whilst $\mathbb{H}(J_1(mx))$, is an empty graph so $A(\mathbb{H}(J_1(mx))) = \emptyset$, implying $\epsilon(\mathbb{H}(J_1(mx))) = 0$. Thus the result holds.

For $n = 2$, the Jaco graph $J_2(mx)$ has the prime Jaconian vertex v_1 and $d^+(v_1) = 1$. $\mathbb{H}(J_2(mx))$ is the null graph on vertex v_2 so $A(\mathbb{H}(J_2(a))) = \emptyset$, implying $\epsilon(\mathbb{H}(J_2(mx))) = 0$. Thus the result holds.

Now assume it holds for all vertices v_i, $i \le k - 1$. Thus vertex v_k has attained its in-degree $d^-(v_k)$. To attain $d(v_k) = \Delta(J_n(mx))$, exactly $\Delta(J_n(mx)) - d^-(v_k) = d^+(v_k)$ arcs can be linked additionally. So the result holds for vertices $v_1, v_2, v_3, \cdots, v_k$.

Clearly we also have $A(\mathbb{H}(J_n(mx))) \subset A(J_n(mx))$. Hence, $\epsilon(J_n(mx)) = \epsilon(\mathbb{H}(J_n(mx))) + d^+(v_1) + d^+(v_2) + d^+(v_3) + \cdots + d^+(v_k)$.

Since it is known that $\epsilon(\mathbb{H}(J_n(mx))) = 1 + 2 + 3 + \cdots + (n - \Delta(J_n(mx)) - 1)$ the result can be written as $\epsilon(J_n(mx)) = \frac{1}{2}(n-k)(n-k-1) + \sum\limits_{i=1}^{k} d^+(v_i)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.3. *The number of arcs of a Jaco graph $J_n(mx)$ having vertex v_i as the prime Jaconian vertex, can also be expressed recursively as*

$$\epsilon(J_{n+1}(mx)) = \begin{cases} \epsilon(J_n(mx)) - i + n & \text{if } d(v_i) = mi \\ \epsilon(J_n(mx)) - i + (n+1) & \text{if } d(v_i) < mi \end{cases}$$

Proof. Consider the Jaco graph $J_n(mx)$ having vertex v_i as the Jaconian vertex.

Case 1: If $d(v_i) = mi$ and the vertex $v_{(n+1)}$ is added to construct $J_{(n+1)}(f(x))$ only the arcs $(v_{(i+1)}, v_{(n+1)}), \ldots, (v_n, v_{(n+1)})$ can be linked additionally. This amounts to $(n-i)$ arcs.

Case 2: If $d(v_i) < mi$ and the vertex $v_{(n+1)}$ is added to construct $J_{(n+1)}(mx)$ only the arcs $(v_i, v_{(n+1)}), (v_{(i+1)}, v_{(n+1)}), \ldots, (v_n, v_{(n+1)})$ can be linked additionally. This amounts to $(n-i+1)$ arcs. $\qquad\qquad\square$

Lemma 3.4 (see [7]). *We find for $m = 1$ and the series $(a_n)_{n \in \mathbb{N}_0}$ defined by*

$$a_0 = 0, \quad a_1 = 1, \quad a_{n \geq 2} = \min\{k < n : k + a_k \geq n\},$$

that Lemma 1.1 *changes to:*

(a) $d^+(v_n) + d^-(v_n) = n$.

(b) $d^-(v_{n+1}) \in \{d^-(v_n), d^-(v_n) + 1\}$.

(c) *If* $(v_i, v_k) \in A(J_\infty(x))$ *and* $i < j < k$, *then* $(v_j, v_k) \in A(J_\infty(x))$.

(d) $d^+(v_n) = a_n$, $n \geq 2$.

Corollary 3.5. *Note that* (a) *and* (c) *above entail that* $d^+(v_{n+1}) = n + 1 - d^-(v_{n+1}) \in \{n - d^-(v_n), n - d^-(v_n) + 1\}$ *and that* (d) *then implies that the series* (a_n) *is well defined and ascending, more specifically,* $a_{n+1} \in \{a_n, a_n + 1\}$, $(n \in \mathbb{N}_0)$.

Lemma 3.6. *Let $i \in \mathbb{N}$. Then $d^+(v_{i+d^+(v_i)}) = i = d^+(v_{i+d^+(v_{i+d^+(v_{i-1})})})$.*

Proof. Let $i + d^+(v_i) = k$. Certainly, $i + d^+(v_i) \geq k$, so $d^+(v_k) \leq i$. From Lemma 3.4 it follows that $i - 1 + d^+(v_{i-1}) \leq i - 1 + d^+(v_i) < i + d^+(v_i)$ so $d^+(v_k) \geq i$. Let $\ell = i + d^+(v_{i-1})$. Since $d^+(v_i) \geq d^+(v_{i-1})$ we have, $d^+(v_\ell) \leq i$ and since, $i - 1 + d^+(v_{i-1}) < \ell$ we have, $d^+(v_\ell) = i$. $\qquad\square$

The next theorem with proof is repeated here. It appears in [6, 7] which are announcement papers only. Applicable reading related to Fibonacci numbers and Zeckendorf representations is found in [5, 9].

Theorem 3.7 (Bettina's Theorem (also see[6, 7])[1]). *Let $\mathbb{F} = \{f_0, f_1, f_2, \ldots\}$ be the set of Fibonacci numbers and let $n = f_{i_1} + f_{i_2} + \ldots + f_{i_r}, n \in \mathbb{N}$ be the Zeckendorf representation of n. Then*

$$d^+(v_n) = f_{i_1-1} + f_{i_2-1} + \ldots + f_{i_r-1}.$$

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[1]The first author wishes to dedicate this theorem to Dr. Bettina Wilkens, Department of Mathematics, University of Botswana

Proof. Through induction we have that first of all, $1 = f_2$ and $d^+(v_1) = 1 = f_1$. Let $2 \leq n = f_{i_1} + f_{i_2} + \ldots + f_{i_r}$ and let $k = f_{i_1-1} + f_{i_2-1} + \ldots + f_{i_r-1}$. If $i_r \geq 3$, then $k = f_{i_1-1} + f_{i_2-1} + \ldots + f_{i_r-1}$ is the Zeckendorf representation of k, such that induction yields $d^+(v_k) = k = f_{i_1-2} + f_{i_2-2} + \ldots + f_{i_r-2}$. Since $k + d^+(v_k) = f_{i_1-1} + f_{i_1-2} + f_{i_2-1} + f_{i_2} - 2 + \ldots + f_{i_r-1} + f_{i_r-2} = f_{i_1} + f_{i_2} + \ldots + f_{i_r} = n$, read with Lemma 3.6 yields $d^+(v_n) = k$.

Finally consider $n = f_{i_1} + f_{i_2} + \ldots + f_{i_r}$, $i_r = 2$. Note that $n > 1$ implies that $i_{r-1} \geq 4$ and that the Zeckendorf representation of $n-1$ given by $n-1 = f_{i_1} + f_{i_2} + \ldots + f_{i_{r-1}}$. Let $k = d^+(v_{n-1})$. Through induction we have that, $k = f_{i_1-1} + f_{i_2-1} + \ldots + f_{i_{r-1}-1}$, and since $i_{r-1} \leq 4$, this is the Zeckendorf representation of k. Accordingly, $d(v_k) = f_{i_1-2} + f_{i_2-2} + \ldots + f_{i_{r-1}-2}$, and $k + d^+(v_k) = f_{i_1-1} + f_{i_1-2} + f_{i_2-1} + f_{i_2} - 2 + \ldots f_{i_{r-1}-1} + f_{i_{r-1}-2} = n-1$. It follows that $d^+(v_n) > k = d^+(v_{n-1})$. From Corollary 3.5 it follows that $d^+(v_n) = k + 1 = (f_{i_1-1} + f_{i_2-1} + \ldots + f_{i_{r-1}-1}) + f_1 = f_{i_1-1} + f_{i_2-1} + \ldots + f_{i_r-1}$. $\square$

Since $d^-(v_n) = n - d^+(v_n)$, the number of edges is also given by $\epsilon(J_n(1)) = \frac{1}{2}n(n+1) - \sum\limits_{i=1}^{n} d^+(v_i)$. Furthermore, for $n \geq 2$ we have $d^+(v_1) = 1$, so we rather consider $\epsilon(J_n(1)) = (\frac{1}{2}(n(n+1) - 1) - \sum\limits_{i=2}^{n} d^+(v_i)$. Bettina's theorem can be applied to the last term to determine the number of arcs of $J_n(x)$.

Illustration 2. For the Jaco Graph $J_{15}(x)$ we have

$$\epsilon(J_{15}(1)) = \frac{1}{2} \cdot 15 \cdot (15 + 1) - 1 - \sum\limits_{i=2}^{15} = 119 - \sum\limits_{i=2}^{15} d^+(v_i).$$

Now, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_4 + f_2 = 4$, $f_5 = 5$, $f_5 + f_2 = 6$, $f_5 + f_3 = 7$, $f_6 = 8$, $f_6 + f_2 = 9$, $f_6 + f_3 = 10$, $f_6 + f_4 = 11$, $f_6 + f_4 + f_2 = 12$, $f_7 = 13$, $f_7 + f_2 = 14$ and $f_7 + f_3 = 15$.

From Bettina's Theorem it follows that;

$$\sum\limits_{i=2}^{15} d^+(v_i) = f_2 + f_3 + (f_3 + f_1) + f_4 + (f_4 + f_1) + (f_4 + f_2) + f_5 + (f_5 + f_1) + (f_5 + f_2)$$
$$+ (f_5 + f_3) + (f_5 + f_3 + f_1) + f_6 + (f_6 + f_1) + (f_6 + f_2)$$
$$= 5f_1 + 4f_2 + 4f_3 + 3f_4 + 5f_5 + 3f_6$$
$$= f_1 + 5f_3 + 5f_5 + 3f_7 = 75.$$

So, $\epsilon(J_{15}(1)) = 119 - 75 = 44$.

Now we present a result for a special case in respect of $f(x) = mx + c$.

Theorem 3.8. *For the Jaco graph* $J_{f(f(1)+1)+1)+1}(f(x))$ *we have that:*

$$\epsilon(J_{f(f(1)+1)+1)+1}(f(x))) = \epsilon(K_{f(1)+1}) + \epsilon(K_{f(f(1)+1))-f(1)}) + \frac{1}{2}mf(1) \cdot (f(1) - 1),$$

alternatively,

$$\epsilon(J_{m^2+m(c+1)+2}(f(x))) = \epsilon(K_{(m+c)+1}) + \epsilon(K_{m^2+m(c-1)-c+2}) + \frac{1}{2}m(m+c) \cdot ((m+c) - 1).$$

Proof. Clearly the Jaco graph $J_{f(1)+1}(f(x))$ is the complete graph $K_{f(1)+1} = K_{m+c+1}$, so the first term including the first alternative term, follows. Clearly the induced subgraph $\langle v_i : f(1) + 1 \le i \le f(f(1)+1)+1 \rangle$ is the complete graph $K_{(f(f(1)+1)+1)-(f(1)+1))} = K_{m^2+m(c-1)-c+2}$, so the second term including the second alternative term, follows.

Consider the graph J' to be the graph obtained by appending the two complete graphs above at the common vertex $v_{f(1)+1}$. To obtain the Jaco graph $J_{f(f(1)+1)+1)+1}(f(x))$ from graph J' we have to add all defined arcs from vertices $v_j \in \{v_i : 2 \le i \le f(1)\}$ to vertices $v_k \in \{v_\ell : f(1) + 2 \le \ell \le f(f(1)+1)\}$. Choose any such vertex v_j and we have $d_{J'}(v_j) = f(1) = m + c$. Relabel the vertex v_j to carry index $j - 1$ and denote the vertex v_{j-1}^*. So now we have that $d_{J'}(v_{j-1}^*) = m + c$. Since $d_{J_{f(f(1)+1)+1)+1}(f(x))}(v_j) = mj + c$, the additional $mj + c - (m+c) = m(j-1)$ arcs must be added.

Hence, $\sum_{i=2-1}^{m+c-1} mi = m \cdot \sum_{i=2-1}^{m+c-1} i$ arcs are required. This simplifies to $m \cdot \sum_{i=1}^{m+c-1} i = m \cdot (\frac{1}{2}(m+c)(m+c-1)$ arcs. By this the result is settled. □

3.1 Linear Function Corresponding to a Linear Jaco Graph

It is noted that if for a sufficiently large linear Jaco Graph $J_n(f(x))$ we have two vertices v_i, v_j for which $d(v_i) = f(i)$ and $d(v_j) = f(j)$ then the linear function $f(x)$ can be derived by solving the simultaneous equations:

$$mi + c = d(v_i), \quad mj + c = d(v_j).$$

The smallest linear Jaco graph for which this is possible is for $J_{f(2)+1}(f(x))$ hence, knowing that $d(v_2) = f(2)$ in the given linear Jaco graph.

Proposition 3.9. *If for a linear Jaco graph we have that $d(v_i) = f(i)$ and $d(v_{i+1}) = f(i+1)$ then for maximum i', j' for which the arcs $(v_i, v_{i'}), (v_{i+1}, v_{j'})$ exist, we have $j' - i' \in \{m, m+1\}$.*

Proof. Let $d(v_i) = f(i)$ and $d(v_{i+1}) = f(i+1)$ in a sufficiently large linear Jaco graph $J_n(f(x))$. Since $d(v_{i+1}) - d(v_i) = m(i+1) + c - mi - c = m$ and from Lemma 1.1(b) we have $d^-(v_{i+1}) \in \{d^-(v_i), d^-(v_i) + 1\}$, the result follows. □

Each positive integer k can be written as $k + 1$ sums of non-negative integers $m + c$, $m \ge 0, c \ge 0$. If in Definition 1.1 we relax the lower limit on m and allow $m \ge 0$ we say for a given k that the linear Jaco graphs corresponding to the functions $f_i(x) = m_i x + c_i$, $m_i + c_i = k$ and $1 \le i \le k + 1$, are f-related Jaco graphs.

For $m = 0$ and $c \ge 0$ we have two special classes of disconnected linear Jaco graphs. For $c = 0$ the Jaco graph $J_n(0)$ is a null graph (*edgeless graph*) on n vertices. For $c = k > 0$, the Jaco graph $J_n(k) = \bigcup_{\lfloor \frac{n}{k+1} \rfloor\text{-copies}} K_{k+1} \cup K_{n-(k+1)\cdot\lfloor \frac{n}{k+1} \rfloor}$.
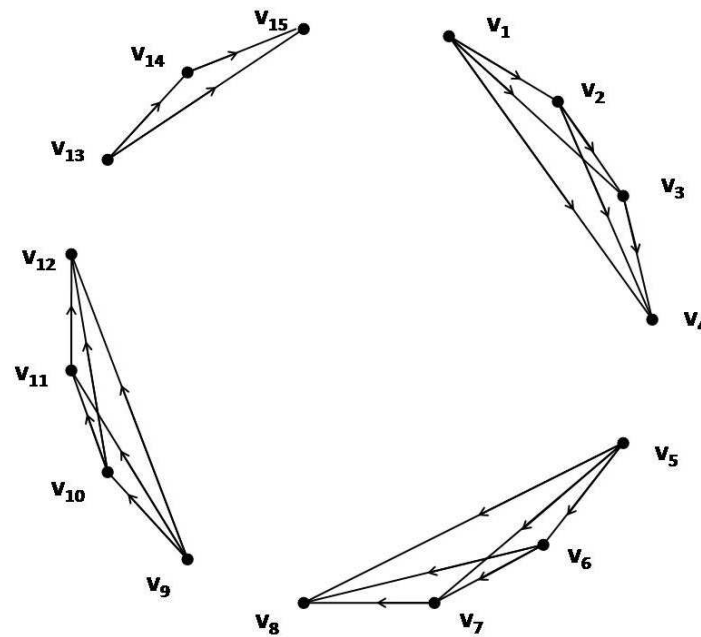
Figure 2

Illustration 3. *Figure 2 depicts the linear Jaco graph* $J_{15}(3) = K_4 \cup K_4 \cup K_4 \cup K_3$.

We note that a complete graph K_n is a linear Jaco graph to any linear function $f(x) = mx + c$ if $n \leq m + c + 1$. And, if K_n is a f-related linear Jaco graph, the complete graph corresponds to any of the $m + c + 1$ defined linear functions. It implies that we need at least a two-component subgraph of complete graphs say $K_s \cup K_t$ to derive the unique linear function $f(x) = k - 1$, $k = \max\{s, t\}$.

4. Conclusion

Thus far all studies of Jaco graphs were indeed on linear Jaco graphs. We largely generalised the results announced in [6, 7]. Also note that the Jaco graphs $J_n(1)$ and $J_n(a)$ as discussed in [6, 7] have a slightly different meaning to that defined in this paper. This might make comparative reading somewhat confusing. The generalisation suggests uniform notation. Amongst the many open problems that exist within the family of Jaco graphs the authors view, finding a closed formula for the number of arcs (edges) of a Jaco graph the most challenging and interesting. Perhaps Theorem 3.8 lays the foundation to further seek such closed formula.

It is proposed that the generalisation of the concepts to polynomial Jaco graphs $J_n(f(x))$ where $f(x) = \sum_{i=1}^{t} a_i x^i + c$, $a_i, x \in \mathbb{N}$ and $c \in \mathbb{N}_0$, will be worthy to study. Definition 1.1 will have to change to define the orientation of edges resulting from $f(x) < 0$. Also, given a sufficiently large polynomial Jaco graph, deriving the polynomial function can be formalised.

It is meant to be an *introductory paper* to encourage further research.

Competing Interests

The authors declare that they have no competing interests.

Author's Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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[Remark. The concept of Jaco Graphs followed from a dream during the night of 10/11 January 2013 which was the first dream Kokkie (first author) could recall about his daddy after his daddy passed away in the peaceful morning hours of 24 May 2012, shortly before the awakening of Bob Dylan, celebrating Dylan's 71st birthday.]