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Research Article

# Production Inventory Model for Time Dependent Holding Cost, Selling Price Demand Rate, Controllable Deterioration Rate, and Multiple Market Demand with Shortages

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**Abstract.** For current competitive market conditions, the production system must produce the ideal product. A production system's ability to produce a flawless final product depends critically on the quality of its raw materials. Using an excellent production-inventory model of a producer of a decaying item distributing goods to several markets with varying selling seasons, this article has investigated this issue. We have provided a solution approach to determine the best creation plan for finished goods and the best production plan for raw materials. For decaying items, a deterministic inventory model is created where shortages are permitted. In this model, the mandate rate is constant, and holding cost is a linear occupation of time. The purpose of this paper is to create a production-inventory model of instantaneous deteriorating items with multiple-market requests and controllable deterioration rates. Highlights of the model are to lower the overall cost of an inventory; this article has taken an analytical method. Lastly, a sensitivity analysis was conducted to investigate how various model parameters would affect the best course of action. Furthermore, we have elucidated mathematical models and assessed affectability to validate the identified models. Based on the model analysis, business enterprises can optimize the entire inventory price of deteriorating items inventory with a controllable deterioration rate. The model's solution is shown to be fairly stable. to determine the best time to restock raw supplies and the best way to produce finished goods.

**Keywords.** Inventory, Deteriorating items, Shortages, Time-varying holding cost, Preservation technology, Multiple Market demand

**Mathematics Subject Classification (2020).** 90B05, 90C30, 90B035

# 1. Introduction

There are three sections in the introduction. The purpose of the research is covered in the first section, and a reported literature review is covered in the second. The description of the research contributions appears in the third and final section.

## 1.1 Motivation (General Problem Description)

The most captivating and extensively studied subject in production, as well as operation management, is inventory. Because it has an impact on our daily lives, inventory is crucial. It may be found in homes, businesses, and social settings alike. Flexibility is offered by inventory, but it is not free. Products accumulated to use to meet future demand are referred to as inventory. The term inventory refers to the physical stock of goods and materials in any business. It includes the stock of goods available for sale as well as the raw materials used to make the goods available for sale. Inventory is a crucial asset for any firm. Its prime objective is to minimize the cost of the firm by maximizing profit. All types of goods or services used by any business organization to sell in the market to earn a profit are called Inventory. However, the assets that are used to earn that profit are not kept in the category of inventory. For example, if an item is produced in a production plant with the help of a machine, then the final product will be treated as inventory and the machine used for it is an asset and not inventory.

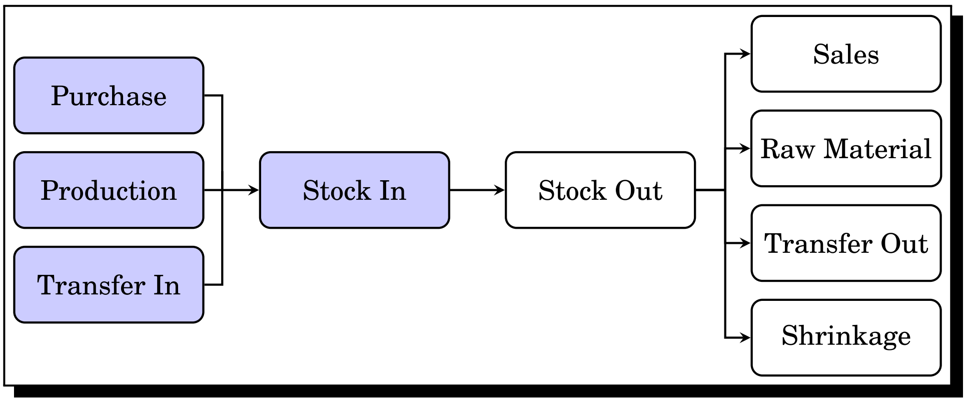


**Figure 1.** Process representation of inventory

Inventory in the construction industry is not only the manufactured product, the product that is suitable for sale but also all the raw materials used in the preparation of each form of concrete, which is also a form of inventory. For example, after preparing sugar, and jaggery, these will also be called inventories, and to make them, sugarcane, etc. will go into inventory as raw material. In today's global business environment, one of the most important components involved in the success of any business is inventory management. The fundamental role of inventory in any business is to bridge the gap between demand and supply and facilitate equilibrium. Inventory control and inventory visibility are two very important elements in any business that cannot be ignored as they act as cost drivers and the balance sheet is directly affected by them. The most timely and reliable inventory method is the load-up methodology, which has been around since the 2nd decade of the past century. It might be agreed upon that the 1950s to 2000s were an excellent period for inventory study. When the inventory problem was expressed theoretically and quantitatively, it was extended. A few inquiries reveal that requests for changes in conditions were made on their own, independently of time and expense. A few analysts have considered how interest, cost, and time together affect the best possible arrangements.

### 1.2 Literature Review

Here, we highlight the key exploration areas that apply to this paper. This subsection looks at the crucial research that has been done in the area of the provided stock models: (1) Deterioration, (2) Preservation technique, (3) Time-dependent holding cost, (4) Shortage facilities, (5) Multiple market demand.

Deterioration is the ongoing process of everything, varying in intensity depending on the item. Numerous inventory models exist for decomposing objects; the most important of these takes into account a steady rate of deterioration over an extended period, practically speaking. Examples of such items are unstable fluids and horticulture items. Deterioration has a significant impact on many inventory models. Products such as food items, medications, and radioactive materials are examples of models where substantial degradation can occur during the units' appropriate stockpiling period. Anyway, very few studies have looked at the effects of investing in slowing down the rate of item deterioration. The primary creation portion size was precisely constructed by Mishra and Singh [18] at a steady and fluctuating rate of deterioration. The production models demand that everything be produced to the highest standards and that nothing be subpar. An imperfect manufacturing system was studied by Khara *et al*. [15], and the system generated both perfect and defective products. Harvests from factories are often inspected using a 100% transmission method, and any defective goods are separated from the rest of the production. Product defects are either rectified or offered for sale at a discounted rate; all flawless products are retailed. The effects of assembling out-of-order items on a production model were examined by Benkherouf *et al*. [3]. They surmised that once the regular creation cycle was completed, a constant rate of revamps caused damage to objects. Scientists and academics have recently looked at inventory concerns using a variety of methods, including Dizbin and Tan [8], Singh and Mukherjee [29], and Bera and Jana [4]. The machine creates a defective product in this defective, degrading production system, endangering superiority decline. Certain authors took non-instantaneous degradation into account. For example, Singh *et al*. [35] spoke about how a control issue determines when to develop an accessible component, which restricts the usual cost of the framework in a inventory management considering demand and supply uncertainties constant state. The state-subordinate edge strategy is the best control strategy. Chandramohan *et al*. [7] have taken into consideration cost and time-subordinate interest for non-instantaneous decaying items. The problem addressed by Singh *et al*. [33] was figuring out the best replenishment strategy for a steady rate of degradation with a stock-dependent request that relaxes the fatal stage of ending inventory.

In actuality, the rate at which an item deteriorates is managed by different preservation strategies and slowed down by different efforts including adjustments to procedures and acquisition of specialized equipment, among other things. *Preservation Technology* (PT) is a crucial factor to consider when it comes to the long-term preservation of the product and how it can slow down the rate of deterioration. PT can do this primarily by reducing financial losses, improving customer service, and stepping up marketing efforts. A small number of researchers looked into the impact of investing in preservation technologies. Mishra *et al*. [19], He and Huang [12], and Singh *et al*. [34] conducted analyses. A production model under an unknown preservation methodology approach was evaluated by Sean *et al*. [28]. Deteriorating objects with the inventory-hooked-up call is currently the greatest score replenishment protection as well as preservation era investment for a non-right, according to Bardhan *et al*. [2]. Mashud *et al*. [17] made investments in partial rain check systems, pre-arranged settlements, and non-momentary crumbling. The results and affectability analysis demonstrate how the absolute stock model benefit is positively impacted by controllable fossil fuel byproducts and degradation. A problematic integrated industrial system dealing with fractional backorder for a defective item and protective conservation was described by Taleizadeh [38]. Regression techniques have been used by Singh *et al*. [32] to create an analytics model for the information flow value on manufacturing coordination among members of a three-layer, multi-channel and multi-echelon supply chain, consisting of manufacturers, distributors and retailers.

Keeping products or inventory in a warehouse comes with a cost known as inventory holding costs. One liability that drives up operating expenses and reduces profit margins for organizations is stored inventory. Rent for space, insurance, depreciation, and security are some of the costs related to maintaining goods. As these costs rise, businesses should use demand sensing and demand planning. These technologies help businesses keep the right amount of inventory. Holding expenses fluctuate from time to time. Depending on the state of the market. Accordingly, we use time-dependent linear holding costs to control the competition in the market. This study takes this road by taking into account various market demands with varying creation levels. Holding expenses can never be constant in a market environment. It alters linearly an inventory system is effective when the cost of holding a product is connected to its length of holding and is dependent on time. Sarkar and Lee [26] showed that the two manufacturers who create related things at different speeds and with different packaging strategies assume important roles inside the inventory network. Sarkar *et al*. [27] developed the effect of flexible conception rate, which is similar to the effect of time-dependent holding cost. Alfares and Ghaithan [1] show that time-dependent holding cost assumes a considerable part. What portion of the store network model would be affected by the time-and request-dependent generation rate that determines holding costs for both the producers and anyone who would need to supply essential items? This raises an essential inspection hole. According to Pervin *et al*. [23] and Hariom *et al*. [10], a time-dependent interest method is capable of demonstrating the suitability of our suggested model when it comes to appreciation for time. The main purpose of the article is to investigate the renewal options that the optimal retailer should have for deteriorating goods, comprising of time-dependent. According to Cárdenas-Barrón *et al.*'s analysis [5], holding costs may have an indivisible impact on output. More specifically, it increases over time since a large capacity area necessitates more expensive stockroom offices.

In a similar vein, several academics have studied numerous markets and multiple variates of demands. On the other hand, take into account a production inventory model for deteriorating things, as different markets require different rates of creation and preservation innovation in tandem. In their explanation of a standard EPQ (Economic Production Quantity) model with deteriorating raw materials, Nasr *et al*. [20] looked into the effects of the production progression's deterioration. The systematic raw material for several production cycles is the case of the EPQ corrupt case with no decline. A production inventory model was examined by Duari and Chakraborti [9], whereas demand is determined by the selling price as well as a variety of market wants. They calculated the level of quality degradation beginning from the item's age of appearance in stock. Discrete conveyance systems, enforced spaces, and a multi-item production model were deduced by Nobil *et al*. [21] and Singh *et al*. [30]. They made improvements to their model using a genetic computation. Remanufacturing cycles, according to Uthayakumar and Tharani [41], may prevent removal expenses and lessen environmental concerns. To combat this issue, a financial creation amount model that weakens items with modification and diverse cremation arrangements has been presented. Finally, some of the comparable works are listed in Table 1.

**Table 1.** Recent and related work to inventory literature

| Authors | Deterioration (Constraints) | Holding Cost | Demand Function | Preservation | Shortage |
|---|---|---|---|---|---|
| Karimi and Sadjadi [14] | Discrete | Constant | Time-dependent | No | Allowed |
| Hariom *et al*. [11] | Constant | Constant | Disruption | No | No |
| He *et al*. [13] | Constant | Constant | Constant | No | No |
| Malik and Sharma [16] | Constant | Constant | Linear | No | Allowed |
| Sajadifar and Hashempour [25] | Constant | Constant | Constant | No | Allowed |
| Singhal and Singh [36] | Weibull distribution | Linear increasing | ——— | No | Not Allowed |
| Chakraborty *et al*. [6] | Constant over time | Constant | Stock dependent | No | Allowed |
| Pal *et al*. [22] | Constant | Constant | Time-dependent | No | Not Allowed |
| This paper | Controllable | Linear | Constant | Yes | Allowed |

This study provides a multiple-market inventory model for decomposing objects that has a consistent, predictable rate of deterioration for all objects and a time-shifting holding cost. Failed attempts at fractional accumulating are acceptable in this test. Nevertheless, a transporter has a cut-off limit, and we are not allowed to assemble an infinite quantity of things. This is a legitimate assumption, as there is not a vehicle that has an infinite limit; hence, the generated model becomes more sensible. The time horizon is finite and unchanging, but the decision variables that determine the start of the deficit season are distinct.

The term 'shortage' describes a situation in which necessary items are either absent or cannot be obtained in sufficient quantities. To put things more simply, let's assume that a company distributes real goods to customers whose demands fluctuate suddenly or drastically, and the company is unable to fulfill the order due to unforeseen circumstances. At that moment, there are shortages for the customers of this particular item. For some types, shortages are essential, especially if there is an installation delay. If a scarcity occurs, the company provides an installment delay. It can receive orders from customers. It is often assumed that unsatisfied interest is either lost or accumulates in inventory models with shortages. Contemplation of deficit, which suggests a discrepancy that is delayed obtained, is another important factor to take into account when dealing with inventory concerns. Taleizadeh *et al*. [39] and Thinakaran *et al*. [40] dealt with shortage-related inventory issues. They determine the ideal degree of scarcity while considering how to maximize system revenues. Under the assumption of fractional multiplying, to find the best solution for the EPQ inventory model with deficiency, a single logarithmic organization approach without subordinates was proposed. Deficiency, or the lack that the creation framework experiences due to delayed buys or losses is another important factor to consider when dealing with inventory concerns. In order to maximize system revenues, they determine the ideal degree of shortage. Singh *et al*. [31] investigated a production model under the assumption of fractional multiplying, with subordinate request creation as well as a constant declining rate. Suggested a single logarithmic organization approach devoid of subordinates to get the best solution for the deficient EPQ inventory model. Inventory is the amount of stock that is held in trust and is used as a single mechanism system for multiple item defects and incomplete backorders. A few studies have addressed the impacts of partial backlog and scarcity with time-dependent demand (Prasad and Mukherjee [24]). In their study, Srividhya and Muniappan [37] compare the effectiveness of ensemble deep learning-based forecasting techniques to forecast future demand in the online retail market. By combining each model's best performance, ensemble learning has the potential to increase prediction accuracy when compared to single-model learning.

### 1.3 For the Research Gap

The first multiple-market demand model has been described by He *et al*. [13]. After discussing the unique opportunity to boost a deteriorating item's manufacturer's productivity via generating variations in decaying products at changed markets, Sajadifar and Hashempour [25] expanded this work by employing this model with shortage facilities. Subsequent study on numerous markets as well as multi-variate with constant holding costs and constant deterioration rates is presented in Table 1. It is anticipated that the holding cost in an inventory model will be constant and known. Holding costs, however, might not always remain the same. But the majority of the time, inventory is really kept in stockrooms to meet client demands or for later usage. Because maintaining inventory is so important, factor holding cost thus receives the greater portion of the connected analysts' attention. Additionally, holding costs are quite high and dependent on perishable items such as vegetables, organic goods, medications, and erratic fluids. Therefore, variable holding cost is the most important factor to know about the state of the market.

However, time-dependent holding cost with a controlled deterioration rate is not used by anyone. Thus, we adopt a time-dependent holding cost along with a regulated deterioration rate. The following serves as the paper's true motivation. The time-subordinate holding cost technique is one of the various requirements in EOQ models that makes perfect sense in a real-world scenario. This research examines an order-level lot size model for degrading items in an inventory system with time-dependent and constant demand. Inventory holding costs can be reduced, customer service can be enhanced, and business competitiveness can be raised by first ensuring that there is no inventory shortage case and then addressing inventory shortage cases.

To determine the best replenishment plan for raw materials that encounter demand and corrosion in the event of an inventory shortage, this document has been structured with production plans included initially. The structure of the paper is mentioned here: The study's objectives, notations, symbolizations, and underlying assumptions are presented in Section 2 in order to define the suggested inventory model. The cost estimation and mathematical formulation of the produced finished product inventory system with the shortfall are covered in Section 3. Theoretical formulation of the cost intention of raw materials and manufactured goods is derived in Section 4. In Section 5, the outcome was examined using a numerical guide to show the intended model. The sensitivity analysis and graphical depiction of the key parameter are presented in Section 6. Managerial implications are examined in Section 7. The overall findings are finally summarized in section 8 along with a recommendation for more investigation. In order to solve the model, minimize the overall cost of inventory, and maximize total profit, we employ both numerical and analytical methods in this paper.

## 2. Assumptions and Notations

The notations and presumptions mentioned below form the foundation of the mathematical model.

### 2.1 Notations

The following are the cost parameters set by the manufacturer:

| | |
|---|---|
| $c_p$ | Unit production cost of deteriorating item |
| $p$ | Production rate |
| $c_b$ | Backlogging cost per unit |
| $c_{fc}$ | Purchase cost of finished products |
| $h_p$ | Unit holding cost finished products per unit time. The holding cost is a linear function of time, i.e., $=(h_p+h_{pp}t)$; $h_p>0$ and $h_{pp}>0$ |
| $\theta_2$ | Constant deterioration rate of finished products |
| $\theta_p$ | Resultant deterioration rate, $\theta_p=\theta_2 e^{-\alpha\xi}$ |
| $k_0$ | Setup cost |
| $I_i(t)$ | Inventory level in the $i$th interval $i=1,2,3,\dots,u,u+1,\dots,T$ |

The parameters of raw materials cost by the manufacturer are as follows:

| | |
|---|---|
| $s_r$ | Ordering cost |
| $c_r$ | Unit price of raw materials |
| $h_r$ | Holding cost of raw material per unit time for the manufacturing |
| $\theta_1$ | Constant deterioration rate of raw materials |
| $\xi$ | PT cost for lowering deterioration rate to preserve the product $\xi>0$ |
| $\theta_r$ | Resultant deterioration rate, $\theta_r=\theta_1 e^{-\alpha\xi}$of raw materials |
| $q_i$ | Lot-size per delivery from supplier to manufacture |

| | |
|---|---|
| $n_r$ | Number of raw materials deliveries from supplier to manufacture |
| $f$ | Unit usage of raw materials per finished product |
| $c_p$ | Purchased cost per unit finished product |

### 2.2 Assumptions

This requirement has been met in order to formulate the issue:

(1) Production rate $p$is greater than any selling price demand rate.

(2) Production rate is deterministic and constant.

(3) The planning horizon is finite.

(4) A single product, a single manufacturer and multiple –markets demand.

(5) Demand rate is constant and known.

(6) Lead time is zero or negligible.

(7) The rates at which materials and completed goods deteriorate are predictable and under control.

(8) The backlog during a stock-out period is based on how long it will take to receive the next replenishment through a subcontract. The rate of deterioration is managed through the application of preservation techniques.

## 3. Mathematical Modal and Solution

Every cycle begins with the first opening market and concludes with the last closing market in accordance with contemporary market criteria. Every market has a unique selling price and demand rate. Every cycle begins with the entire market's demand building up in order of magnitude. There has been discussion of 2 models: Model 1 (no shortfall permitted) and Model 2 (with shortage permitted). In both models, production starts with a 0-level stock cycle at the beginning. Both production and demand are absent.

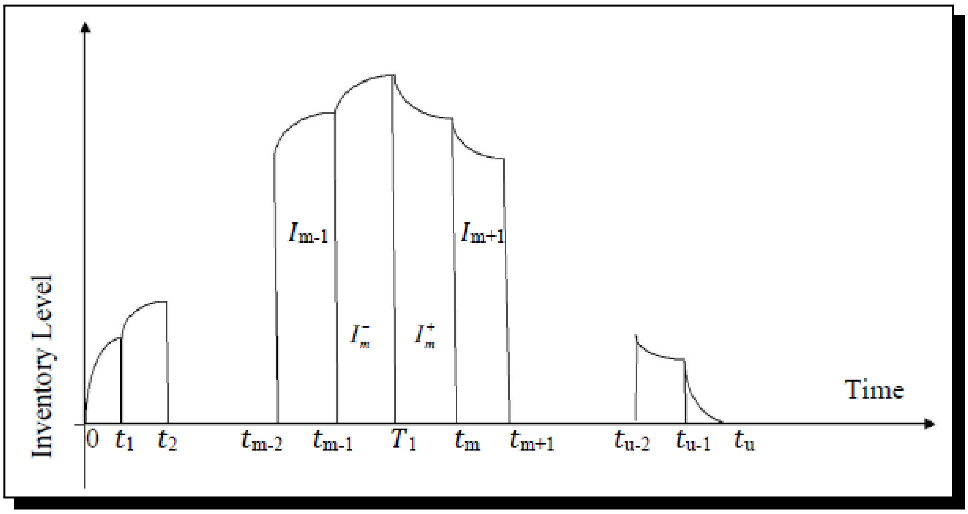


**Figure 2.** Production inventory level without shortage

Inventory builds up steadily while manufacturing goes on, meeting demand until the time $T_1$ at which production is halted. The inventory also declines till the levels of inventory reach 0 after time $T_1$ up to time $t_u = T$ because of both consumption and" deterioration.

### 3.1 Manufacture's Finished Products Inventory Model

### Phase 1

The rate of conversion of inventory throughout a positive stock period $[t_{k-1}, t_k]$ and $[t_{m-1}, T_1]$ is governed by the following differential equations.

$$\frac{dI_k(t)}{dt} + \theta_p I_k(t) = p - (d_k - as_k), \qquad t_{k-1} \le t \le t_k, k = 1,2,3,\dots,m-1, \tag{3.1}$$

$$\frac{dI_m^-(t)}{dt} + \theta_p I_m^-(t) = p - (d_m - as_m), \quad t_{m-1} \le t \le T_1, \tag{3.2}$$

with limit circumstance

$$I_1(0) = 0, \;\; I_k(t_{k-1}) = I_{k-1}(t_{k-1}) \;\text{ and }\; I_m^-(t_{k-1}) = I_{m-1}(t_{k-1}). \tag{3.3}$$

### Phase 2

The rate of change of inventory during positive stock period $[t_1, t_m]$ and $[t_{j-1}, t_j]$ is governed by the following differential equations:

$$\frac{dI_m^+(t)}{dt} + \theta_p I_m^+(t) = -(d_m - as_m), \quad T_1 \le t \le t_m, \tag{3.4}$$

$$\frac{dI_j(t)}{dt} + \theta_p I_j(t) = -(d_j - as_j), \qquad t_{j-1} \le t \le t_j, \; j = m+1, m+2, \dots, u, \tag{3.5}$$

with boundary condition

$$I_{j-1}(t_{j-1}) = I_j(t_{j-1}), \;\; I_m^+(t_{m-1}) = I_m(T_{m-1}) \;\text{ and }\; I_u(t_u) = 0. \tag{3.6}$$

Solution of the Phase 1 from the differential equations (3.1) and (3.2) using boundary condition (3.3) are as follow:

$$I_k(t) = \frac{p - (d_k - as_k) - pe^{-\theta_p t}}{\theta_p} + \sum_{i=1}^{k} \frac{\{(d_i - d_{i-1}) - a(s_i - s_{i-1})\}}{\theta_p} e^{-\theta_p (t - t_{i-1})},$$
$$t_{k-1} \le t \le t_k, \; k = 1,2,3,\dots,m-1, \tag{3.7}$$

$$I_m^-(t) = \frac{p - (d_m - as_m) - pe^{-\theta_p t}}{\theta_p} + \sum_{i=1}^{m} \frac{\{(d_i - d_{i-1}) - a(s_i - s_{i-1})\}}{\theta_p} e^{-\theta_p (t - t_{i-1})},$$
$$t_{m-1} \le t \le T_1. \tag{3.8}$$

Solution of the Phase 2 from the differential equations (3.4) and (3.5) using equation (3.6) are as follow:

$$I_m^+(t) = \frac{-(d_m - as_m)}{\theta_p} + \frac{d_u e^{-\theta_p (t - t_u)}}{\theta_p} - \sum_{i=m+1}^{u} \frac{\{(d_i - d_{i-1}) - a(s_i - s_{i-1})\}}{\theta_p} e^{-\theta_p (t - t_{i-1})}, \;\; T_1 \le t \le t_m, \tag{3.9}$$

$$I_j(t) = \frac{-(d_j - as_j)}{\theta_p} + \frac{d_u e^{-\theta_p (t - t_u)}}{\theta_p} - \sum_{j=j+1}^{u} \frac{\{(d_i - d_{i-1}) - a(s_i - s_{i-1})\}}{\theta_p} e^{-\theta_p (t - t_{i-1})},$$
$$t_{j-1} \le t \le t_j, \; j = m+1, m+2, \dots, u. \tag{3.10}$$

When the production stop, which is $T_1$ using the condition, $I_m^-(T_1)=I_m^+(T_1)$ is as follow:

$$\frac{p-(d_m-as_m)-pe^{-\theta_p t}}{\theta_p}+\sum_{i=1}^{m}\frac{\{(d_i-d_{i-1})-a(s_i-s_{i-1})\}}{\theta_p}e^{-\theta_p(t-t_{i-1})}$$
$$=\frac{-(d_m-as_m)}{\theta_p}+\frac{d_u e^{-\theta_p(t-t_u)}}{\theta_p}-\sum_{i=m+1}^{u}\frac{\{(d_i-d_{i-1})-a(s_i-s_{i-1})\}}{\theta_p}e^{-\theta_p(t-t_{i-1})}. \tag{3.11}$$

After simplification and neglecting the higher power of $\theta_p$, then, we get

$$T_1=\frac{1}{\theta_p}\ln\Big[\frac{(d_u-as_u)e^{(\theta_p)t_u}-\sum_{i=1}^{u}((d_i-d_{i-1})-a(s_i-s_{i-1}))e^{(\theta_p)t_{i-1}}}{p}+1\Big]$$
$$=\frac{1}{p}\Big(A+\frac{\theta_p B}{2}\Big), \tag{3.12}$$

where $A=\sum_{i=1}^{u}(d_i-as_i)(t_i-t_{i-1})$ and $B=\sum_{i=1}^{u}(d_i-as_i)(t_i^2-t_{i-1}^2)$.

Equations (3.12) have no relation with $m$ so if the value of $t_i$ and $(d_i-as_i)$ are known then the optimal production time $T_1$ can be found directly by using equation (3.12).

## 3.2 Cost Calculation of Finished Products

(1) Set up cost $TS_c=k_0$. (3.13)

(2) Time depend Holding Cost of finished products as follow:

$$TH_p=\Big[h_p\Big(\sum_{k=1}^{m-1}I_k(t)+I_m^-(t)+I_m^+(t)+\sum_{j=m+1}^{u}I_j(t)\Big)+h_{pp}t\Big(\sum_{k=1}^{m-1}I_k(t)+I_m^-(t)+I_m^+(t)+\sum_{j=m+1}^{u}I_j(t)\Big)\Big]$$
$$=\Big[\frac{h_pB}{2}+h_{pp}\Big\{-\frac{B}{2\theta_p}-C\Big(\frac{A}{p\theta_p}+\frac{B}{2p}\Big)+\frac{D}{\theta_p^2}+\frac{A^2BE}{p^2}+\frac{A^3}{2p^2}+\frac{3A^2B\theta_p}{4p^2}$$
$$+(d_u-as_u)\Big(\frac{\theta_pBt_u}{2p}+\frac{At_u}{p}-\frac{AB}{p^2}\Big)\Big\}\Big], \tag{3.14}$$

where

$$C=\sum_{i=1}^{u}\{(d_i-d_{i-1})-a(s_i-s_{i-1})\}t_{i-1},\quad D=\sum_{i=1}^{u}\{(d_i-d_{i-1})-a(s_i-s_{i-1})\}(t_i-t_{i-1})$$

and

$$E=\sum_{i=1}^{u}\{(d_i-d_{i-1})-a(s_i-s_{i-1})\}.$$

(3) The deterioration cost of finished products

$$TD_p=c_p\Big\{pT_1-\sum_{i=1}^{u}(d_i-as_i)(t_i-t_{i-1})\Big\}=c_p\Big\{A+\frac{\theta_pB}{2}-A\Big\}=\frac{c_p\theta_pB}{2}. \tag{3.15}$$

(4) Preservation cost of finished products

$$PT_c=\xi t_u. \tag{3.16}$$

Total cost of finished products

$$TC_p=TS_c+TH_p+TD_p+PT_c$$
$$=\Big[k_0+\Big\{\frac{h_pB}{2}+h_{pp}\Big(-\frac{B}{2\theta_p}-C\Big(\frac{A}{p\theta_p}+\frac{B}{2p}\Big)+\frac{D}{\theta_p^2}+\frac{A^2BE}{p^2}+\frac{A^3}{2p^2}$$
$$+\frac{3A^2B\theta_p}{4p^2}+(d_u-as_u)\Big(\frac{\theta_pBt_u}{2p}+\frac{At_u}{p}-\frac{AB}{p^2}\Big)\Big)\Big\}+\frac{c_p\theta_pB}{2}+\xi t_u\Big]. \tag{3.17}$$

### 3.3 Manufactories' Warehouse Raw Materials Inventory Model

Because of deterioration and consumption of demand at a time $t = \frac{T_1}{n_r}$ that can be defined, the raw material inventory level drops to zero:

$$\frac{dI_r(t)}{dt} + \theta_r I_r(t) = -fp, \quad 0 \le t \le \frac{T_1}{n_r}.$$

Using the boundary condition

$$I_r\left(\frac{T_1}{n_r}\right) = 0. \tag{3.18}$$

We have

$$I_r = \frac{fp}{\theta_r}[e^{-\theta_r\left(t-\frac{T_1}{n_r}\right)} - 1] = \frac{fp}{\theta_r}\left[e^{\theta_r\frac{T_1}{n_r}}\left(1-\theta_r t + \frac{(\theta_r t)^2}{2} - \dots\right) - 1\right], \quad 0 \le t \le \frac{T_1}{n_r}. \tag{3.19}$$

Using an additional boundary condition $I_r(0) = q_r$. The lot size-per-delivery $q_r$ between supplier to manufacturer is shown in Figure 3 as

$$q_r = \frac{fp}{\theta_r}[e^{\theta_r\left(\frac{T_1}{n_r}\right)} - 1] = fp\left(\frac{T_1}{n_r} + \frac{1}{2}\frac{\theta_r T_1^2}{n_r^2}\right). \tag{3.20}$$

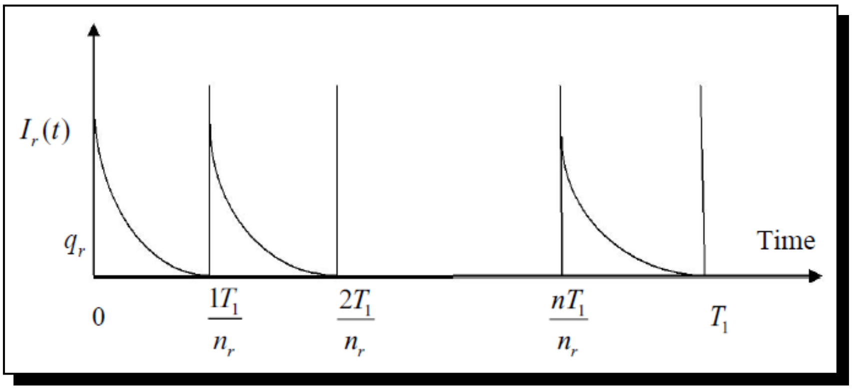


**Figure 3.** Raw materials of inventory system

### 3.4 Cost Calculation of Raw Materials

(1) Raw material's ordering cost

$$TO_r = s_r n_r. \tag{3.21}$$

(2) Raw material's holding cost

$$\begin{aligned} THr &= n_r h_r \int_0^{\frac{T_1}{n_r}} t I_r(t) dt \\ &= n_r h_r \int_0^{\frac{T_1}{n_r}} \frac{tfp}{\theta_r}\{e^{\theta_r\left(\frac{T_1}{n_r}-t\right)} - 1\} dt \\ &= \frac{fh_r}{3p^2 n_r^2}\left[\frac{A^3}{2} + \frac{3A^2B\theta_p}{4} + \frac{A^4\theta_r}{pn_r}\right]. \end{aligned} \tag{3.22}$$

(3) Raw material's deterioration cost

$$TD_r = c_r(n_r q_r - fpT_1)$$

$$
\begin{aligned}
&= c_r\Big(\frac{n_r f p}{\theta_r}(e^{\theta_r\left(\frac{T_1}{n_r}\right)}-1)-fpT_1\Big)\\
&= \frac{f c_r\theta_r}{2pn_r}[A^2+\theta_p AB]\\
&= \frac{f c_r\theta_r A^2}{2pn_r}. \qquad (3.23)
\end{aligned}
$$

(4) Raw materials PT (Preservation Technology) cost:

$$
\begin{aligned}
PTC_r &= \xi\frac{T_1}{n_r}\\
&= \frac{\xi}{pn_r}\Big(A+\frac{B\theta_p}{2}\Big)\\
&= \frac{\xi A}{pn_r}+\frac{\xi B\theta_p}{2pn_r} \qquad (3.24)
\end{aligned}
$$

Total cost $TC_r = TO_r + TH_r + TD_r + PTC_r$,

$$
TC_r = \Big[s_r n_r + \frac{fh_r}{3p^2n_r^2}\Big\{\frac{A^3}{2}+\frac{3A^2B\theta_p}{4}+\frac{A^4\theta_r}{pn_r}\Big\}+\frac{fc_r\theta_r A^2}{2pn_r}+\frac{\xi A}{pn_r}+\frac{\xi B\theta_p}{2pn_r}\Big]. \qquad (3.25)
$$

**The Model for Integrated Inventory and Its Solution**

The combined overall cost for completed goods and raw materials, $TC = TC_p + TC_r$ is

$$
\begin{aligned}
TC = \Big[k_0 &+ \frac{h_pB}{2}+h_{pp}\Big\{-\frac{B}{2\theta_p}-C\Big(\frac{A}{p\theta_p}+\frac{B}{2p}\Big)+\frac{D}{\theta_p^2}+\frac{A^2BE}{p^2}+\frac{3A^2B\theta_p}{4p^2}\\
&+\frac{A^3}{2p^2}+(d_u-as_u)\Big(\frac{\theta_pBt_u}{2p}+\frac{At_u}{p}-\frac{AB}{p^2}\Big)\Big\}+\frac{c_p\theta_pB}{2}+\xi t_u+s_rn_r\\
&+\frac{fh_r}{3pn_r^2}\Big\{\frac{A^3}{2}+\frac{3A^2B\theta_p}{4}+\frac{A^4\theta_r}{pn_r}\Big\}+\frac{fc_r\theta_rA^2}{2pn_r}+\frac{\xi A}{pn_r}+\frac{\xi B\theta_p}{2pn_r}\Big]. \qquad (3.26)
\end{aligned}
$$

## 3.5 Model 2: Modal with Shortage Allowed

Model 2 has worked similarly to Model 1, with the $i$th market-related demand rate $(d_i - as_i)$ being provided in the period $[t_{i-1}, t_i]$, for $i = 1,2,3,\dots,n$ where $t_u$ only the time when it reaches a short time interval $[t_{i-1}, t_i]$ for $i = u+1, u+2, \dots, n$ demands arrives with the rate $d_{i-1}$. This indicates that inventory is zero at this time $t_u$ and will always be positive before that time.

### 3.5.1 Manufacture's Finished Products Inventory Model

In Model 2 also production begins at time zero and just at this time the inventory goes to accumulate until the end of production time that is $T_1$. After time $T_1$, the inventory declines due to both consumption and deterioration as shown in Figure 3. For obtaining the most benefit, the manufacture has to determine the value $T_1$. The only difference with Model 1 is since shortages are allowed in Model 2 and fulfilled by backlogging. The producer will compensate those demands that have purchased the backlogged demands via a subcontract after the time $t_u$ up to time $T = t_{u+1}$ at the end of each cycle. Shortage will reach to maximum value. Phase 1 and Phase 2 of Model 2, all the calculation are same as those in Model 1, since in both the model at time $t = t_u$ the inventory reduced to zero.

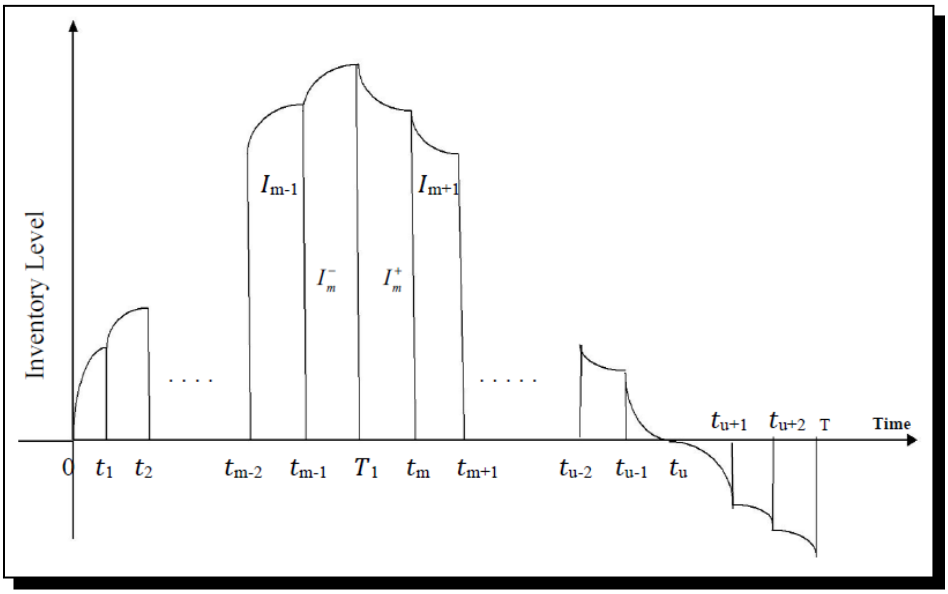


**Figure 4.** Final production inventory level with shortage under backlogging

Only differences is that in (Model 1) the total time period of a cycle $T = t_u$ where the inventory reduce to zero only but in Model 2 the total time period of a cycle $T = t_{u+1}$ when the shortage will reached to its maximum values and the insufficient demand is practically backlogged by the manufacturer via a subcontract.

Hence the inventory level in the time interval $t_u \le t \le T$ is given as:

$$I_{z+1} = -(d_z - as_z)(t - t_z) + \sum_{i=u}^{z-1}(d_i - as_i)(t_{i+1} - t_i), \quad t_z \le t \le t_{z+1},\ z = u+1, u+2, \dots, n. \quad (3.27)$$

Maximum shortage has been full field

$$I_{z+1} = \int_{t_z}^{t_{z+1}} \Big(-(d_z - as_z)(t - t_z) - \sum_{i=u}^{z-1}(d_i - as_i)(t_{i+1} - t_i)\Big), \ \ t_z \le t \le t_{z+1}, z = u+1, u+2, \dots, n,$$

$$I_{z+1} = -\Big(\frac{(d_z - as_z)}{2}(t_{z+1} - t_z)^2 + (t_{z+1} - t_z)\sum_{i=u}^{z-1}(d_i - as_i)(t_{i+1} - t_i)\Big), \quad t_z \le t \le t_{z+1}. \quad (3.28)$$

### 3.5.2 Cost Calculation of Shortage Model

Similar the Model 1 the holding cost $TH_p$, the deterioration cost $TD_p$ and the preservation technology cost $PTC_p$ of finished products are same but the backlogging cost of finished products $TS_p$ and purchase cost of finished products $TB_p$ are as follow:

The purchase cost of finished products

$$TS_p = c_{fc} \sum_{z=u}^{n} I_{z+1} = -c_{fc}\Big(\sum_{i=u}^{z-1}\sum_{z=u}^{n}(d_i - as_i)(t_{i+1} - t_i)(t_{z+1} - t_z) + \sum_{z=u}^{z=n}\frac{(d_z - as_z)}{2}(t_{z+1} - t_z)^2\Big). \quad (3.29)$$

The backlogging cost of finished products

$$TB_p = -c_b \sum_{z=u}^{n} I_{z+1} = c_b\Big(\sum_{i=u}^{z-1}\sum_{z=u}^{n}(d_i - as_i)(t_{i+1} - t_i)(t_{z+1} - t_z) + \sum_{z=u}^{z=n}\frac{(d_z - as_z)}{2}(t_{z+1} - t_z)^2\Big). \quad (3.30)$$

Hence total cost of finished products is

$$TC_p = k_0 + TH_p + TD_p + PTC_p + TS_p + TB_p$$

$$= \Big[k_0 + \frac{h_pB}{2} + h_{pp}\Big\{-\frac{B}{2\theta_p} - C\Big(\frac{A}{p\theta_p} + \frac{B}{2p}\Big) + \frac{D}{\theta_p^2} + \frac{A^2BE}{p^2} + \frac{A^3}{2p^2} + \frac{3A^2B\theta_p}{4p^2}$$
$$+ (d_u - as_u)\Big(\frac{\theta_pBt_u}{2p} + \frac{At_u}{p} - \frac{AB}{p^2}\Big)\Big\} + \frac{c_p\theta_pB}{2} + \xi t_u + (c_b - c_{fc})$$
$$\cdot\Big(\sum_{i=u}^{z-1}\sum_{z=u}^{n}(d_i - as_i)(t_{i+1} - t_i)(t_{z+1} - t_z) + \sum_{z=u}^{z=n}\frac{(d_z - as_z)}{2}(t_{z+1} - t_z)^2\Big)\Big]. \tag{3.31}$$

## 4. Objective (Cost Calculation of Finished Products and Raw Materials)

The total cost of finished products goods for Model 2 and total cost of raw materials is:

$$TC = TC_p + TC_r$$
$$= \Big[k_0 + \frac{h_pB}{2} + h_{pp}\Big\{-\frac{B}{2\theta_p} - C(\frac{A}{p\theta_p} + \frac{B}{2p}) + \frac{D}{\theta_p^2} + \frac{A^2BE}{p^2} + \frac{A^3}{2p^2}$$
$$+ \frac{3A^2B\theta_p}{4p^2} + \Big(\frac{\theta_pBt_u}{2p} + \frac{At_u}{p} - \frac{AB}{p^2}\Big)(d_u - as_u)\Big\} + \frac{c_p\theta_pB}{2} + (c_b - c_{fc})$$
$$\cdot\Big(\sum_{i=u}^{z-1}\sum_{z=u}^{n}(d_i - as_i)(t_{i+1} - t_i)(t_{z+1} - t_z) + \sum_{z=u}^{z=n}\frac{(d_z - as_z)}{2}(t_{z+1} - t_z)^2\Big)$$
$$+ \xi t_u + s_rn_r + \frac{fh_r}{3p^2n_r^2}\Big\{\frac{A^3}{2} + \frac{3A^2B\theta_p}{4} + \frac{A^4\theta_r}{pn_r}\Big\} + \frac{fc_r\theta_rA^2}{2pn_r} + \frac{\xi A}{pn_r} + \frac{\xi B\theta_p}{2pn_r}.\Big]. \tag{4.1}$$

The objective of the study is to determine the optimal value of preservation cost $\xi^*$ for both the model that minimizes the total cost $TC$ is as follows: Put $\theta_p = \theta_2 e^{-\alpha\xi}$ and $\theta_r = \theta_1 e^{-\alpha\xi}$ then equation reduces to $TC$ follow as:

$$TC = \Big[k + \frac{h_pB}{2} + h_{pp}\Big\{-\frac{Be^{\alpha\xi}}{2\theta_2} - C\Big(\frac{Ae^{\alpha\xi}}{p\theta_2} + \frac{B}{2p}\Big) + \frac{De^{2\alpha\xi}}{\theta_2^2} + \frac{A^2BE}{p^2} + \frac{A^3}{2p^2} + \frac{3A^2B\theta_2e^{-\alpha\xi}}{4p^2}$$
$$+ (d_u - as_u)\Big(\frac{\theta_2e^{-\alpha\xi}Bt_u}{2p} + \frac{At_u}{p} - \frac{AB}{p^2}\Big)\Big\} + \frac{C_p\theta_2e^{-\alpha\xi}B}{2} + \xi t_u + (C_B - C_{fc})$$
$$\cdot\Big(\sum_{i=u}^{z-1}\sum_{z=u}^{n}(d_i - as_i)(t_{i+1} - T_i)(T_{z+1} - T_z) + \sum_{z=u}^{z=n}\frac{(d_z - as_z)}{2}(T_{z+1} - T_z)^2\Big)$$
$$+ s_rn_r + \frac{fh_r}{3p^2n_r^2}\Big\{\frac{A^3}{2} + \frac{3A^2B\theta_2e^{-\alpha\xi}}{4} + \frac{A^4\theta_1e^{-\alpha\xi}}{pn_r}\Big\} + \frac{fc_r\theta_1e^{-\alpha\xi}A^2}{2pn_r} + \frac{\xi A}{pn_r} + \frac{\xi B\theta_2e^{-\alpha\xi}}{2pn_r}\Big]. \tag{4.2}$$

Differentiate with respect to $\xi$:

$$\frac{\partial TC}{\partial\xi} = \Big[h_{pp}\Big(-\frac{\alpha Be^{\alpha\xi}}{2\theta_2} - \frac{\alpha Ae^{\alpha\xi}C}{p\theta_2} + \frac{2\alpha e^{2\alpha\xi}D}{\theta_2^2} - \frac{3\alpha A^2B\theta_2e^{-\alpha\xi}}{4p^2} - \frac{(d_u - as_u)T_u\alpha B\theta_2e^{-\alpha\xi}}{2p}\Big)$$
$$- \frac{\alpha c_pB\theta_2e^{-\alpha\xi}}{2} + t_u - \frac{\alpha fh_rA^2B\theta_2e^{-\alpha\xi}}{12p^2n_r^2} - \frac{\alpha fh_rA^4\theta_1e^{-\alpha\xi}}{3p^3n_r^3} - \frac{\alpha fc_rA^2\theta_1e^{-\alpha\xi}}{2pn_r} + \frac{A}{pn_r}$$
$$- \frac{\alpha\xi B\theta_2e^{-\alpha\xi}}{2pn_r} + \frac{B\theta_2e^{-\alpha\xi}}{2pn_r}\Big]. \tag{4.3}$$

Again, differentiate cost function with respect to $\xi$, then, we get

$$\begin{aligned}\frac{\partial^2 TC}{\partial \xi^2} &= \Big[h_{pp}\Big(-\frac{\alpha^2 BE^{\alpha\xi}}{2\theta_2}-\frac{\alpha^2 Ae^{\alpha\xi}C}{p\theta_2}+\frac{4\alpha^2e^{2\alpha\xi}D}{\theta_2^2}+\frac{3\alpha^2A^2B\theta_2e^{-\alpha\xi}}{4p^2}+\frac{(d_u-as_u)T_u\alpha^2B\theta_2e^{-\alpha\xi}}{2p}\Big)\\ &\quad+\frac{\alpha^2C_pB\theta_2e^{-\alpha\xi}}{2}+\frac{\alpha^2fh_rA^2B\theta_2e^{-\alpha\xi}}{12p^2n_r^2}+\frac{\alpha^2fh_rA^4\theta_1e^{-\alpha\xi}}{3p^3n_r^3}\\ &\quad+\frac{\alpha^2fc_rA^2\theta_1e^{-\alpha\xi}}{2pn_r}+\frac{\alpha^2\xi B\theta_2e^{-\alpha\xi}}{2pn_r}-\frac{\alpha B\theta_2e^{-\alpha\xi}}{2pn_r}-\frac{\alpha B\theta_2e^{-\alpha\xi}}{2pn_r}\Big]\\ &=\Big[h_{pp}\Big\{\frac{\alpha^2e^{\alpha\xi}}{\theta_2}\Big(\frac{2De^{\alpha\xi}}{\theta_2}-\frac{AC}{p}\Big)+\frac{\alpha^2e^{\alpha\xi}}{\theta_2}\Big(\frac{2e^{\alpha\xi}D}{\theta_2}+\frac{B}{2}\Big)\\ &\quad+\frac{3\alpha^2A^2B\theta_2e^{-\alpha\xi}}{4p^2}+\frac{(d_u-as_u)T_u\alpha^2B\theta_2e^{-\alpha\xi}}{2p}\Big\}+\frac{\alpha^2fh_rA^2B\theta_2e^{-\alpha\xi}}{12pn_r}\\ &\quad+\alpha B\theta_2e^{-\alpha\xi}\Big(\frac{\alpha c_p}{2}-\frac{1}{pn_r}\Big)+\frac{\alpha^2fh_rA^4\theta_1e^{-\alpha\xi}}{3p^2n_r^2}+\frac{\alpha^2fc_r\theta_1A^2e^{-\alpha\xi}}{2pn_r}+\frac{\alpha^2\xi B\theta_2e^{-\alpha\xi}}{2pn_r}\Big] \quad (4.4)\\ &>0 \quad \text{if } \Big(\frac{2De^{\alpha\xi}}{\theta_2}-\frac{AC}{p}\Big)>0 \text{ and } \Big(\frac{\alpha c_p}{2}-\frac{1}{pn_r}\Big)>0.\end{aligned}$$

The optimal value of $\xi^*$ will be calculated using MATHEMATICA 9 from equation (4.3). The next objective of the study is to determine the optimal value of total number of devilries $n_r^*$, the Manufactories' warehouse raw materials inventory model is same for Model 1 and Model 2 of the manufacturers' warehouse raw material inventory are similar. Thus, the value of $n_r^*$, that reduces $TC$, here $n_r$ a separate variable can be defined as follows differentiate about $n_r$.

The ideal value of a total number of deliveries $n_r^*$ is differentiate concerning $n_r$ then,

$$\frac{\partial TC}{\partial n_r}=\Big[s_r-\frac{h_rf}{6p^2n_r^3}\Big(A^3+\frac{3\theta_pA^2B}{2}\Big)-\frac{h_rfA^4\theta_r}{3p^3n_r^4}-\frac{fc_r\theta_rA^2}{2pn_r^2}-\frac{\xi}{pn_r^2}\Big(A+\frac{B\theta_p}{2}\Big)\Big]. \quad (4.5)$$

Again, differentiate cost function with respect to $n_r$ then we get second derivate

$$\begin{aligned}\frac{\partial^2 TC}{\partial n_r^2}&=\Big[\frac{h_rf}{2p^2n_r^3}\Big(A^3+\frac{3\theta_pA^2B}{2}\Big)+\frac{4h_rfA^4\theta_r}{3p^3n_r^4}+\frac{fc_r\theta_rA^2}{pn_r^3}+\frac{2\xi}{pn_r^3}\Big(A+\frac{B\theta_p}{2}\Big)\Big] \quad (4.6)\\ &>0.\end{aligned}$$

It is clear from equation (4.6), $\frac{\partial^2 TC}{\partial n_r^2}>0$, the ideal value of the total number of deliveries $n_r^*$ would be computed utilizing MATHEMATICA 9 from equation (4.5).

## 5. Numerical Analysis of the Proposed Model

The three-market scenario and the proposition-1 criteria are now being reviewed. The data below has been used as an example to support the validity of Models 1 and 2. In market one, the selling season is valued at \$8, in market two, at \$20, and in market three, at \$12. The demand rates for market one, market two, and market three are \$20, \$38, and \$18 units/week, respectively. \$150 units are produced each week at this rate. The first market's selling season runs from week one to week eight, the 2nd market from week eight to twenty, and the 3rd market from week twenty to thirty. The raw material degradation rate is \$0.2 units per week, the completed product deterioration rate is \$0.3 units/week, and the raw material usage rate is \$1.2 units/unit of completed product. The unit price of raw materials is \$5, the ordering cost is \$400/order, the production set-up cost is \$600, the holding fees are \$0.1 per week for raw materials and \$0.15/week for finished products and $h_{pp} = 0.25$. The conservation parameter $a = \$0.5$ and $\alpha = 0.2$ the unit production cost are \$10 correspondingly. The weekly cost of the final goods shortage is \$2, while the weekly cost of the completed product purchase is \$1.6.

Equations (3.12), (3.20), (4.3), (4.5), (4.1) and (4.2) have been used to calculate the ideal time of production $T_1^*$, lot size per delivery between supplier to manufacturer $q_r^*$, ideal $PT$ value cost $\xi^*$, ideal number of raw material deliveries between the supplier to manufacturer $n_r^*$, and ideal total cost $TC^{**}$ with/without $PT$, correspondingly. The results are displayed in Table 2.

**Table 2.** The dependent variables, total costs, and decision variables are at their optimal values

| $n_r^*$ | With preservation technology | | | | | Without preservation technology | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\xi^*$ | $T_1^*$ | $q_r^*$ | Optimal Cost $TC^*$ | | $T_1^*$ | $q_r^*$ | Optimal Cost $TC^*$ | |
| | | | | Without shortage | With shortage | | | Without shortage | With shortage |
| 2 | 6.397 | 8.43 | 854.53 | 746749 | 749811 | 20.33 | 5970.06 | 788772 | 791834 |
| 3 | 6.415 | 8.43 | 547.31 | 746602 | 749664 | 20.33 | 2589.16 | 788725 | 791787 |

## 5.1 Sensitivity Analyses of The Proposed Model

Impact of modifications to distinct parameters of the suggested model (1 and 2): The sensitivity analysis is performed by taking into account increases or decreases of 10% and 20% in each of the aforementioned parameters while maintaining the same values for the remaining parameters. The sensitivity study is performed by modifying the parameter $p, f, (d_1 - as_1)$, $(d_2 - as_2)$, $(d_3 - as_3)$, $c_r, c_p, c_b$, and $c_{fc}$. (Suppose $(d_1 - as_1) = d_{11}$, $(d_2 - as_2) = d_{22}$, $(d_3 - as_3) = d_{33}$. Table 2 demonstrates the sensitiveness of several parameters on the ideal value of $\xi^*, n_r^*, T_1^*$ and $q_r^*$, with/without shortage $TC^*$. The analysis showed the facts mentioned here:

**Table 3.** Sensitivity analysis for effect of changes for various parameter

| Parameter | | With preservation technology | | | | | | Without preservation technology | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\xi^*$ | $n_r^*$ | $T_1^*$ | $q_r^*$ | Without and with shortage | | $n_r^*$ | $T_1^*$ | $q_r^*$ | Without and with shortage | |
| | | | | | | $TC^*$ | $TC^*$ | | | | $TC^*$ | $TC^*$ |
| $p$ | 20 | 4.00 | −8.60 | −16.63 | −2.68 | 44.69 | −30.76 | −8.82 | −16.67 | −4.89 | 42.09 | −29.51 |
| | 10 | 2.13 | −4.59 | −9.04 | −1.47 | 21.27 | −17.47 | −4.57 | −9.10 | −2.48 | 20.22 | −16.76 |
| | −10 | −2.38 | 5.42 | 11.12 | 1.84 | −19.16 | 23.60 | 5.48 | 11.07 | 2.66 | −18.52 | 22.64 |
| | −20 | −5.02 | 12.09 | 25.00 | 4.20 | −36.23 | 46.57 | 12.05 | 24.94 | 4.57 | −35.27 | 44.28 |
| $f$ | 20 | −0.20 | 8.75 | 0.00 | 20.00 | −0.01 | 0.01 | 9.34 | 0.00 | 20.00 | −0.02 | 0.02 |
| | 10 | −0.05 | 5.42 | 0.00 | 10.00 | −0.01 | 0.01 | 4.71 | 0.00 | 10.00 | −0.01 | 0.01 |
| | −10 | 0.05 | −4.59 | 0.00 | −10.00 | 0.01 | −0.01 | −4.95 | 0.00 | −10.00 | 0.01 | −0.01 |
| | −20 | 0.11 | −9.93 | 0.00 | −20.00 | 0.02 | −0.01 | −10.36 | 0.00 | −20.00 | 0.02 | −0.02 |

(*Contd.* Table)

| Parameter | | With preservation technology | | | | | | Without preservation technology | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\xi^*$ | $n_r^*$ | $T_1^*$ | $q_r^*$ | Without and with shortage | | $n_r^*$ | $T_1^*$ | $q_r^*$ | Without and with shortage | |
| | | | | | | $TC^*$ | $TC^*$ | | | | $TC^*$ | $TC^*$ |
| $d_{11}$ | 20 | −0.67 | 2.08 | 2.70 | 3.15 | −9.40 | 10.38 | 3.55 | 1.82 | 3.72 | −9.15 | 10.07 |
| | 10 | −0.36 | 0.75 | 1.40 | 1.57 | −4.86 | 5.12 | 2.00 | 0.89 | 1.85 | −4.73 | 4.96 |
| | −10 | 0.26 | −0.59 | −1.33 | −1.56 | 5.21 | −4.96 | −1.86 | −0.93 | −1.82 | 5.06 | −4.81 |
| | −20 | 0.73 | −1.92 | −2.75 | −3.12 | 10.81 | −9.78 | −3.79 | −1.87 | −3.61 | 10.48 | −9.48 |
| $d_{22}$ | 20 | −2.84 | 6.08 | 11.48 | 13.59 | −27.98 | 38.57 | 11.66 | 11.46 | 24.90 | −27.43 | 37.69 |
| | 10 | −1.45 | 3.41 | 5.79 | 6.74 | −15.52 | 18.23 | 5.86 | 5.71 | 11.89 | −15.17 | 17.84 |
| | −10 | 1.51 | −3.26 | −5.72 | −6.62 | 19.57 | −16.25 | −5.73 | −5.76 | −10.87 | 18.99 | −15.92 |
| | −20 | 3.22 | −5.93 | −11.53 | −13.12 | 44.62 | −30.62 | −11.52 | −11.51 | −20.80 | 43.09 | −30.04 |
| $d_{33}$ | 20 | −0.67 | 2.75 | 5.79 | 6.84 | −28.36 | 39.39 | 4.71 | 6.69 | 13.99 | −27.49 | 37.77 |
| | 10 | −0.05 | 0.08 | 0.45 | 0.56 | −13.90 | 19.97 | 2.39 | 3.30 | 6.81 | −15.24 | 17.91 |
| | −10 | 0.42 | −1.26 | −2.87 | −3.37 | 20.23 | −16.74 | −2.25 | −3.34 | −6.46 | 19.25 | −16.08 |
| | −20 | 0.73 | −2.59 | −5.84 | −6.72 | 46.71 | −31.66 | −4.95 | −6.69 | −12.59 | 44.01 | −30.44 |
| $c_r$ | 20 | 0.05 | 4.08 | 0.00 | 0.00 | −0.01 | 0.01 | 5.09 | 0.00 | 0.00 | −0.01 | 0.01 |
| | 10 | 0.05 | 2.08 | 0.00 | 0.00 | 0.00 | 0.00 | 2.39 | 0.00 | 0.00 | −0.01 | 0.01 |
| | −10 | −0.05 | −1.26 | 0.00 | 0.00 | 0.00 | 0.00 | −2.64 | 0.00 | 0.00 | 0.01 | −0.01 |
| | −20 | −0.05 | −3.93 | 0.00 | 0.00 | 0.01 | −0.01 | −5.34 | 0.00 | 0.00 | 0.01 | −0.01 |
| $c_p$ | 20 | −2.69 | 0.00 | 0.00 | 0.00 | −0.18 | 0.18 | 0.00 | 0.00 | 0.00 | −0.62 | 0.62 |
| | 10 | −1.45 | 0.00 | 0.00 | 0.00 | −0.09 | 0.09 | 0.00 | 0.00 | 0.00 | −0.31 | 0.31 |
| | −10 | 1.66 | 0.00 | 0.00 | 0.00 | 0.09 | −0.09 | 0.00 | 0.00 | 0.00 | 0.31 | −0.31 |
| | −20 | 3.53 | 0.00 | 0.00 | 0.00 | 0.18 | −0.18 | 0.00 | 0.00 | 0.00 | 0.63 | −0.62 |
| $b_f$ | 20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.41 | 0.00 | 0.00 | 0.00 | 0.00 | 0.39 |
| | 10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.19 |
| | −10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | −0.20 | 0.00 | 0.00 | 0.00 | 0.00 | −0.19 |
| | −20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | −0.41 | 0.00 | 0.00 | 0.00 | 0.00 | −0.39 |
| $c_{fc}$ | 20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | −0.33 | 0.00 | 0.00 | 0.00 | 0.00 | −0.31 |
| | 10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | −0.16 | 0.00 | 0.00 | 0.00 | 0.00 | −0.15 |
| | −10 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.16 | 0.00 | 0.00 | 0.00 | 0.00 | 0.15 |
| | −20 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.33 | 0.00 | 0.00 | 0.00 | 0.00 | 0.31 |

Optimal worth of $n_r^*$ slightly change in the value of parameters $d_{11}$ and $d_{33}$, ascetically $c_r$ and decidedly with $p, f$ and $d_{22}$.

Optimum worth of $q_r^*$ slightly change in the value of parameters $p$ and $d_{11}$, abstemiously $d_{33}$ highly with $f$ and $d_{22}$.

Finest value of $TC^*$ (without shortage) deviations extremely in the worth of parameters $p, d_{22}$ and $d_{33}$ ascetically to the value of $d_{11}$ whereas very slightly $c_r$ and $c_p$.

Finest value of $TC^*$ (with shortage) deviations highly in the value of parameters $p, d_{22}$ and $d_{33}$ moderately to the value of $d_{11}$ whereas slightly with $c_r, c_p, c_b$ and $c_{fc}$.

### 5.2 Graphical Analysis of The Proposed Model

Figures 5 and 6, respectively, provide the graphical depiction of the ideal $TC^*$ with respect to the number of raw material deliveries with/without $PT$ that is the $TC^*$ convexity with respect to $n_r^*$:

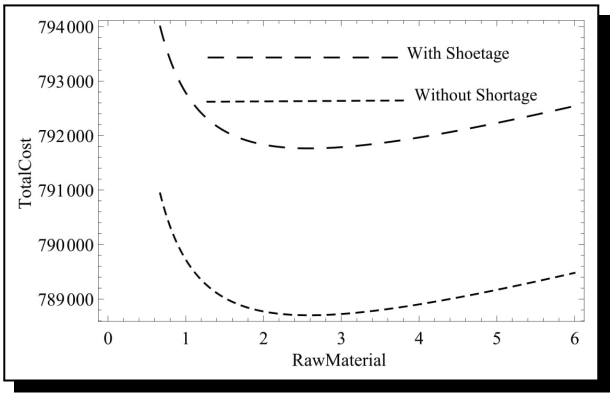


**Figure 5.** Without preservation technology

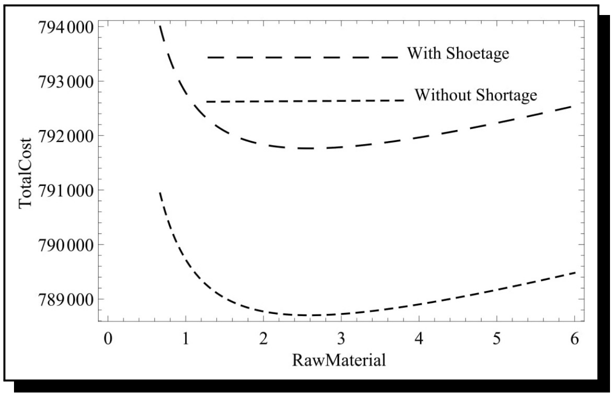


**Figure 6.** With preservation technology

### 5.3 Solution Algorithm of This Proposed Model

The solution algorithm of our proposed model is given below:

*Step* 1: Calculate different type of costs using equation (3.7) to (3.10) and (3.16) to (3.20).

*Step* 2: Calculate total inventory cost using equation (3.22).

*Step* 3: Find first decision variable of $\xi$, first derivative of Total cost $TC$ with respect to $\xi$ and calculate value of $\xi$. We find positive value of $\xi$. The value of $\xi$ is 3.516 from equation (4.3).

*Step* 4: Calculate the second decision variable $n_r$, first derivative of Total cost $TC$ with respect to $n_r$. We find positive value of $n_r = 1.625$ from (4.5).

*Step* 5: Using the mathematical software MATHEMATICA 9.0, we find decision variable and optimal cost.

*Step* 6: After that we plots different graphs of with respect to decision variables to validate the proposed model.

*Step* 7: Change values of different parameters with rate of +40%, +20%, −20%, −40% and find different results. From these results we construct sensitivity analysis.

*Step* 8: Result.

## 6. Managerial Implication for Business and Industry

Successful logistics operations in business resulted in greater rates of production, reduced expenses, better inventory resistor, more creative utilization of warehouse space, happier suppliers and customers, and an all-around better customer experience. These results provide a methodical technique for the procurement, storage, and sale of inventory, including both completed goods and raw materials (components) (products). In the business world, having the appropriate stock at the right amounts, in the right location at the right time, at a fair cost, and the right price are critical values. The enhancement of purchaser service in manufacturing industries is the primary driving force behind this issue. The retail industry benefits from this study. It will be utilized for a variety of products, including home goods, stylish clothing, and electronic components. Some realistic inventory features, like electronic parts, stylish clothing, household goods, fruits, fish, etc., are included in the suggested model. Consequently, the retailers operating on the aforementioned components will benefit from the managerial implications derived from the numerical findings of our suggested model. Success in business was logically correlated with improved customer and supplier satisfaction, increased productivity, decreased expenses, greater rates of production, better inventory control, and creative utilization of warehouse space.

## 7. Conclusion

To maximize profits, there is a greater chance that the producer may sell the products in several timelines, places, and global markets during distinct sales seasons. By taking benefit of the variations in the selling season of degrading things at several marketplaces, a maker of deteriorating items has a unique potential to increase profitability. The approach is very applicable to industries whose holding costs are contingent on time. The sensitivity analysis verifies the different process variables. It has been noted that the model's solution is relatively stable. This paper presents a way that makers of degrading things can use to identify the best production and inventory plan. The method initially uses preservation technology, and then the results are described both with and without preservation technology. In this case, the producer manufactures in one place and distributes to several markets throughout various selling seasons. It has been demonstrated that the technique aids in maximizing earnings and minimizing expenses. The collected findings demonstrate the model's stability and validity.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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