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Elegant Labeled Graphs

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Abstract. An *elegant labeling* f of a graph G with m edges is an injective function from the vertices of G to the set $\{0, 1, 2, \dots, m\}$ such that when each edge xy is assigned the label $f(x) + f(y) (\text{mod } m+1)$, the resulting edge labels are distinct and non zero.

In this paper we prove the following results

(i) The graph P_n^2 is elegant, for all $n \geq 1$.
(ii) The graphs $P_m^2 + \overline{K}_n$, $S_m + S_n$ and $S_m + \overline{K}_m$ are elegant, for all $m, n \geq 1$.
(iii) Every even cycle $C_{2n} :< a_0, a_1, \dots, a_{2n-1}, a_0 >$ with $2n - 3$ chords $a_0a_2, a_0a_3, \dots, a_0a_{2n-2}$ is elegant, for all $n \geq 2$.
(iv) The graph $C_3 \times P_m$ is elegant, for all $m \geq 1$.

1. **Introduction**

An *elegant labeling* f of a graph G with m edges is an injective function from the vertices of G to the set $\{0, 1, 2, \dots, m\}$ such that when each edge xy is assigned the label $f(x) + f(y) (\text{mod } m+1)$, the resulting edge labels are distinct and nonzero.

The kth power P_n^k of P_n, is the graph obtained from P_n by adding edges between all vertices u and v of P_n with $d(u, v) \leq k$. Grace [3] has shown that the graph P_n^2 is harmonious. Kang *et al.* [5] have shown that P_n^2 is graceful. In this direction, we prove that the graph P_n^2 is elegant, for all $n \geq 1$.

The *join* of disjoint graphs G and H, denoted $G + H$, is the graph obtained from G and H by joining each vertex of G to every vertex of H. Chang *et al.* [2] have shown that the graph $S_m + K_1$ is harmonious, where S_m is a star graph on m vertices. Graham and Sloane [4] have proved that the graphs $P_n + K_1$ and $P_n + \overline{K}_2$ are graceful and harmonious. Here we prove that the graphs $P_m + \overline{K}_n$, $S_m + S_n$ and $S_m + \overline{K}_n$ are elegant, for all $m, n \geq 1$.

Koh and Punnim [6] have proved that the cycles with 3-consecutive chords are graceful. Here we show that every even cycle $C_{2n} : \langle a_0, a_1, a_2, \dots, a_{2n-1}, a_0 \rangle$ with $2n - 3$ chords $a_0a_2, a_0a_3, \dots, a_0a_{2n-2}$ is elegant, for all $n \geq 2$.

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The *product* of graphs G and H denoted $G \times H$ is the graph with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. Consider $C_3 \times P_m$. We denote the graph $C_3 \times P_m$ as shown in Figure 1 with $V(C_3 \times P_m) = \{v_{11}, v_{12}, \ldots, v_{1m}, v_{21}, v_{22}, \ldots, v_{2m}, v_{31}, v_{32}, \ldots, v_{3m}\}$.

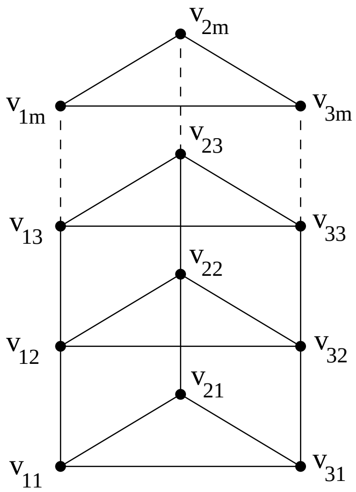

Figure 1. The graph $C_3 \times P_m$

Here we prove that the graph $C_3 \times P_m$ is elegant, for all $m \geq 1$.

2. Elegant Labeled Graphs

Theorem 2.1. *The graph P_n^2 is elegant, for all $n \geq 1$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_{n-1})$ and $G = P_n^2$. Clearly $|V(G)| = n$ and $|E(G)| = 2n - 3 = M$.

Define $f : V(G) \to \{0, 1, 2, \ldots, M\}$ by

$$f(v_0) = 0, \quad f(v_i) = 2(n-1) - i, \quad \text{if } 1 \leq i \leq n-1.$$

Clearly f is injective, the label of the edge $v_i v_{i+1}$ is $M - 2i$, $1 \leq i \leq n-2$ and the label of the edge $v_i v_{i+2}$ is $M - 2i - 1$, $0 \leq i \leq n-3$. Hence f is an elegant labeling of G. □

Theorem 2.2. *The graph $P_m^2 + \overline{K}_n$ is elegant, for all $m, n \geq 1$.*

Proof. Let $P_m = (u_1, u_2, \ldots, u_m)$ and let $V(\overline{K}_n) = \{v_1, v_2, \ldots, v_n\}$ and $G = P_m^2 + \overline{K}_n$. Clearly $|V(G)| = m + n$ and $|E(G)| = m(n+2) - 3 = M$.

Define $f : V(G) \to \{0, 1, 2, \ldots, M\}$ by

$$f(u_1) = 0,$$

$$f(u_i) = m(n+2) - (i+1), \quad \text{if } 2 \leq i \leq m$$

$$f(v_j) = jm, \quad \text{if } 1 \le j \le n.$$

Clearly f is injective, the label of the edge u_iu_{i+1} is $M-2(i-1)$, $1 \le i \le m-1$, the label of the edge u_iu_{i+2} is $M-2i+1$, $1 \le i \le m-2$, the label of the edge u_1v_j is jm, $1 \le j \le n$ and the label of the edge u_iv_j is $[m(n+j+2)-(i+1)] (\text{mod } M+1)$, $2 \le i \le m$ and $1 \le j \le n$.

Hence f is an elegant labeling of G. □

Theorem 2.3. *The graph $S_m + S_n$ is elegant, for all $m, n \ge 1$.*

Proof. Let S_m and S_n be two stars with $V(S_m) = \{u_1, u_2, \dots, u_m\}$ and $V(S_n) = \{v_1, v_2, \dots, v_n\}$ such that $deg\ u_1 = m$, $deg\ u_i = 1$, for $2 \le i \le m$ and $deg\ v_1 = n$, $deg\ v_i = 1$, for $2 \le i \le n$. Consider $G = S_m + S_n$. Clearly $|V(G)| = m+n$ and $|E(G)| = m(n+1)+n-2 = M$.

Define $f : V(G) \to \{0, 1, 2, \dots, M\}$ by

$$f(u_1) = 0,$$

$$f(u_i) = M-(i-2), \quad \text{if } 2 \le i \le m,$$

$$f(v_1) = n(m+1)-1,$$

$$f(v_j) = (j-1)(m+1), \quad \text{if } 2 \le j \le n.$$

Clearly f is injective, the label of the edge u_1u_i is $M-(i-2)$, $2 \le i \le m$, the label of the edge u_1v_1 is $n(m+1)-1$, the label of the edge u_1v_j is $(j-1)(m+1)$, $2 \le j \le n$, the label of the edge u_iv_1 is $[2n(m+1)+m-(i+1)]$ (mod $M+1$), $2 \le i \le m$, the label of the edge u_iv_j is $[(m+1)(n+j)-(i+1)]$ (mod $M+1$), $2 \le i \le m$ and $2 \le j \le n$ and the label of the edge v_1v_j is $[m(n+j-1)+n+j-2]$ (mod $M+1$), $2 \le j \le n$. Hence f is an elegant labeling of G. □

Theorem 2.4. *The graph $S_m + \overline{K}_n$ is elegant, for all $m, n \ge 1$.*

Proof. Let S_m be a star with $V(S_m) = \{u_1, u_2, \dots, u_m\}$ such that $\deg u_1 = m$, $\deg u_i = 1$, for $2 \le i \le m$ and let $V(\overline{K}_n) = \{v_1, v_2, \dots, v_n\}$. Consider $G = S_m + \overline{K}_n$. Clearly $|V(G)| = m+n$ and $|E(G)| = m(n+1)-1 = M$.

Define $f : V(G) \to \{0, 1, 2, \dots, M\}$ by

$$f(u_1) = 0,$$

$$f(u_i) = M-(i-2), \quad \text{if } 2 \le i \le m$$

$$f(v_j) = jm, \quad \text{if } 1 \le j \le n.$$

Clearly f is injective, the label of the edge u_1u_i is $M-(i-2)$, $2 \le i \le m$, the label of the edge u_1v_j is jm, $1 \le j \le n$ and the label of the edge u_iv_j is $[m(n+j+1)-(i-1)]$ (mod $M+1$), $2 \le i \le m$ and $1 \le j \le n$. Hence f is an elegant labeling of G. □

Theorem 2.5. *Every even cycle* C_{2n} : $\langle a_0, a_1, \ldots, a_{2n-1}, a_0 r\rangle$ *with* $2n-3$ *chords* $a_0a_2, a_0a_3, \ldots, a_0a_{2n-2}$ *is elegant, for all* $n \geq 2$.

Proof. Let G be an even cycle C_{2n} : $\langle a_0, a_1, \ldots, a_{2n-1}, a_0\rangle$ with $2n-3$ chords $a_0a_2, a_0a_3, \ldots, a_0a_{2n-2}$, for $n \geq 2$. Clearly $|V(G)| = 2n$ and $|E(G)| = 4n-3 = M$.

Define $f : V(G) \to \{0, 1, 2, \ldots, M\}$ by

$$f(a_0) = 0, \quad f(a_i) = 2i-1, \quad \text{if } 1 \leq i \leq 2n-1.$$

Clearly f is injective, the label of the edge a_0a_i is $2i-1$, $1 \leq i \leq 2n-1$, the label of the edge a_ia_{i+1} are $4i$, $1 \leq i \leq n-1$ and $4i \pmod{M+1}$, $n \leq i \leq 2n-2$. Hence f is an elegant labeling of G. □

Theorem 2.6. *The graph* $C_3 \times P_m$ *is elegant, for all* $m \geq 1$.

Proof. Let $C_3 \times P_m = V(v_{11}, v_{12}, v_{13}, \ldots, v_{1m}, v_{21}, v_{22}, \ldots, v_{2m}, v_{31}, v_{32}, \ldots, v_{3m})$ and $G = C_3 \times P_m$. Clearly $|V(G)| = 3m$ and $|E(G)| = 3(2m-1) = M$.

Define $f : V(G) \to \{0, 1, 2, \ldots, M\}$ by

$$f(v_{31}) = 0$$

$$f(v_{ki}) = 3(2m-i) + \delta_{ki},$$

where

$$\delta_{ki} = -\frac{1+(-1)^i}{2}, \quad \text{if } k = 1 \text{ and } 1 \leq i \leq m$$

$$\delta_{ki} = (-1)^i, \quad \text{if } k = 2 \text{ and } 1 \leq i \leq m$$

$$\delta_{ki} = \frac{1-(-1)^i}{2}, \quad \text{if } k = 3 \text{ and } 1 \leq i \leq m.$$

Clearly f is injective, the label of the edge $v_{1i}v_{2i}$ is $[6(2m-i)-\alpha] \pmod{(M+1)}$, $1 \leq i \leq m$, where $\alpha = 1$ or 0 depends on i is odd or even, the label of the edge $v_{21}v_{31}$ is $6m-4$, the label of the edge $v_{2i}v_{3i}$ is $[6(2m-i)+\alpha] \pmod{(M+1)}$, $2 \leq i \leq m$, where $\alpha = 0$ or 1 depends on i is odd or even, the label of the edge $v_{31}v_{11}$ is $6m-2$, the label of the edge $v_{3i}v_{1i}$ is $[6(2m-i)-\alpha] \pmod{(M+1)}$, $2 \leq i \leq m$, where $\alpha = -1$ or 1 depends on i is odd or even, the label of the edge $v_{1i}v_{1(i+1)}$ is $6(m-i)-2$, $1 \leq i \leq m-1$, the label of the edge $v_{2i}v_{2(i+1)}$ is $6(m-i)-1$, $1 \leq i \leq m-1$, the label of the edge $v_{3i}v_{3(i+1)}$ is $6(m-i)$, $1 \leq i \leq m-1$. Hence f is an elegant labeling of G. □

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