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Research Article

Quotient and Homomorphism in Ternary Semihyperrings

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Abstract. Ternary semihyperring is an algebraic structure with one binary hyper operation and ternary multiplication. In this paper, more characterizations and some properties of the ternary semihyperring are presented. Moreover, the quotient ternary semihyperrings are constructed via another relation and the isomorphism theorems are proved.

Keywords. Ternary semihyperring, Quotient ternary semihyperring, Homomorphism, Isomorphism

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1. Introduction

Marty [9] introduced Hyperstructure theory at the *8th Congress of Scandinavian Mathematicians* where he defined the notion of hyperoperation on groups. In a classical algebraic structure, the binary operation of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the hyperoperation of two elements, is a subset of the same set. If this hyperoperation sends two elements to a singleton set then the hyperoperation coincides with the classical binary operation.

In literature, a number of different hyperstructure theories are widely studied since these represent a suitable and natural generalization of classical algebraic structures such as groups, rings and modules and for their applications to many areas of pure and applied mathematics and computer science.

In 2004, Davvaz [3] proved the isomorphism theorems in ternary semihyperrings provided that the hyperideals considered in the isomorphism theorems are normal. A book of Davvaz and Leoreanu-Fotea [5] entitled *Hyperring Theory and Applications*, presented applications in pure and applied mathematics, probability, computer science, geometry, physics and chemistry.

Ternary algebra on the other hand was introduced in 1932 when Lehmer [7] studied certain ternary system called triplexes which is a generalization of abelian groups and of the ternary ring investigated by Lister [8] in 1971. The regular hyperrings were studied by Asokkumar and Velrajan [2]. In 2009, Davvaz [4] introduces the concept of ternary semihyperring and investigate some properties of its fuzzy hyperideals. hyperideals. In 2012, Anvariyeh and Mirkavili [1] studied Canonical (m,n)-hypermodules over Krasener hyperrings.

In this paper, the quotient ternary semihyperrings are constructed via another relation and the isomorphism theorems are proved.

2. Preliminaries and Basic Definitions

Definition 2.1 ([3])**.** Let H be a non-empty set and $\circ : H \times H \to \wp*(H)$ be a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H. The couple $(H,\circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$ we have

$$A \circ B = \bigcap_{a\in A, b\in B} a \circ b,\ A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Example 2.2. Define a hyperoperation '$\oplus$' on N by $x \oplus y = \{x, y\}$, for all $x, y \in N$. Then $(N,\oplus)$ is a hypergroupoid.

Definition 2.3 ([9, 10])**.** Let R be a nonempty set. A *ternary operation* on R is a map $f : R \times R \times R \to R$.

Example 2.4. Consider the set Z^- of negative integers and '$\cdot$' is the usual multiplication on Z. Then '$\cdot$' is a ternary operation in Z^- which is not binary, since Z^- is not closed under the binary product.

Definition 2.5. A non-empty set H is called ternary semihyperring if for all $h_1,h_2,h_3,h_4,h_5 \in H$ and $(H,\oplus)$ is a commutative semi hyper group and the ternary multiplication [ ] satisfies the following conditions:

(i) $[[h_1h_2h_3]h_4h_5] = [h_1[h_2h_3h_4]h_5] = [h_1h_2[h_3h_4h_5]]$,

(ii) $[(h_1 \oplus h_2)h_3h_4] = [h_1h_3h_4] \oplus [h_2h_3h_4]$,

(iii) $[h_1(h_2 \oplus h_3)h_4] = [h_1h_2h_4] \oplus [h_1h_3h_4]$,

(iv) $[h_1h_2(h_3 \oplus h_4)] = [h_1h_2h_3] \oplus [h_1h_2h_4$.

Definition 2.6. A ternary hyper semi ring H is said to have a zero element if there exist an element $0 \in H$ such that for all $h_1,h_2 \in H$,

$$[0h_1h_2] = [h_10h_2] = [h_1h_20] = \{0\}.$$

Definition 2.7. An element e of ternary hyper semi ring H is called an identity if

$$[h_1h_1e] = [h_1eh_1] = [eh_1h_1] = \{h_1\}, \quad \text{for all } h_1 \in H.$$

Definition 2.8 ([2])**.** A non empty additive subsemihyper group I of a ternary semihyperring H is called:

(i) a left hyper ideal of H if $[HHI] \subseteq I$,

(ii) a lateral hyper ideal of H if $[HIH] \subseteq I$,

(iii) a right hyper ideal of H if $[IHH] \subseteq I$.

If I is both left as well as right hyper ideal of H, then I is called a two sided hyper ideal of H. If I is a left, a lateral, a right hyper ideal of H then I is called a hyper ideal of H.

Lemma 2.9 ([6, 10])**.** *Let R be a ternary semihyperring. Then,*

(i) *if I, J are hyper ideals of R, then $I+J$ is a hyper ideal of R.*

(ii) *if I, J, and K are hyper ideals of R, then IJK is a hyper ideal of R.*

(iii) *the intersection of arbitrary hyper ideals of R is a hyper ideal of R.*

(iv) *if I is a hyper ideal of R and $x, y \in R$, then $xyI = \{xyi \mid i \in I\} = I$.*

3. Main Results

Lemma 3.1. *Let R be a canonical hypergroup and $x, y, z \in R$. If $z \in x + y$, then $-z \in -(x+y)$.*

Proof. Let $z \in x + y$. Since $(R,+)$ is a canonical hypergroup, by Definition 2.5(i) and (v), $y \in -x+z = z+(-x)$. Applying, Definition 2.5(i) and (v) again, $-x \in -z+y = y+(-z)$ and therefore, $-z \in -y+(-x) = -x-y = -(x+y)$. □

Lemma 3.2 (Hyper ideal Criterion)**.** *Let R be a ternary semihyperring. A nonempty subset I of R is a right (resp., left and lateral) hyper ideal of R if and only if for all $x, y \in I$ and $a, b \in R$,*

(i) *$x + y \subseteq I$,*

(ii) *$abi \in I$ (resp., $iab \in I$ and $aib \in I$).*

Proof. It is clear that $(I,+)$ is a canonical hypergroup if and only if condition (i) is satisfied. Thus, by Definition 2.8, I is a hyper ideal of R. □

Remark 3.3. Let H be a ternary semihyperring and R, a subternary semihyperring of R. If I is a hyper ideal of H such that $I \subseteq R$, then I is a hyper ideal of R.

The next remark follows directly from Lemma 3.2 and Remark 3.3.

Remark 3.4. Let R be a ternary semihyperring and I, J be right (resp., left and lateral) hyper ideal of R. Then $I \cap J$ is a hyper ideal of I and J. Also, I and J are hyper ideals of $I+J$.

Theorem 3.5. *Let J, K be right (resp., left and lateral) hyper ideals of a ternary semihyperring R. Then $J+K$ is the smallest hyper ideal containing J and K.*

Proof. By Remark 3.4, $J \subseteq J+K$ and $K \subseteq J+K$. Thus, $J \cup K \subseteq J+K$. Now, let I be a hyper ideal of R such that $J \cup K \subseteq I$ and let $x \in J+K$. Then, there exists $j \in J$ and $k \in K$ such that $x \in j+k$. Since I is a hyper ideal, by Lemma 3.2(i), $j+k \subseteq I$. Therefore, $x \in I$. Consequently, $J+K \subseteq I$. □

The next result is the construction of the the quotient class of a ternary semihyperring.

Theorem 3.6. *Let $(H,+,\cdot)$ be a ternary semihyperring and I a hyperideal of H. We define the relation ρ by $a\ \rho b$ if and only if $a\in b+I$. Then ρ is an equivalence relation on H.*

Proof. Let $a,b,c \in H$. Then $0 \in a-a$. So $0 \in I$, since I is a hyperideal of H. Therefore, $a \in \{a\} = a+0 \subseteq a+I$. Hence $a\rho a$.
Therefore, ρ is reflexive. Suppose $a\rho b \in R$. Then $a \in b+I$.
Thus, $\exists\ i \in I$ such that $a \in b+i$. By Lemma 3.1, $-a \in -(b+i) = -b-i$. Thus, by Lemma 3.1, $-b \in -a+i$. Consequently, by Lemma 3.1, $b = -(-b) \in -(-a+i) = a-i$. Since I is a hyperideal, $-i \in I$. Hence, $b \in a+I$. Therefore, $b\rho a$. So, ρ is symmetric.
Suppose that $a\rho b$ and $b\rho c$. Then $a \in b+I$ and $b \in c+I$. Thus $a \in b+i_1$ and $b \in c+i_2$ for some $i_1, i_2 \in I$. Hence $a \in b+i_1 \subseteq x \in c+i_2 x+i_1 = (c+i_2)+i_1 = c+(i_1+i_2) \subseteq c+I$. Therefore, $a\rho c$. Hence ρ is transitive. □

Definition 3.7. Let R be a ternary semihyperring. The equivalence class of $a \in R$ under ρ is defined as $\{x \in R/x\rho a\}$. The set R/I is the set of all equivalence classes of R.

Remark 3.8. Let H be a ternary semihyperring and I a hyper ideal of H, and $a,b \in R$. Then $a+I = b+I$ if and only if $a \in b+I$.
Thus $a,b \in R$ and $a+I = b+I$ if and only if $a \in b+I$ or $b \in a+I$.

Theorem 3.9. *Let $(R,+,[\ ])$ be a ternary semihyperring and I be a hyper ideal of R. Then R/I is a ternary semihyperring with the hyperoperation and ternary multiplication defined as follows.*

$$(a+I)\oplus(b+I) = \{x+I \mid x \in a+b\}$$

and

$$(a_1+I)*(a_2+I)*(a_3+I) = [a_1\ a_2\ a_3]+I.$$

Proof. Let $a+I$, $b+I$, $c+I$, $d+I$, $e+I \in R/I$. Then

$$\begin{aligned}(a+I)\oplus(b+I) &= \{r+I \mid r \in a+b\}\\ &= \{r+I \mid r \in b+a\}\\ &= (b+I)\oplus(a+I).\end{aligned}$$

Thus, $\oplus$ is a commutative hyperoperation on R/I. Also,

$$\begin{aligned}[(a+I)\oplus(b+I)]\oplus(c+I) &= \bigcup_{x+I\in(a+I)\oplus(b+I)} (x+I)\oplus(c+I)\\ &= \bigcup_{x\in a+b} \{y+I \mid y \in x+c\}\\ &= \bigcup_{x\in a+b} \{y+I \mid y \in (a+b)+c\}\\ &= \bigcup_{x\in a+b} \{y+I \mid y \in a+(b+c)\}\\ &= \bigcup_{x+I\in(a+I)\oplus(b+I)} (a+I)\oplus\{x+I | x \in b+c\}\\ &= (a+I)\oplus[(b+I)\oplus(c+I)].\end{aligned}$$

Thus $\oplus$ is an associative hyperoperation on R/I.

Now, for every $a+I \in R/I$,

$$(a+I)\oplus(0+I) = \{x+I \mid x \in a+0\} = \{x+I \mid x \in \{a\}\} = \{a+I\}.$$

Hence, I is the neutral element in R/I. Also, for every $a+I \in R/I$, $-a+I \in R/I$ and $(a+I)\oplus(-a+I) = \{x+I \mid x \in a-a\}$. Since $0 \in a-a$, $I \in (a+I)\oplus(-a+I)$. Therefore, every element of R/I has an inverse element in R/I.

Therefore, R/I is a additive semihypergroup.

Also,

$$\begin{aligned}
[(a+I)*(b+I)*(c+I)]*(d+I)*(e+I) &= [(abc+I)]*(de+I)\\
&= [(abc)de+I]\\
&= [a(bcd)e+I]\\
&= [(a+I)*(bcd+I)*(e+I)]\\
&= (a+I)*[(b+I)*(c+I)*(d+I)]*(e+I)\\
&= (a+I)*(b+I)*[(c+I)*(d+I)*(e+I)].
\end{aligned}$$

Finally,

$$\begin{aligned}
[(a+I)\oplus(b+I)]*(c+I)*(d+I) &= \{x+I \mid x \in a+b\}*(c+I)*(d+I)\\
&= \{xcd+I \mid x \in a+b\}\\
&= \{r+I \mid r \in (a+b)cd\}\\
&= \{r+I \mid r \in acd+bcd\}\\
&= (acd+I)*(bcd+I)\\
&= [(a+I)*(c+I)*(d+I)]\oplus[(b+I)*(c+I)*(d+I)].
\end{aligned}$$

Conditions (iii), (iv) are proved similarly. Therefore, $(R/I,\oplus,*)$ is a ternary semihyperring. □

Definition 3.10. Let I be a hyper ideal of a ternary semihyperring R. The ternary semihyperring R/I is called the factor or quotient ternary semihyperring.

Theorem 3.11. *Let I and J be hyper ideals of a ternary semihyperring R such that $I \subseteq J$. Then J/I is a hyper ideal of R/I.*

Proof. Immediately follows from Remark 3.3 since $J/I \subseteq R/I$. □

Definition 3.12. Let $(R_1,+,\cdot)$ and $(R_2,\oplus,*)$ be two ternary semihyperrings. A mapping $\varphi : R_1 \to R_2$ is called a homomorphism if the following are satisfied:

(i) $\varphi(a+b) = \varphi(a)\oplus\varphi(b)$, for all $a,b \in R_1$;

(ii) $\varphi(a_1\cdot a_2\cdot a_3) = \varphi(a_1)*\varphi(a_2)*\varphi(a_3)$, for all $a_1,a_2,a_3 \in R_1$;

(iii) $\varphi(0_{R_1}) = 0_{R_2}$.

Definition 3.13. Let φ be a homomorphism from a ternary semihyperring R_1 into a ternary semi hyperring R_2. Then

(i) the set $\{x \in R \mid \varphi(x) = 0\}$ is called $\operatorname{Ker}\varphi$,

(ii) the set $\{\varphi(x) \mid x \in R_1\}$ is called the Image of φ and is denoted by $\mathrm{Im}\,\varphi$.

Lemma 3.14. *If $\varphi : R_1 \to R_2$ is a homomorphism, then $\varphi(-x) = -\varphi(x)$, for all $x \in R_1$.*

Proof. Let $x \in R_1$. Since φ is a homomorphism $\varphi(x-x) = \varphi(x)+\varphi(-x)$. Since $0 \in x-x$, it follows that $\varphi(0) = 0 \in \varphi(x)+\varphi(-x)$. Therefore, by $\varphi(-x) = -\varphi(x)$. □

Theorem 3.15. *Let φ be a homomorphism from a ternary semihyperring $(R_1,+,\cdot)$ into a ternary semihyperring $(R_2,\cdot,*)$. Then $\mathrm{Ker}\,\varphi$ is a hyper ideal of R_1.*

(i) *If φ is an epimorphism and I is a hyper ideal of R_1, then $\varphi(I)$ is a hyper ideal of R_2.*

(ii) *If B is a hyper ideal of R_2, then $f^1(B) = \{a \in R_1 \mid f(a) \in B\}$ is a hyper ideal of R_1 containing $\mathrm{Ker}\,f$.*

Proof. (a) Let $x, y \in \mathrm{Ker}\,\varphi$. Then $\varphi(x-y) = \varphi(x)-\varphi(y) = 0-0 = \{0\}$. Therefore, $x-y \subseteq \mathrm{Ker}\,\varphi$. On the other hand, let $a_1, a_2 \in R$ and $r \in \mathrm{Ker}\,\varphi$.

Then

$$\varphi(a_1 \cdot a_2 \cdot r) = \varphi(a_1) * \varphi(a_2) * \varphi(r) = \varphi(a_1) * \varphi(a_2) * 0 = 0.$$

Thus, $a_1 \cdot a_2 \cdot r \in \mathrm{Ker}\,\varphi$. Therefore by Lemma 3.2, $\mathrm{Ker}\,\varphi$ is a hyper ideal of R_1.

(ii) Let $y_1, y_2 \in \varphi(I)$. Then there exist $x_1, x_2 \in I$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Thus, by Lemma 3.14,

$$y_1 - y_2 = \varphi(x_1) - \varphi(x_2) = \varphi(x_1 - x_2) \subseteq \varphi(I).$$

Let $y_1, y2 \in R_2$ and $r \in \mathrm{Im}\,\varphi$. Since φ is an epimorphism, there exist x_1, x_2, $xinR_1$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y \in 2$, and $\varphi(x) = r$. Thus,

$$y_1 * y_2 * r = \varphi(x_1) * \varphi(x_2) * \varphi(x) = \varphi(x_1 \cdot x_2 \cdot x) \in \varphi(I).$$

Therefore, by Lemma 3.2 $\varphi(I)$ is a hyper ideal of R_2.

(iii) Let $a, b \in f^{-1}(B)$. Then $f(a), f(b) \in B$. Since f is a homomorphism and B is a hyper ideal, $f(a+b) = f(a) \oplus f(b) \in B$. Hence, $a+b \in f^{-1}(B)$. Also, for all $i \in f^{-1}(B)$ and $x, y \in R_1$, $f(xyi) = f(x)f(y)f(i) \in B$. Thus, $xyi \in f^{-1}(B)$. Similarly, $xiy, ixy \in f^{-1}(B)$. Therefore, by Theorem **??**, $f^{-1}(B)$ is a hyper ideal of R_1. Finally, if $x \in \mathrm{Ker}\,f$, then $f(x) = 0 \in B$. Hence, $x \in f^{-1}(B)$. Therefore, $\mathrm{Ker}\,f \subseteq f^{-1}(B)$. □

Theorem 3.16. *Let R be a ternary semihyperring and I a hyper ideal of R. Then the map $\pi : R \to R/I$ defined by $\pi(r) = r + I$ is an epimorphism with kernel I.*

Proof. Let $x, y \in R_1$ such that $x = y$. Then $x + I = y + I$.

Thus $\pi(x) = \pi(y)$, which means that π is a well defined map.

Now,

$$\begin{aligned}
\pi(x+y) &= \{\pi(r) \mid r \in a+b\} \\
&= \{r + I \mid r \in a+b\} \\
&= (x+I) \oplus (y+I) \\
&= \pi(x) \oplus \pi(y)
\end{aligned}$$

and

$$\begin{aligned}\pi(xyz) &= [xyz]+I\\ &= (x+I)*(y+I)*(z+I)\\ &= \pi(x)*\pi(y)*\pi(z).\end{aligned}$$

Hence, π is a homomorphism. Also, let $x+I \in R/I$.
Then $\varphi(x) = x+I$. Thus π is an epimorphism. Finally, by Remark 3.8 ,

$$\begin{aligned}\operatorname{Ker}\pi &= \{x \in R_1 \mid \pi(x) = I\}\\ &= \{x \in R_1 \mid x+I = I\}\\ &= \{x \in R_1 \mid x \in I\}\\ &= I.\end{aligned}$$

The function π in Theorem 3.16 is called the *canonical ternary semihyperring epimorphism*. □

Theorem 3.17. *Let $f : R_1 \to R_2$ be a homomorphism of ternary semihyperrings and I be a hyperideal of R_1 contained in the kernel of f. Then*

(i) *there exists a unique homomorphism $\psi : R_1/I \to R_2$ defined by $\psi(x+I) = f(x)$;*

(ii) $\operatorname{Im} f = \operatorname{Im}\psi$;

(iii) $\operatorname{Im} f = \operatorname{Im}\psi$;

(iv) *ψ is an isomorphism if and only if f is an epimorphism and* $\operatorname{Ker} f = I$.

Proof. Let $x+I,\ y+I,\ z+I \in R_1/I$ and $f : R_1 \to R_2$ be a homomorphism.

(i) Suppose that $x+I = y+I$. Then, by Remark 3.8 $x \in y+I$. Thus, $x \in y+i$ for some $i \in I$. Hence,

$$f(x) \in f(y+i) = f(y)+f(i) = f(y)+0 = \{f(y)\}.$$

Therefore, $f(x) = f(y)$ and so $\varphi(x+I) = \varphi(y+I)$. Thus, φ is a well defined map. Now,

$$\begin{aligned}\psi((x+I)\oplus(y+I)) &= \psi(\{c+I \mid c \in x+y\})\\ &= \{f(c) \mid c \in x+y\}\\ &= f(x+y)\\ &= f(x)+f(y)\\ &= \psi(x+I)\oplus\psi(y+I)\end{aligned}$$

and

$$\begin{aligned}\psi((x+I)(y+I)(z+I)) &= \psi(xyz+I)\\ &= f(xyz)\\ &= f(x)*f(y)*f(z)\\ &= \psi(x+I)*\psi(y+I)*\psi(z+I).\end{aligned}$$

Hence, φ is a ternary semihyperring homomorphism and φ is unique, since it is uniquely determined by f.

(ii) It is clear that

$$\operatorname{Im}\psi = \{y \in R_2 \mid \psi(x+I) = y,\ \exists\ x+I \in R/I\}$$

$$= \{y \in R_2 \mid f(x) = y\}$$
$$= \operatorname{Im} f$$

and

$$\begin{aligned}\operatorname{Ker}\psi &= \{x + I \mid \psi(x+I) = 0\}\\ &= \{x + I \mid f(x) = 0\}\\ &= \{x + I \mid x \in \operatorname{Ker} f\}\\ &= (\operatorname{Ker} f)/I.\end{aligned}$$

(iii) Suppose that ψ is an isomorphism. Then, by (ii), $\operatorname{Im} f = \operatorname{Im}\psi = R_2$. Thus, f is an epimorphism. On the other hand, let $x \in \operatorname{Ker} f$. Then $f(x) = 0 = f(0)$. Hence, $\varphi(x+I) = \varphi(0+I)$. Since φ is injective, $x + I$. Thus, by Remark 3.8 , $x \in I$. Therefore, $\operatorname{Ker} f \subseteq I$. By hypothesis, $I \subseteq \operatorname{Ker} f$. So $I = \operatorname{Ker} f$. Conversely, by (i), ψ is a homomorphism.
Let $x+I, y+I \in R/I$ such that $\psi(x+I) = \psi(y+I)$. Then $f(x) = f(y)$. Thus, $0 \in f(x) - f(y) = f(x-y)$ by Definition 2.3(iv). Since $f(0) = 0$, there exists $t \in x - y$ such that $f(t) = 0$. Hence $t \in \operatorname{Ker} f = I$. Then $x \in -(-y) + t = y + t \subseteq y + I$. By Remark 3.8 , $x + I = y + I$. Therefore ψ is injective. Now, let $y \in R_2$. Since f is an epimorphism, there exists $x \in R_1$ such that $f(x) = y$. Thus, $\psi(x+I) = f(x) = y$. Hence ψ is surjective. Therefore, ψ is an isomorphism. □

Theorem 3.18 (First Isomorphism Theorem)**.** *Let $(R_1, +, \cdot)$ and $(R_2, \oplus, *)$ be ternary semihyperrings and let $\psi : R_1 \to R_2$ be a homomorphism. Then ψ induces an isomorphism of ternary semi hyperrings $R_1/\operatorname{Ker}\psi \cong \operatorname{Im}\psi$.*

Proof. Let $\varphi : R_1 \to \operatorname{Im}\psi$. Then φ is an epimorphism. Let $I = \operatorname{Ker}\psi$. By Theorem 3.17, $R_1/\operatorname{Ker}\psi \cong \operatorname{Im}\psi$. □

Theorem 3.19 (Second Isomorphism Theorem)**.** *Let R be a Krasner ternary semihyperring and I, J be hyper ideals of R, then $I/(I \cap J) \cong (I + J)/J$.*

Proof. Let I and J be hyper ideals of R. By Remark 3.4 and Theorem 3.5, $I \cap J$ is a hyper ideal of I and J is a hyper ideal of $I + J$, respectively. Thus, by Theorem 3.9, $(I+J)/I$ and $I/(I \cap J)$ are quotient ternary semihyperrings. Define a map $\psi : J \to (I + J)/I$ by $\varphi(j) = j + I$. Let $a, b \in J$ such that $a = b$. Then $a + I = b + I$. Therefore, $\psi(a) = \psi(b)$ and ψ is a well defined map. Let $a, b, c \in J$. Then

$$\begin{aligned}\psi(a+b) &= \{\psi(x) \mid x \in a + b\}\\ &= \{x + I \mid x \in a + b\}\\ &= (a+I) \oplus (b+I)\\ &= \psi(a) + \psi(b)\end{aligned}$$

and

$$\begin{aligned}\psi(abc) &= abc + I\\ &= (a+I)(b+I)(c+I)\\ &= \psi(a)\psi(b)\psi(c).\end{aligned}$$

Therefore, ψ is a homomorphism. Also,

$$\begin{aligned}\operatorname{Ker}\psi &= \{x \in J \mid \psi(j) = 0\}\\ &= \{x \in J \mid x + I = I\}\\ &= \{x \in J \mid x \in I\}\\ &= I \cap J\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}\psi &= \{\psi(x) \in (J+I)/I \mid x \in J\}\\ &= \{x + I \in (J+I)/I \mid x \in J\}\\ &= (I+J)/I.\end{aligned}$$

Therefore, by Theorem 3.18, $I/(I \cap J) \cong (I+J)/J$. □

Theorem 3.20 (Third Isomorphism Theorem)**.** *Let I and J be hyper ideals of a ternary semihyperring R. If $I \subseteq J$, then $(R/I)/(J/I) \cong R/J$.*

Proof. By Theorem 3.9, J/I is a hyper ideal of R/I. Thus by Theorem 3.11 $R/I/J/I$ is a quotient ternary semihyperring. Define a map $\phi : R/I \to R/J$ by $\phi(x+I) = x+J$ and let $x+I, y+I \in R/I$ such that $x+I = y+I$.

By Remark 3.8 , $x \in y+I$. Thus, $xy+i$, $i \in I$. Since $I \subseteq J$, $i \in J$. Therefore, $x \in y+i \subseteq y+J$. It follows that $x \in y+J$. Hence, by Remark 3.8 , $x+J = y+J$, which means that $\phi(x) = \phi(y)$. Therefore, ϕ is well-defined. Let $a, b, c \in R$. Then

$$\begin{aligned}\phi[(a+I) \oplus (b+I)] &= \phi\{x+I \mid x \in a+b\}\\ &= \{\phi(x+I) \mid x \in a+b\}\\ &= \{x+J \mid x \in a+b\}\\ &= (a+J) \oplus (b+J)\\ &= \phi(a+I) \oplus (b+I)\end{aligned}$$

and

$$\begin{aligned}&\phi[(a+I)*(b+I)*(c+I)]\\ &\qquad = \phi(abc+I)\\ &\qquad = ([abc]+J)\\ &\qquad = (a+J)*(b+J)*(c+J)\\ &\qquad = \phi(a+I)\phi(b+I)\phi(c+I).\end{aligned}$$

Therefore, ϕ is a homomorphism.

Also,

$$\begin{aligned}\operatorname{Ker}\phi &= \{x+I \in R/I \mid \phi(x+I) = J\}\\ &= \{x+I \in R/I \mid x+J = J\}\\ &= \{x+I \mid x \in J\}\\ &= J/I\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}\phi &= \{\phi(x+I) \mid x+I \in R/I\}\\ &= \{x+J \mid x \in R\}\\ &= R/J.\end{aligned}$$

Therefore, by Theorem 3.18, $(R/I)/(J/I) \cong R/J$. □

Theorem 3.21 (Correspondence Theorem). *If I and J are hyper ideals of a ternary semihyperring R, then there is a one-to-one correspondence between the set of hyper ideals of R which contains I and the set of all hyperideals of R/I, given by $J \to J/I$. Hence, every hyperideal in R/I is of the form J/I, where J is a hyperideal of R containing I.*

Proof. Let I be a hyper ideal of R. Then, by Theorem 3.16, $f: R \to R/I$ is an epimorphism with kernel I. Now, if J is a hyper ideal of R, then by Theorem 3.11, $f(J) = J/I$ is a hyper ideal of R/I and if K is a hyper ideal of R/I, then by Theorem 3.15(iii), $f^{-1}(\mathrm{Ker})$ is a hyper ideal of R. Thus, the map φ from the set of hyper ideals of R which contains I to the set of hyper ideals of R/I is a well defined map. Now, if K is a hyper ideal of R/I, then $\{0+I\} = \{I\} \subseteq K$. Thus $I = \operatorname{Ker} f = f^{-1}(I) \subseteq f^{-1}(K)$. Hence, $\varphi(f^{-1}(K)) = f(f^{-1}(K)) = K$ and φ is surjective. Next, claim that if J is a hyper ideal of R containing I, then $f^{-1}(f(J)) = J$. Clearly, $J \subseteq f^{-1}(f(J))$. Let $x \in f^{-1}(f(J))$. Then, $f(x) \in f(J)$, hence $f(x) = f(j)$ for some $j \in J$. Hence, $x+I = j+I$. By Remark 3.8 , $x \in k+I$. Since $I \subseteq J$, $x \in k+I \subseteq J$. Thus, $x \in J$. This proves the claim. Now, Let X and Y be hyper ideals of R containing I such that $\varphi(X) = \varphi(Y)$. Then $f(X) = f(Y)$ and $f^{-1}(f(X)) = f^{-1}(f(Y))$ which implies $X = Y$ by our assumption. Hence, φ is injective. Finally, it is clear that $f(J) = J/I$. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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