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On Mixed Type Duality for Multiobjective Programming Containing Support Functions

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Abstract. A mixed type vector dual to a multiobjective programming problem containing support functions is formulated and various duality results are proved under generalized invexity conditions. Special cases are generated from our results.

1. Introduction

In [5], Husain *et al.* considered the following multiobjective programming containing support functions

(NP) $\quad$ Minimize$(f^1(x)+S(x|C^1),\dots,f^p(x)+S(x|C^p))$

subject

$$g^j(x)+S(x|D^j)\leqq 0,\quad j=1,2,\dots,m.$$

Where

(i) $f^i:R^n\to R$ and $g^j:R^n\to R$, $j=1,2,\dots,m$ are differentiable functions and
(ii) $S(\cdot|C^i)$, $i=1,2,\dots,p$ and $S(\cdot|D^j)$, $j=1,2,\dots,m$ are support functions of a compact convex set C^i, $i=1,2,\dots,p$ and D^j, $j=1,2,\dots,m$ in R^n, to be defined later.

The following Wolfe type dual to the problem (NP) is presented [5]:

(WND)
$$\text{Maximize}\Bigg(f^1(u)+u^Tz^1+\sum_{j=1}^m y^j(g^j(u)+u^Tw^j),\dots,$$
$$f^p(u)+u^Tz^p+\sum_{j=1}^m y^j(g^j(u)+u^Tw^j)\Bigg)$$

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subject to

$$\sum_{i=1}^{p}\lambda^i\nabla(f^i(u)+u^Tz^i)+\sum_{j=1}^{m}y^j\nabla(g^j(u)+u^Tw^j)=0,$$

$$z^i\in C^i,\quad i=1,2,\dots,p,$$

$$w^j\in D^j,\quad j=1,2,\dots,m,$$

$$y\geqq 0,$$

$$\lambda>0,\ \sum_{i=1}^{p}\lambda^i=1.$$

The problem (WND) is a dual to (NP) assuming that

$$\sum_{i=1}^{p}\lambda^i(f^i(\cdot)+(\cdot)^Tz^i)+\sum_{j=1}^{m}y^j(g^j(\cdot)+(\cdot)^Tw^j)$$

is pseudoinvex with respect to η. The authors in [5] further weakened the invexity required in Wolfe type by constructing the following Mond-Weir type vector dual.

The Mond-Weir vector type dual is the following to (NP):

(M-WNP) Maximize $(f^1(u)+u^Tz^1,\dots,f^p(u)+u^Tz^p)$

subject to

$$\sum_{i=1}^{p}\lambda^i\nabla(f^i(u)+u^Tz^i)+\sum_{j=1}^{m}y^j(g^j(u)+u^Tw^j)=0,$$

$$z^i\in C^i,\quad i=1,2,\dots,p,$$

$$w^j\in D^j,\quad j=1,2,\dots,m,$$

$$\sum_{j=1}^{m}y^j(g^j(u)+u^Tw^j)\geqq 0,$$

$$\lambda>0,y\geqq 0.$$

Husain *et al.* [5] established usual duality theorems under the hypotheses that $\sum_{i=1}^{p}\lambda^i\nabla(f^i(\cdot)+(\cdot)^Tz^i)$ is pseudoinvex and $\sum_{j=1}^{m}y^j(g^j(\cdot)+(\cdot)^Tw^j)$ is quasi-invex with respect to the same η.

In this paper, we propose in the spirit of Husain and Jabeen [4] and Xu [7], a mixed type dual to (NP) to combine the problems (WND) and (M-WNP) and establish various duality theorems under generalized invexity conditions. Special cases are discussed to show that our results extend some earlier results in the literature.

2. **Pre-requisites**

Before stating our multiobjective nonlinear problem, we mention the following conventions for vectors x and y in n-dimensional Euclidian space R^n to be used throughout the analysis of this research.

$$x < y \iff x_i < y_i, \quad i = 1,2,\ldots,n.$$
$$x \leqq y \iff x_i \leqq y_i, \quad i = 1,2,\ldots,n.$$
$$x \leq y \iff x_i \leq y_i, \quad i = 1,2,\ldots,n, \text{ but } x \neq y$$

$x \nleq y$, is the negation of $x \leq y$

For $x, y \in R$, $x \leq y$ and $x < y$ have the usual meaning.

Before presenting our mixed type dual (Mix D), we mention some definitions of invexity and generalized invexity for easy reference.

Definition 2.1 (*Invexity*)**.** The function $\phi : R^n \to R$ is said to be invex with respect to η at $\bar{x}$ if there exists a vector function $\eta(x,\bar{x}) \in R^n$, such that for all x and $\bar{x} \in R^n$

$$\phi(x) - \phi(\bar{x}) \geqq \eta(x,\bar{x})^T \nabla\phi(\bar{x}).$$

Definition 2.2 (*Pseudoinvex*)**.** The function $\phi : R^n \to R$ is said to be pseudoinvex with respect to η at $\bar{x}$ if there exists a vector function $\eta(x,\bar{x}) \in R^n$, such that for all x and $\bar{x} \in R^n$

$$\eta(x,\bar{x})^T \nabla\phi(\bar{x}) \geqq 0$$

implies

$$\phi(x) \geqq \phi(\bar{x}).$$

Definition 2.3 (*Quasi-invex*)**.** The function $\phi : R^n \to R$ is said to be quasi-invex with respect to η at $\bar{x}$ if there exists a vector function $\eta(x,\bar{x}) \in R^n$, such that for all x and $\bar{x} \in R^n$

$$\phi(x) \leqq \phi(\bar{x})$$

implies

$$\eta(x,\bar{x})^T \nabla\phi(\bar{x}) \leqq 0.$$

Definition 2.4 (*Support function*)**.** Let K be a compact set in R^n, then the support function of K is defined by

$$S(x|K) = \max\{x^T v \in K\}.$$

A support function, being convex everywhere finite, has a subdifferential in the sense of convex analysis. The subdifferential of $s(x|K)$ is given by

$$\partial S(x|K) = \{z \in K | z^T x = S(x|K)\}.$$

For a set K, the normal cone to K at a point $x \in K$ is defines by

$$N_k(x) = \{y | y^T(z - x) \leq 0, \text{ for all } z \in K\}.$$

When K is a compact convex set, y is in $N_k(x)$ if and only if $S(y|K) = x^T y$ i.e., x is a subdifferential of s at y.

Definition 2.5 (*Efficient solution*)**.** A feasible solution $\bar{x}$ is efficient for (NP) if there exist no other feasible x for (VPE) such that for some $i \in P = \{1, 2, \ldots, p\}$,

$$f^i(x) + S(x|C^i) < f^i(\bar{x}) + S(\bar{x}|C^i)$$

and

$$f^j(x) + S(x|C^j) \leqq f^j(\bar{x}) + S(\bar{x}|C^j) \quad \text{for all} \quad j \in P, j \neq i.$$

In order to prove the strong duality theorem we will invoke the following lemma due to Chankong and Haimes [1]. In the subsequent analysis we shall denote the set of feasible solutions of the problem (NP) by X.

Lemma 2.6. *A point $\bar{x} \in X$ is an efficient for* (NP)*, if and only if $\bar{x} \in X$ solves the following problem:*

$(\mathrm{P_k}(\bar{\mathrm{x}}))$ Minimize $f^k(x) + s(S|C^k)$

subject to

$$f^i(x) + S(x|C^i) \leqq f^i(\bar{x}) + S(\bar{x}|C^i) \ \forall \ i \in P$$

$$g^j(x) + S(x|D^j) \leqq 0, \quad j = 1, 2, \ldots, m.$$

3. **Mixed type duality**

We formulate the following type dual (Mix D) to (NP):

(Mix D) $$\text{Maximize}\Bigg(f^1(u) + u^T z^1 + \sum_{j \in J_\circ} y^j(g^j(u) + u^T w^j), \ldots, f^p(u) + u^T z^p + \sum_{j \in J_\circ} y^j(g^j(u) + u^T w^j)\Bigg)$$

subject to

(1) $$\sum_{i=1}^{p} \lambda^i \nabla (f^i(u) + u^T z^i) + \sum_{j=1}^{m} y^j \nabla (g^j(u) + u^T w^j) = 0,$$

(2) $$\sum_{j \in J_\alpha} y^j(g^j(u) + u^T w^j) \geqq 0, \quad \alpha = 1, 2, \ldots, r,$$

(3) $$z^i \in C^i, \qquad i = 1, 2, \ldots, p,$$

(4) $$w^j \in D^j, \quad j = 1, 2, \ldots, m,$$

(5) $$y \geqq 0,$$

(6) $$\lambda \in \Lambda,$$

where $\Lambda = \left\{ \lambda \in R^p \middle| \lambda > 0, \sum_{i=1}^{p} \lambda^i = 1 \right\}$.

where $J_\alpha \subseteq M = \{1, 2, \ldots, m\}$, $\alpha = 0, 1, 2, \ldots, r$ with $\bigcup_{\alpha=0}^{r} J_\alpha = M$ and $J_\alpha \bigcap J_\beta = \phi$, if $\alpha \neq \beta$. If $J_\circ = M$, then (Mix D) becomes Wolfe type dual considered in [5], if $J_\circ = \phi$ and $J_\alpha = M$ for some $\alpha \in \{1, 2, \ldots, r\}$, then (Mix D) becomes Mond-Weir type dual considered in [5].

Theorem 3.1 (*Weak duality*). *Let $\bar{x}$ be feasible for* (NP) *and* $(u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$ *be feasible for* (Mix D). *If for all feasible* $(x, u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$, $\sum_{i=1}^{p} \lambda^i \nabla(f^i(\cdot) + (\cdot)^T z^i) + \sum_{j \in J_\circ} y^j (g^j(\cdot) + (\cdot)^T w^j)$ *is pseudoinvex and* $\sum_{j \in J_\alpha} y^j (g^j(\cdot) + (\cdot)^T w^j)$, $\alpha = 1, 2, \ldots, r$ *is quasi-invex with respect to* η, *then the following cannot hold.*

$$f^i(x) + s(x|C^i) \leq f^i(u) + u^T z^i + \sum_{j \in J_\circ} y^j (g^j(u) + u^T w^j) \tag{7}$$

for all $i \in \{1, \ldots, p\}$, *and*

$$f^k(x) + s(x|C^k) < f^k(u) + u^T z^k + \sum_{j \in J_\circ} y^j (g^j(u) + u^T w^j) \tag{8}$$

for some k.

Proof. Suppose that (7) and (8) hold. Then in view of $\lambda > 0$ and $\sum_{i=1}^{p} \lambda^i = 1$, (7) and (8) together with $x^T z^i \leqq s(x|C^i)$, $i = 1, 2, \ldots, p$ and $x^T w^j \leqq s(x|D^j)$, $j = 1, 2, \ldots, m$ and the feasibility for (NP) and (Mix D) imply

$$\begin{aligned} &\sum_{i=1}^{p} \lambda^i (f^i(x) + x^T z^i) + \sum_{j \in J_\circ} y^j (g^j(x) + x^T w^j) \\ &\quad < \sum_{i=1}^{p} \lambda^i (f^i(u) + u^T z^i) + \sum_{j \in J_\circ} y^j (g^j(u) + u^T w^j) \end{aligned}$$

This in view of the pseudoinvexity of

$$\sum_{i=1}^{p} \lambda^i (f^i(\cdot) + (\cdot)^T z^i) + \sum_{j \in J_\circ} y^j (g^j(\cdot) + (\cdot)^T w^j)$$

with respect to η, implies,

$$\eta^T(x, u) \left(\sum_{i=1}^{p} \lambda^i \nabla (f^i(u) + u^T z^i) + \sum_{j \in J_\circ} y^j \nabla (g^j(u) + u^T w^j) \right) < 0 \tag{9}$$

Since $\bar{x}$ is feasible for (VP), $(u, y, z^1, \ldots, z^p, w^1, \ldots, w^m, \lambda)$ is feasible for (Mix D), and $x^T w^j \leqq s(x|D^j)$, $j = 1, 2, \ldots, m$, we have

$$\sum_{j \in J_\alpha} y^j (g^j(x) + x^T w^j) \leqq \sum_{j \in J_\alpha} y^j (g^j(u) + u^T w^j), \quad \alpha = 1, 2, \ldots, r.$$

This in view of quasi-invexity of $\sum_{j \in J_\alpha} y^j (g^j(\cdot) + (\cdot)^T w^j)$, $\alpha = 1, 2, \ldots, r$ with respect to η, gives

$$\eta^T(x, u) \left(\sum_{j \in J_\alpha} y^j \nabla (g^j(x) + x^T w^j) \right) \leqq 0, \quad \alpha = 1, 2, \ldots, r$$

Hence

$$\eta^T(x, u) \nabla \left(\sum_{j \in M - J_\circ} y^j (g^j(x) + x^T w^j) \right) \leqq 0, \quad \alpha = 1, 2, \ldots, r \tag{10}$$

Combining (9) and (10), we have

$$\eta^T(x,u)\left(\sum_{i=1}^{p}\lambda^i\nabla(f^i(u)+u^Tz^i)+\sum_{j=1}^{m}y^j\nabla(g^j(u)+u^Tw^j)\right)<0 \tag{11}$$

From the equality constraint of (Mix D), we have

$$\eta^T(x,u)\left(\sum_{i=1}^{p}\lambda^i\nabla(f^i(u)+u^Tz^i)+\sum_{j=1}^{m}y^j\nabla(g^j(u)+u^Tw^j)\right)=0 \tag{12}$$

The relation (12) contradicts (11). Hence the conclusion of the theorem is true.

Theorem 3.2 (*Strong duality*). *Let $\bar{x}$ be an efficient solution of* (NP) *and for at least one i, $i \in \{1,2,\dots,p\}$, $\bar{x}$ satisfies the regularity condition [3] for the problem* ($P_k(\bar{x})$). *Then there exist $\lambda \in R^p$ with $\lambda^T = (\bar{\lambda}^1,\dots,\bar{\lambda}^i,\dots,\bar{\lambda}^p)$, $\bar{y} \in R^m$ with $\bar{y}^T = (\bar{y}^1,\dots,\bar{y}^i,\dots,\bar{y}^m)$, $z^i \in R^n$, $i = \{1,2,\dots,p\}$ and $w^j \in R^n$, $j = 1,2,\dots,m$ such that $(x,u,y,z^1,\dots,z^p,w^1,\dots,w^m,\lambda)$ is feasible for* (Mix D) *and the objectives of* (NP) *and* (Mix D) *are equal.*

Further, if the hypotheses of Theorem 1 are satisfied, then $(x,u,y,z^1,\dots,z^p,w^1,\dots,w^m,\lambda)$ is an efficient solution of (Mix D).

Proof. Since $\bar{x}$ is an efficient solution for ($P_k(\bar{x})$), this implies that there exists $\xi \in R^p$, $\upsilon \in R^m$ with $\bar{\upsilon}^T = (\bar{\upsilon}^1,\dots,\bar{\upsilon}^i,\dots,\bar{\upsilon}^m)$ and $z^i \in R^n$, $i = \{1,2,\dots,p\}$ such that

$$\bar{\xi}^k\nabla(f^k(x)+\bar{x}^T\bar{z}^k)+\sum_{\substack{i=1\\ i\neq k}}^{p}\bar{\xi}^i\nabla(f^i(x)+\bar{x}^T\bar{z}^i)$$

$$+\sum_{j=1}^{m}y^j\nabla(g^j(x)+x^Tw^j)=0, \tag{13}$$

$$\sum_{j=1}^{m}\bar{\upsilon}^j\nabla(g^j(x)+x^Tw^j)=0, \tag{14}$$

$$\bar{x}^T\bar{z}^i=S(\bar{x}|C^i),\quad i=1,2,\dots,p, \tag{15}$$

$$\bar{x}^T\bar{w}^j=S(\bar{x}|D^j),\quad j=1,2,\dots,m, \tag{16}$$

$$z^i\in C^i,\quad i=1,2,\dots,p, \tag{17}$$

$$w^j\in D^j,\quad j=1,2,\dots,m, \tag{18}$$

$$\xi>0,\ \bar{\upsilon}\geqq 0 \tag{19}$$

Dividing (13), (14) and (19) by $\sum_{i=1}^{p}\xi^i \neq 0$, and putting $\bar{\lambda}^i = \frac{\bar{\xi}^i}{\sum_{i=1}^{p}\xi^i}$ and $\bar{y}^i = \frac{\bar{\upsilon}^i}{\sum_{i=1}^{p}\xi^i}$, we have

$$(20)\qquad \sum_{i=1}^{p}\bar{\lambda}^i\nabla(f^i(\bar{x})+\bar{x}^T\bar{z}^i)+\sum_{j=1}^{m}\bar{y}^j\nabla(g^j(x)+\bar{x}^T\bar{w}^j)=0$$

$$(21)\qquad \sum_{j=1}^{m}\bar{y}^j\nabla(g^j(x)+\bar{x}^T\bar{w}^j)=0$$

$$(22)\qquad \lambda>0,\ \sum_{i=1}^{p}\lambda^i=1,\quad \bar{y}\geqq 0$$

The equation (21) implies

$$(23)\qquad \sum_{j\in J_\circ}\bar{y}^j(g^j(x)+\bar{x}^T\bar{w}^j)=0$$

and

$$(24)\qquad \sum_{j\in J_\alpha}\bar{y}^j(g^j(x)+\bar{x}^T\bar{w}^j)=0,\ \ \alpha=1,2,\ldots,r$$

The relation (20), (22) and (24) imply that $(x,u,y,z^1,\ldots,z^p,w^1,\ldots,w^m,\lambda)$ is feasible for (Mix D).

$$f^i(\bar{x})+\bar{x}^T\bar{z}^i+\sum_{j\in J_\circ}\bar{y}^j(g^j(x)+\bar{x}^T\bar{w}^j)=f^i(\bar{x})+S(\bar{x}|C^i),\quad i=1,2,\ldots,p.$$

This implies the objective of the primal and dual problems are equal.

Further, in view of the assumptions Theorem 1, the efficiency of $\bar{x}$ for (NP) is immediate.

Theorem 3.3 (*Converse duality*)**.** *Let* $(\bar{x},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda})$ *be an efficient solution for* (Mix D). *Assume that*

(A_1) f *and* g *are twice continuously differentiable,*
(A_2) $\nabla f^i(\bar{x})+\bar{z}^i+\sum_{j\in J_\circ}\bar{y}^j(\nabla g^j(\bar{x})+\bar{w}^j)$ *are linearly independent,*
(A_3) $\nabla^2(\lambda^T f^i(\bar{x})+\bar{y}^T g(\bar{x}))$ *is positive or negative definite.*

Further, if the assumptions of Theorem 1 are met, then $\bar{x}$ is an efficient solution.

Proof. Since $(\bar{x},\bar{y},\bar{z}^1,\ldots,\bar{z}^p,\bar{w}^1,\ldots,\bar{w}^m,\bar{\lambda})$ be an efficient solution of (Mix D), then there exist $\tau\in R^p$, $\beta\in R^n$, $\gamma\in R$ for each γ constraints, $\eta\in R^p$ with $\eta^T=(\eta^1,\ldots,\eta^i,\ldots,\eta^p)$ and $\mu\in R^m$ such that the following Fritz-John optimality conditions [2] are satisfied,

$$-\sum_{i=1}^{p}\tau^i\left(\nabla(f^i(\bar{x})+\bar{x}^T\bar{z}^i)+\sum_{j\in J_\circ}\bar{y}^j\nabla(g(\bar{x})+\bar{x}^T\bar{w}^j)\right)$$

$$(25)\qquad +\beta^T\nabla^2(\lambda^T f(\bar{x})+\bar{y}^T g(\bar{x}))-\gamma\sum_{\alpha=1}^{r}\sum_{j\in J_\circ}\bar{y}^j\nabla(g^j(x)+\bar{x}^T\bar{w}^j)=0$$

$$(26)\qquad -(\tau^T e)(g^j(\bar{x})+\bar{x}^T\bar{w}^j+\beta^T\nabla(g^j(\bar{x})+\bar{x}^T\bar{w}^j))-\mu^j=0,\quad j\in J_\circ$$

(27) $-\gamma(g^j(\bar{x}) + \bar{x}^T\bar{w}^j + \beta^T\nabla(g^j(\bar{x}) + \bar{x}^T\bar{w}^j)) - \mu^j = 0, \;\; j \in J_\alpha, \, \alpha = 1, \ldots, r$

(28) $\beta^T\nabla(f(\bar{x}) + \bar{x}^T\bar{z}^i) + \sum_{j\in J_\circ}\bar{y}^j(\nabla g^j(\bar{x}) + \bar{w}^j) - \eta^i = 0$

(29) $(\lambda^i\beta - \tau^i\bar{x}) \in N_{C^i}(\bar{z}^i), \quad i = 1, \ldots, p$

(30) $(\beta - (\tau^T e)\bar{x})\bar{y}^j \in N_{D^j}(\bar{w}^j), \quad j \in J_\circ$

(31) $(\beta - \gamma\bar{x})\bar{y}^j \in N_{D^j}(\bar{w}^j), \quad j \in J_\alpha, \; \alpha = i, \ldots, r$

(32) $\mu^T\bar{y} = 0$

(33) $\eta^T\lambda = 0$

(34) $\gamma\sum_{j\in J_\alpha}\bar{y}^j\nabla(g^j(\bar{x}) + \bar{x}^T\bar{w}^j) = 0, \quad \alpha = 1, \ldots, r$

(35) $(\tau, \mu, \eta, \gamma) \geqq 0$

(36) $(\tau, \beta, \mu, \eta, \gamma) \neq 0$

Since $\lambda > 0$, (33) implies $\eta = 0$. Consequently (28) implies

(37) $\left(\nabla(f^i(\bar{x}) + \bar{x}^T\bar{z}^i) + \sum_{j\in J_\circ}\bar{y}^j(\nabla g^j(\bar{x}) + \bar{w}^j)\right)\beta = 0$

Using the equality constraint of (Mix D) in (25), we have

$$-\sum_{i=1}^{p}(\tau^i - \gamma\lambda^i)\left(\nabla f^i(\bar{x}) + \bar{z}^i + \sum_{j\in J_\circ}\bar{y}^T(\nabla g^j(\bar{x}) + \bar{w}^j)\right)$$

(38) $$+\beta^T\nabla^2(\lambda^T f(\bar{x}) + \bar{y}g(\bar{x})) = 0$$

Postmultiplying (38) by β and then using (37), we have

$$\beta^T\nabla^2(\lambda^T f(\bar{x}) + \bar{y}^T g(x))\beta = 0$$

This because of (A_3), yields

(39) $\beta = 0$

Using (39) along with (A_2), we have

(40) $\tau^i - \gamma\lambda^i = 0, \quad i = 1, 2, \ldots, p$

Suppose $\gamma = 0$, then from (40) we have $\tau = 0$. Consequently we have from (26) and (27), $\mu = 0$.
Thus $(\tau, \beta, \mu, \eta, \gamma) = 0$, contradicting (36).
Hence $\gamma > 0$ and $\tau > 0$.
In view of (39), (29), (30) and (31) we have,

(41) $\bar{x}^T\bar{z}^i = S(\bar{x}|C^i), \quad i = 1, 2, \ldots, p$

(42) $\bar{x}^T\bar{w}^j = S(\bar{x}|D^j), \quad j = 1, 2, \ldots, m$

From (26) and (27) along with (42) and (35), we have

$$g^j(x) + s(\bar{x}|D^j) \leqq 0, \quad j = 1, 2, \ldots, m$$

This implies the feasibility of $\bar{x}$ for (VP).

From (26) and (32), we have

$$\sum_{j \in J_\circ} \bar{y}^j \nabla (g^j(\bar{x}) + \bar{x}^T \bar{w}^j) = 0$$

In view of this together with (41), we have

$$f^i(\bar{x}) + \bar{x}^T \bar{z}^i + \sum_{j \in J_\circ} \bar{y}^j (g^j(\bar{x}) + \bar{x}^T \bar{w}^j)^i = f^i(\bar{x}) + S(\bar{x}|C^i), \;\; i = 1, 2, \ldots, p$$

This establishes the equality of objective values of (NP).

This in view of the hypothesis of Theorem 1 gives the efficiency of $\bar{x}$ for (NP).

4. **Special cases**

In this section, we specialize our problem (NP) and its mixed dual problems (Mix D). As discussed in [6] we may write $S(x|C^i) = (x^T B^i x)^{\frac{1}{2}}, i = 1, \ldots, p$ and $S(x|D^j) = (x^T E^j x)^{\frac{1}{2}}$, $j = 1, \ldots, m$ and the matrices B^i, $i = 1, \ldots, p$ and E^j, $j = 1, \ldots, m$ are positive semidefinite. Putting these in our problems, we have

$(\text{NP})_1$ Minimize$(f^1(x) + (x^T B^1 x)^{\frac{1}{2}}, \ldots, f^p(x) + (x^T B^p x)^{\frac{1}{2}})$

subject to

$$g^j(x) + (x^T E^j x)^{\frac{1}{2}} \leqq 0, \quad j = 1, 2, \ldots, m$$

For the dual (Mix D) problem, we get

$(\text{Mix D})_1$ Maximize$\left(f^1(u) + u^T B^i z^1 + \sum_{j \in J_\circ} y^j (g^j(u) + u^T E^j w^j)\right)$

$$\left(f^p(u) + u^T B^p z^p + \sum_{j \in J_\circ} y^j (g^j(u) + u^T E^j w^j)\right)$$

subject to

$$\sum_{i=1}^{p} \lambda^i \nabla (f^i(u) + u^T B^i z^i) + \sum_{j=1}^{m} y^j (g^j(u) + u^T D^j w^j) = 0,$$

$$\sum_{j \in J_\alpha} y^j (g^j(u) + u^T D^j w^j) \geqq 0, \quad \alpha = 1, 2, \ldots, r,$$

$$z^T B^i z \leq 1, \quad i = 1, 2, \ldots, p,$$

$$(w^j)^T E^j w^j \leq 1, \quad j = 1, 2, \ldots, m,$$

$$\lambda > 0, \;\; y \geqq 0.$$

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