Journal of Informatics and Mathematical Sciences
Vol. 6, No. 1, pp. 45–54, 2014
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

On Some Convergence Theorems of Double Laplace Transform

Research Article

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Abstract. Starting with convergence, absolute convergence and uniform convergence of double Laplace transform is presented. Finally, a Volterra Integro-Partial Differential Equation is solved by using double Laplace transform.

Keywords. Double Laplace transform; Inverse Laplace transform; Integro-Partial differential equation; Partial derivatives

MSC. 44A10; 44A30

Received: October 11, 2014 **Accepted:** November 21, 2014

1. Introduction

The double Laplace transform [8] of a function $f(x,t)$ defined in the positive quadrant of the xt-plane is defined by the equation

$$L_x L_t\{f(x,t)\} = \bar{f}(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)\,dt\,dx \tag{1.1}$$

whenever that integral exist. Here p, s are complex numbers.

The double Laplace transform is useful in solving wave, Laplace, heat and non-homogeneous partial differential equation which occur in many branches of physics, in applied mathematics as well as in engineering. It can be used as a very effective tool in simplifying the calculations in many fields of engineering and mathematics. It also provides a powerful method for analyzing linear systems. It is heavily used in solving differential and integral equations. Estrin and

Higgins [6] in 1951, has given the systematic account of the general theory of an operational calculus using double Laplace transform and illustrated it's application through solution of two problems, one in electrostatics, the other in heat conduction. Coon and Bernstein [3] has done systematic study of double Laplace transform in 1953 and developed its properties including conditions for transforming derivatives, integrals and convolution. Buschman [2] in 1983 used double Laplace transform to solve a problem on heat transfer between a plate and a fluid flowing across the plate. In 1989, Debnath and Dahiya [4] established a set of new theorems concerning multidimensional Laplace transform of some functions and used the technique developed to solve an electrostatic potential problem.

Recently in 2013, Aghili and Zeinali [1] implemented multidimensional Laplace transforms method for solving certain non-homogeneous forth order partial differential equations. In 2014, Eltayeb, Kilicman and Mesloub [5] applied double Laplace transform method to evaluate the exact value of double infinite series.

Despite this, systematic account of the convergence of double Laplace transform is not accessible and understandable to many. Therefore, we revisit the double Laplace transform developed in the middle half of 20th century and present convergence, absolute convergence and uniform convergence of double Laplace transform. In the last section, we use double Laplace transform to solve Volterra Integro-Partial Differential Equation [7].

2. Convergence Theorem of Double Laplace Integral

In this section, we prove the convergence theorem of double Laplace integral.

Theorem 2.1. *Let $\varphi(x,t)$ be a function of two variables continuous in the positive quadrant of the xt-plane. If the integral*

$$\int_0^\infty \int_0^\infty e^{-px-st}\varphi(x,t)dxdt \tag{2.1}$$

converges at $p=p_0$, $s=s_0$ then integral (2.1) *converges for $p>p_0$, $s>s_0$.*

For the proof we will use the following lemmas:

Lemma 2.2. *If the integral*

$$\int_0^\infty e^{-st}\varphi(x,t)dt \tag{2.2}$$

converges at $s=s_0$ then the integral (2.2) *converges for $s>s_0$.*

Proof. Set

$$\alpha(x,t)=\int_0^t e^{-s_0 v}\varphi(x,v)dv, \quad 0<t<\infty. \tag{2.3}$$

Clearly $\alpha(x,0)=0$ and $\lim\limits_{t\to\infty}\alpha(x,t)$ exists because integral $\int_0^\infty e^{-st}\varphi(x,t)dt$ converges at $s=s_0$. By the Fundamental theorem of Calculus

$$\alpha_t(x,t)=e^{-s_0 t}\varphi(x,t)$$

Choose ε_1 and R_1 so that $0 < \varepsilon_1 < R_1$.

$$\int_{\varepsilon_1}^{R_1} e^{-st}\varphi(x,t)dt = \int_{\varepsilon_1}^{R_1} e^{-st}e^{s_0 t}\alpha_t(x,t)dt$$
$$= \int_{\varepsilon_1}^{R_1} e^{-(s-s_0)t}\alpha_t(x,t)dt.$$

By using integration by parts

$$= [e^{-(s-s_0)t}\alpha(x,t)]_{\varepsilon_1}^{R_1} - \int_{\varepsilon_1}^{R_1} e^{-(s-s_0)t}[-(s-s_0)]\alpha(x,t)dt$$

$$\int_{\varepsilon_1}^{R_1} e^{-st}\varphi(x,t)dt = e^{-(s-s_0)R_1}\alpha(x,R_1) - e^{-(s-s_0)\varepsilon_1}\alpha(x,\varepsilon_1) + (s-s_0)\int_{\varepsilon_1}^{R_1} e^{-(s-s_0)t}\alpha(x,t)dt.$$

Now let $\varepsilon_1 \to 0$. Both terms on the right which depend on ε_1 approach a limit and

$$\int_0^{R_1} e^{-st}\varphi(x,t)dt = e^{-(s-s_0)R_1}\alpha(x,R_1) + (s-s_0)\int_0^{R_1} e^{-(s-s_0)t}\alpha(x,t)dt.$$

Now let $R_1 \to \infty$. If $s > s_0$, the first term on the right approaches zero.

$$\int_0^{\infty} e^{-st}\varphi(x,t)dt = (s-s_0)\int_0^{\infty} e^{-(s-s_0)v}\alpha(x,t)dt \quad \text{for } s > s_0. \tag{2.4}$$

The theorem is proved if the integral on the right converges.

By using the Limit test for convergence (see [10]).

For we have,

$$\lim_{t\to\infty} t^2 e^{-(s-s_0)t}\alpha(x,t) = \lim_{t\to\infty}\frac{t^2}{e^{(s-s_0)t}}\left[\lim_{t\to\infty}\alpha(x,t)\right]$$
$$= 0 * \left[\lim_{t\to\infty}\alpha(x,t)\right] = 0 = \text{finite}$$

Therefore, integral on the right of (2.4) converges for $s > s_0$.

Hence the integral $\int_0^\infty e^{-st}\varphi(x,t)dt$ converges for $s > s_0$. □

Lemma 2.3. *If* (a) *integral*

$$h(x,s) = \int_0^{\infty} e^{-st}\varphi(x,t)dt \tag{2.5}$$

converges for $s \geq s_0$ *and* (b) *integral*

$$\int_0^{\infty} e^{-px}h(x,s)dx \tag{2.6}$$

converges at $p = p_0$ *then the integral* (2.6) *converges for* $p > p_0$.

Proof. Let

$$\beta(x,s)=\int_0^x e^{-p_0u}h(u,s)du, \quad 0<x<\infty. \tag{2.7}$$

Therefore $\beta(0,s)=0$ and $\lim_{x\to\infty}\beta(x,s)$ exists because integral $\int_0^\infty e^{-px}h(x,s)dx$ converges at $p=p_0$.

By Fundamental theorem of Calculus

From (2.7), $\beta_x(x,s)=e^{-p_0x}h(x,s)$.

Choose ε_2 and R_2 so that $0<\varepsilon_2<R_2$.

$$\begin{aligned}
&\int_{\varepsilon_2}^{R_2} e^{-px}h(x,s)dx\\
&\quad=\int_{\varepsilon_2}^{R_2} e^{-px}e^{p_0x}\beta_x(x,s)dx\\
&\quad=\int_{\varepsilon_2}^{R_2} e^{-(p-p_0)x}\beta_x(x,s)dx\\
&\quad=[e^{-(p-p_0)x}\beta(x,s)]_{\varepsilon_2}^{R_2}-\int_{\varepsilon_2}^{R_2} e^{-(p-p_0)x}[-(p-p_0)]\beta(x,s)dx\\
&\quad=e^{-(p-p_0)R_2}\beta(R_2,s)-e^{-(p-p_0)\varepsilon_2}\beta(\varepsilon_2,s)+(p-p_0)\int_{\varepsilon_2}^{R_2} e^{-(p-p_0)x}\beta(x,s)dx.
\end{aligned}$$

Now let $\varepsilon_2\to 0$. Both terms on the right which depend on ε_2 approach a limit and

$$\int_0^{R_2} e^{-px}h(x,s)dx=e^{-(p-p_0)R_2}\beta(R_2,s)+(p-p_0)\int_0^{R_2} e^{-(p-p_0)x}\beta(x,s)dx. \tag{2.8}$$

Now let $R_2\to\infty$. If $p>p_0$, the first term on the right approaches zero.

$$\int_0^{R_2} e^{-px}h(x,s)dx=(p-p_0)\int_0^{R_2} e^{-(p-p_0)x}\beta(x,s)dx, \quad \text{for } p>p_0. \tag{2.9}$$

The theorem is proved if the integral on the right converges.

By using the Limit test for convergence (see [10]).

For we have,

$$\begin{aligned}
\lim_{x\to\infty} x^2e^{-(p-p_0)x}\beta(x,s)&=\lim_{x\to\infty}\frac{x^2}{e^{(p-p_0)x}}\left[\lim_{x\to\infty}\beta(x,s)\right]\\
&=0*\left[\lim_{x\to\infty}\beta(x,s)\right]=0=\text{finite}.
\end{aligned}$$

Therefore, integral on the right of (2.9) converges for $p>p_0$.

Hence, the integral $\int_0^\infty e^{-px}h(x,s)dx$ converges for $p>p_0$.

The proof of the Theorem 2.1 is as follows.

$$\begin{aligned}
\int_0^\infty\int_0^\infty e^{-px-st}\varphi(x,t)dxdt&=\int_0^\infty e^{-px}\left\{\int_0^\infty e^{-st}\varphi(x,t)dt\right\}dx\\
&=\int_0^\infty e^{-px}h(x,s)dx
\end{aligned} \tag{2.10}$$

where $h(x,s) = \int_0^\infty e^{-st}\varphi(x,t)dt$.
By Lemma 2.2, integral $\int_0^\infty e^{-st}\varphi(x,t)dt$ converges for $s > s_0$.
Also by Lemma 2.3, integral $\int_0^\infty e^{-px}h(x,s)dx$ converges for $p > p_0$.
Therefore, the integral in RHS of (2.10) converges for $p > p_0$, $s > s_0$.
Hence the integral

$$\int_0^\infty \int_0^\infty e^{-px-st}\varphi(x,t)dxdt$$

converges for $p > p_0$, $s > s_0$.
This completes the proof of the Theorem 2.1. □

Corollary 2.4. *If the integral* (2.1) *diverges at* $p = p_0$, $s = s_0$ *then the integral* (2.1) *diverges at* $p > p_0$, $s > s_0$.

Corollary 2.5. *The region of the convergence of the integral* (2.1) *is the positive quadrant of the* xt*-plane.*

Now we prove absolute convergence of integral (2.1).

Theorem 2.6. *If the integral* (2.1) *converges absolutely at* $p = p_0$, $s = s_0$ *then integral* (2.1) *converges absolutely for* $p \geq p_0$, $s \geq s_0$.

Proof. We know that

$$e^{-px-st}|\varphi(x,t)| \leq e^{-p_0x}, \quad p_0 \leq p < \infty, s_0 \leq s < \infty.$$

Therefore

$$\int_0^\infty \int_0^\infty e^{-px-st}|\varphi(x,t)|dtdx \leq \int_0^\infty \int_0^\infty e^{-p_0x-s_0t}|\varphi(x,t)|dtdx.$$

From given hypothesis,

$$\int_0^\infty \int_0^\infty e^{-p_0x-s_0t}|\varphi(x,t)|dtdx \text{ converges.}$$

Hence, $\int_0^\infty \int_0^\infty e^{-px-st}|\varphi(x,t)|dtdx$ converges for $p \geq p_0$, $s \geq s_0$.
Therefore, integral (2.1) converges absolutely for $p \geq p_0$, $s \geq s_0$. □

3. Uniform Convergence

In this section we prove the uniform convergence of double Laplace transform.

Theorem 3.1. *If* $f(x,t)$ *is continuous on* $[0,\infty)\times[0,\infty)$ *and*

$$H(x,t) = \int_0^x \int_0^t e^{-p_0u-s_0v}f(u,v)dvdu \tag{3.1}$$

is bounded on $[0,\infty)\times[0,\infty)$*, then the double Laplace transform of* f *converges uniformly on* $[p,\infty)\times[s,\infty)$ *if* $p > p_0$, $s > s_0$.

For the proof we will use the following lemmas:

Lemma 3.2. *If* $g(x,t)=\int_0^t e^{-s_0v}f(x,v)dv$ *is bounded on* $[0,\infty)$ *then the Laplace transform of* f *with respect to* s *converges uniformly on* $[s,\infty)$ *if* $s>s_0$.

Proof. If $0\le r\le r_1$,

$$\begin{aligned}\int_r^{r_1} e^{-st}f(x,t)dt &= \int_r^{r_1} e^{-(s-s_0)t}e^{-s_0t}f(x,t)dt\\ &= \int_r^{r_1} e^{-(s-s_0)t}g_t(x,t)dt.\end{aligned}$$

Using integration by parts

$$= e^{-(s-s_0)r_1}g(x,r_1)-e^{-(s-s_0)r}g(x,r)+(s-s_0)\int_r^{r_1} e^{-(s-s_0)t}g(x,t)dt.$$

Therefore, if $|g(x,t)|\le M$ then

$$\begin{aligned}\left|\int_r^{r_1} e^{-st}f(x,t)dt\right| &\le M\left\{e^{-(s-s_0)r_1}+e^{-(s-s_0)r}+(s-s_0)\int_r^{r_1} e^{-(s-s_0)t}dt\right\}\\ &= M\{e^{-(s-s_0)r_1}+e^{-(s-s_0)r}-e^{-(s-s_0)r_1}+e^{-(s-s_0)r}\}\\ &= 2Me^{-(s-s_0)r}, \quad \text{for } s>s_0.\end{aligned}$$

By Cauchy criterion for uniform convergence (see [9]).
$\int_0^\infty e^{-st}f(x,t)dt$ converges uniformly on $[s,\infty)$ if $s>s_0$.
Hence, Laplace transform of f with respect to s converges uniformly on $[s,\infty)$ if $s>s_0$. □

Lemma 3.3. *If* (a) *integral* $g(x,s)=\int_0^\infty e^{-st}f(x,t)dt$ *converges uniformly on* $[s,\infty)$ *if* $s>s_0$, (b) $\alpha(x,s)=\int_0^x e^{-p_0u}g(u,s)du$ *is bounded on* $[0,\infty)$ *then the Laplace transform of* f *with respect to* s *converges uniformly on* $[p,\infty)$ *if* $p>p_0$.

Proof. Proof is similar to Lemma 3.2.

The proof of the Theorem 3.1 is as follows:

$$\begin{aligned}H(x,t) &= \int_0^x\int_0^t e^{-p_0u-s_0v}f(u,v)dvdu\\ &= \int_0^x e^{-p_0u}\left\{\int_0^t e^{-s_0v}f(u,v)dv\right\}du\\ &= \int_0^x e^{-p_0u}g(u,t)du\end{aligned}$$

Where $g(u,t)=\int_0^t e^{-s_0v}f(u,v)dv$ is bounded on $[0,\infty)$.

By Lemma 3.2, Laplace transform of f with respect to s converges uniformly on $[s,\infty)$ if $s>s_0$.

Also by Lemma 3.3, Laplace transform of g with respect to p converges uniformly on $[p,\infty)$ if $p>p_0$.

Hence double Laplace transform of f converges uniformly on $[p,\infty)\times[s,\infty)$ if $p>p_0$, $s>s_0$. □

We now prove the differentiability of double Laplace transform

Theorem 3.4. *If $f(x,t)$ is continuous on $[0,\infty)\times[0,\infty)$ and*

$$H(x,t)=\int_0^x\int_0^t e^{-p_0u-s_0v}f(u,v)dvdu$$

is bounded on $[0,\infty)\times[0,\infty)$, then the double Laplace transform of f is infinitely differentiable with respect to p and s on $[p,\infty)\times[s,\infty)$ if $p>p_0$, $s>s_0$, with

$$\frac{\partial^{m+n}}{\partial p^m\partial s^n}\bar{f}(p,s)=(-1)^{m+n}\int_0^\infty\int_0^\infty e^{-px-st}x^mt^nf(x,t)dtdx. \tag{3.2}$$

For the proof we will use the following lemmas:

Lemma 3.5. *If $g(x,t)=\int_0^t e^{-s_0v}f(x,v)dv$ is bounded on $[0,\infty)$ then the Laplace transform of f is infinitely differentiable with respect to s on $[s,\infty)$ if $s>s_0$, with*

$$\frac{\partial^n}{\partial s^n}\bar{f}(x,s)=(-1)^n\int_0^\infty e^{-st}t^nf(x,t)dt. \tag{3.3}$$

Proof. First we prove that the integrals

$$I_n(x,s)=(-1)^n\int_0^\infty e^{-st}t^nf(x,t)dt,\quad n=0,1,2,3,\dots$$

all converge uniformly on $[s,\infty)$ if $s>s_0$.

If $0<r<r_1$, then

$$\begin{aligned}\int_r^{r_1}e^{-st}t^nf(x,t)dt&=\int_r^{r_1}e^{-(s-s_0)t}t^ng_x(x,t)dt\\&=e^{-(s-s_0)r_1}r_1^ng(x,r_1)-e^{-(s-s_0)r}r^ng(x,r)\\&\quad-\int_r^{r_1}\left\{\frac{d}{dt}[e^{-(s-s_0)t}t^n]\right\}g(x,t)dt.\end{aligned}$$

Therefore, if $|g(x,t)|\le M<\infty$ on $[0,\infty)$ then

$$\left|\int_r^{r_1}e^{-st}t^nf(x,t)dt\right|\le M\left\{e^{-(s-s_0)r}r^n+e^{-(s-s_0)r}r^n+\int_r^\infty\left\{\frac{d}{dt}[e^{-(s-s_0)t}t^n]\right\}dt\right\}.$$

Therefore, since $e^{-(s-s_0)r}r^n$ decreases monotonically on $(0,\infty)$ if $s>s_0$.

$$\left|\int_r^{r_1}e^{-st}t^nf(x,t)dt\right|\le 3Me^{-(s-s_0)r}r^n,\quad 0<r<r_1.$$

By Cauchy criterion for uniform convergence (see [9]).

$I_n(x,s)$ converges uniformly on $[s,\infty)$ if $s>s_0$.

Now, using [9, pp. 18-19] and induction proof, we have (3.3).

That is Laplace transform of f is infinitely differentiable with respect to s on $[s,\infty)$ if $s>s_0$. □

Lemma 3.6. *If* (a) *the integral* $\varphi(x,s)=\int_0^\infty e^{-st}t^n f(x,t)dt$ *converges uniformly on* $[s,\infty)$ *if* $s>s_0$. (b) $h(x,s)=\int_0^x e^{-p_0u}\varphi(x,s)dx$ *is bounded on* $[0,\infty)$ *then the Laplace transform of* φ *is infinitely differentiable with respect to* p *on* $[p,\infty)$ *if* $p>p_0$, *with*

$$\frac{\partial^m}{\partial p^m}\varphi(x,s)=(-1)^m\int_0^\infty e^{-st}t^m\varphi(x,s)dx. \tag{3.4}$$

Proof. Proof is similar to Lemma 3.5.

The proof of the Theorem 3.4 is as follows:

$$\begin{aligned}H(x,t)&=\int_0^x\int_0^t e^{-p_0u-s_0v}f(u,v)dvdu\\ &=\int_0^x e^{-p_0u}\left\{\int_0^t e^{-s_0v}f(u,v)dv\right\}du\\ &=\int_0^x e^{-p_0u}g(u,t)du\end{aligned}$$

where $g(u,t)=\int_0^t e^{-s_0v}f(u,v)dv$ is bounded on $[0,\infty)$.

By Lemma 3.5, Laplace transform of f is infinitely differentiable with respect to s on $[s,\infty)$ if $s>s_0$.

Also by Lemma 3.6, Laplace transform of g is infinitely differentiable with respect to p on $[p,\infty)$ if $p>p_0$.

Hence double Laplace transform of f is infinitely differentiable with respect to p and s on $[p,\infty)\times[s,\infty)$ if $p>p_0$, $s>s_0$. □

4. Double Laplace Transform of Double Integral

We now find the double Laplace transform of double integral.

Theorem 4.1. *If* $L_xL_t\{f(x,t)\}=\bar{f}(p,s)$ *and*

$$g(x,t)=\int_0^x\int_0^t f(u,v)dvdu \tag{4.1}$$

then

$$L_xL_t\left\{\int_0^x\int_0^t f(u,v)dvdu\right\}=\frac{\bar{f}(p,s)}{ps}. \tag{4.2}$$

Proof. Denote $h(x,t)=\int_0^t f(x,v)dv$.

By fundamental theorem of calculus

$$h_t(x,t)=f(x,t) \tag{4.3}$$

and

$$h(x,0)=0. \tag{4.4}$$

Taking double Laplace transform of equation (4.3), we get

$$s\bar{h}(p,s)-\bar{h}(p,0)=\bar{f}(p,s) \tag{4.5}$$

and single Laplace transform of equation (4.4)

$$\bar{h}(p,0)=0.$$

Then equation (4.5) becomes,

$$\bar{h}(p,s)=\frac{\bar{f}(p,s)}{s}. \tag{4.6}$$

From (4.1), $g(x,t)=\int_0^x h(u,t)du$

$$g_x(x,t)=h(x,t) \text{ and } g(0,t)=0,$$

$$p\bar{g}(p,s)-\bar{g}(0,s)=\bar{h}(p,s).$$

Now by using (4.6) and (4.1), we obtain

$$L_xL_t\left\{\int_0^x\int_0^t f(u,v)dvdu\right\}=\frac{\bar{f}(p,s)}{ps}.$$

□

5. Application of Double Laplace Transform in Volterra Integro-Partial Differential Equation

We use the double Laplace transform to solve the problem which is already solved in [7] using Differential transform method.

Example 5.1. Consider the following Volterra Integro Partial Differential Equation:

$$\frac{\partial u(x,y)}{\partial x}+\frac{\partial u(x,y)}{\partial y}=-1+e^x+e^y+e^{x+y}+\int_0^x\int_0^y u(r,t)drdt \tag{5.1}$$

Subject to the initial conditions:

$$u(x,0)=e^x \text{ and } u(0,y)=e^y. \tag{5.2}$$

Applying double Laplace transform of equation (5.1), we get

$$p\bar{u}(p,s)-\bar{u}(0,s)+s\bar{u}(p,s)-\bar{u}(p,0)$$

$$=-\frac{1}{ps}+\frac{1}{(p-1)s}+\frac{1}{p(s-1)}+\frac{1}{(p-1)(s-1)}+\frac{\bar{u}(p,s)}{ps} \tag{5.3}$$

and single Laplace transforms of equation (5.2)

$$\bar{u}(p,0)=\frac{1}{(p-1)} \quad \text{and} \quad \bar{u}(0,s)=\frac{1}{s-1}. \tag{5.4}$$

Substituting (5.4) in (5.3) and simplifying, we obtain

$$\bar{u}(p,s)=\frac{1}{(p-1)(s-1)}. \tag{5.5}$$

By using double inverse Laplace transform, we obtain solution of (5.1) as follows:

$$u(x,t)=e^{x+t}. \tag{5.6}$$

6. Conclusion

In this paper, we presented convergence, absolute convergence and uniform convergence of double Laplace transform. Besides these, we obtained double Laplace transform of double integral and use it to solve Volterra integro-partial differential equation.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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