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Boundedness of a Max-type Fourth Order Difference Equation with Periodic Coefficients

Research Article

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Abstract. The boundedness nature is considered in this paper for positive solutions of a max-type fourth order difference equation with periodic coefficients. A series of sufficient conditions are obtained to ensure the existence of bounded and unbounded solutions to this equation.

Keywords. Max-type difference equation; Boundedness; Periodic coefficient

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1. Introduction

In the past two decades, there has been much interest in studying max-type difference equations. For example, see [1–23] and the references cited therein. Our main aim in this paper is to investigate the boundedness nature of positive solutions of the following max-type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-3}}\right\}, \quad n = 0, 1, \cdots, \tag{1.1}$$

where $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are two periodic sequences of positive real numbers with prime periods l and m respectively, namely,

$$A_{n+l} = A_n, B_{n+m} = B_n, \quad n = 0, 1, \cdots,$$

and the initial values x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary positive real numbers. It is easy to see that every solution $\{x_n\}_{n=-3}^{\infty}$ of (1.1) is a positive sequence.

With respect to the investigations for such max-type difference equation, there are not only theoretical meaningful, but also practical applications. In practice, one finds that the max operator arises in certain models in automatic control theory, for example, see [24, 25]. Theoretically, one wants to know what the affections of delays and periodicity of coefficients to properties of solutions are on earth. For the theoretical studies to the boundedness and periodicity of such kind of equations, there has been some known work. Let us simply recall some brief history for such investigations of max-type difference equations. For the periodicity of the following particular max-type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-1}}\right\}, \quad n = 0, 1, \cdots, \tag{1.2}$$

where the initial conditions x_{-1} and x_{-0} are arbitrary positive real numbers, the authors in [1] first studied the case where $A_n = 1$ for all $n \geq 0$ and $\{B_n\}_{n=0}^{\infty}$ is a periodic sequence with minimal period 2 and showed that every positive solution of (1.2) becomes eventually periodic. The case where $A_n = 1$ for all $n \geq 0$ and $\{B_n\}_{n=0}^{\infty}$ is a periodic sequence with minimal period 3 was investigated in [2] and it was also shown that every solution of (1.2) becomes eventually periodic.

For the boundedness of (1.2), under the condition that $A_n = 1$ for all $n \geq 0$ and $\{B_n\}_{n=0}^{\infty}$ is a periodic sequence with minimal period 3, the authors [3] proved that every positive solution of (1.2) is unbounded if and only if

$$B_{i+1} < 1 < B_i, \quad \text{for some } i \in \{0, 1, 2\}.$$

The authors, in [4], however, derived that every positive solutions of (1.2) is unbounded under this condition that $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers with minimal period p with $p \in \{1, 2, \cdots\}$ and $\{B_n\}_{n=0}^{\infty}$ is also periodic with minimal period $3k$ for $k = 1, 2, \cdots$ such that

$$B_{1+i+3j} < \min\{A_0, A_1, \cdots, A_{p-1}\} \leq \max\{A_0, A_1, \cdots, A_{p-1}\} < B_{i+3j}$$

for some $i \in \{0, 1, 2\}$ and for all $j \in \{0, 1, \cdots, k-1\}$.

In [5], the max-type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_n}, \frac{B_n}{x_{n-2}}\right\}, \quad n = 0, 1, \cdots, \tag{1.3}$$

was proposed to study by Kerbert and Radin, and they found that every positive solution of (1.3) is unbounded when $\{A_n\}_{n=0}^{\infty}$ is periodic with period p and $\{B_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers that is periodic with minimal period $4k$ for $k = 1, 2, \cdots$ such that

$$B_{1+i+4j} < \min\{A_0, A_1, \cdots, A_{p-1}\} \leq \max\{A_0, A_1, \cdots, A_{p-1}\} < B_{i+4j}$$

for some $i \in \{0, 1, 2, 3\}$ and for all $j \in \{0, 1, \cdots, k-1\}$.

In addition, the work [6–22] also demonstrates that the investigations for max-type difference equations are interesting to many authors. So, inspired by the above work, along this line, our main aim in this paper is to consider the boundedness nature for the positive solutions of the max-type difference equation (1.1). Some sufficient conditions are obtained for every positive solution of (1.1) to be bounded and unbounded respectively.

For the sake of convenience of statement, for nonnegative integers a and b, denote

$$N(a) = \{a, a+1, \cdots\} \quad \text{and} \quad N(a,b) = \{a, a+1, \cdots, b\} \quad \text{for } a \le b.$$

For a periodic sequence $\{y_n\}_{n=0}^{\infty}$ with minimal period $p \in N(1)$, put

$$m_{y,p} = \min\{y_0, y_1, \cdots, y_{p-1}\} \quad \text{and} \quad M_{y,p} = \max\{y_0, y_1, \cdots, y_{p-1}\}.$$

In this paper, one mainly considers the following three cases of periodic sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ with prime periods l and m respectively:

(i) $l \in N(1)$ and $m = 5$;

(ii) $l \in N(1)$ and $m = 10$;

(iii) $l \in N(1)$ and $m = 5k$, $k = 3, 4, \cdots$.

As for the case: $m \in N(1)$ and $l = 5k$, $k = 1, 2, 3, 4, \cdots$, the ways and methods used are completely similar and will be omitted here.

2. $l \in N(1)$ and $m = 5$

In this section it is assumed that $\{B_n\}_{n=0}^{\infty}$ is a positive periodic sequence with minimal period 5, and that for some $i \in N(0,4)$,

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i. \tag{2.1}$$

It will be shown that every positive solution of (1.1) is unbounded under the condition (2.1). One first establishes some useful lemmas. In particular, the next five lemmas will show that every positive solution of (1.1) is unbounded at every case, where the solution will consist of subsequences, some of which converge to 0 and some of which diverge to infinity.

Lemma 2.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_4 < m_{A,l} \le M_{A,l} < B_3,$$

then

$$\lim_{n\to\infty} x_{5n} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n+1} = \lim_{n\to\infty} x_{5n+4} = +\infty.$$

Proof. According to (1.1) and the rule of iteration and properties of maximum and minimum for function, one can see

$$\begin{aligned} x_{5n+5} &= \max\left\{\frac{A_{5n+4}}{x_{5n+4}}, \frac{B_{5n+4}}{x_{5n+1}}\right\} \\ &= \max\left\{\frac{A_{5n+4}}{\max\left\{\frac{A_{5n+3}}{x_{5n+3}}, \frac{B_{5n+3}}{x_{5n}}\right\}}, \frac{B_{5n+4}}{\max\left\{\frac{A_{5n}}{x_{5n}}, \frac{B_{5n}}{x_{5n-3}}\right\}}\right\} \\ &= \max\left\{\min\left\{\frac{A_{5n+4}x_{5n+3}}{A_{5n+3}}, \frac{A_{5n+4}x_{5n}}{B_{5n+3}}\right\}, \min\left\{\frac{B_{5n+4}x_{5n}}{A_{5n}}, \frac{B_{5n+4}x_{5n-3}}{B_{5n}}\right\}\right\}. \end{aligned}$$

Now consider the following four cases.

Case 1: $\min\left\{\frac{A_{5n+4}x_{5n+3}}{A_{5n+3}}, \frac{A_{5n+4}x_{5n}}{B_{5n+3}}\right\} = \frac{A_{5n+4}x_{5n+3}}{A_{5n+3}} \le \frac{A_{5n+4}x_{5n}}{B_{5n+3}} \le \frac{M_{A,l}}{B_3}x_{5n}.$

Case 2: $\min\left\{\frac{A_{5n+4}x_{5n+3}}{A_{5n+3}}, \frac{A_{5n+4}x_{5n}}{B_{5n+3}}\right\} = \frac{A_{5n+4}x_{5n}}{B_{5n+3}} \le \frac{M_{A,l}}{B_3}x_{5n}.$

Case 3: $\min\left\{\frac{B_{5n+4}x_{5n}}{A_{5n}}, \frac{B_{5n+4}x_{5n-3}}{B_{5n}}\right\} = \frac{B_{5n+4}x_{5n}}{A_{5n}} \le \frac{B_4}{m_{A,l}}x_{5n}.$

Case 4: $\min\left\{\frac{B_{5n+4}x_{5n}}{A_{5n}}, \frac{B_{5n+4}x_{5n-3}}{B_{5n}}\right\} = \frac{B_{5n+4}x_{5n-3}}{B_{5n}} \le \frac{B_{5n+4}x_{5n}}{A_{5n}} \le \frac{B_4}{m_{A,l}}x_{5n}.$

Throughout the Cases 1-4, let

$$M = \max\left\{\frac{M_{A,l}}{B_3}, \frac{B_4}{m_{A,l}}\right\}.$$

Then, by the known assumption and synthesizing the above cases, one sees that

$$M < 1 \quad \text{and} \quad x_{5n+5} \le Mx_{5n} \le \cdots \le M^{n+1}x_0.$$

It follows from $0 < x_{5n+5} \le M^{n+1}x_0 \to 0$ that

$$\lim_{n\to\infty} x_{5n} = \lim_{n\to\infty} x_{5n+5} = 0.$$

Also note that for all $n \ge 0$,

$$x_{5n+6} = \max\left\{\frac{A_{5n+5}}{x_{5n+5}}, \frac{B_{5n+5}}{x_{5n+2}}\right\} \ge \frac{A_{5n+5}}{x_{5n+5}} \to +\infty,$$

thus

$$\lim_{n\to\infty} x_{5n+1} = \lim_{n\to\infty} x_{5n+6} = +\infty.$$

In addition, notice that

$$x_{5n+9} = \max\left\{\frac{A_{5n+8}}{x_{5n+8}}, \frac{B_{5n+8}}{x_{5n+5}}\right\} \ge \frac{B_{5n+8}}{x_{5n+5}} \to +\infty,$$

then

$$\lim_{n\to\infty} x_{5n+4} = \lim_{n\to\infty} x_{5n+9} = +\infty.$$

Therefore, the proof is over. □

The following Lemmas 2.2–2.5 may be derived, whose proofs are similar to the proof of Lemma 2.1 and will be omitted.

Lemma 2.2. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1). *Suppose that*

$$B_3 < m_{A,l} \le M_{A,l} < B_2,$$

then

$$\lim_{n\to\infty} x_{5n+4} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n} = \lim_{n\to\infty} x_{5n+3} = +\infty.$$

Lemma 2.3. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_2 < m_{A,l} \le M_{A,l} < B_1,$$

then

$$\lim_{n\to\infty} x_{5n+3} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{5n+2} = \lim_{n\to\infty} x_{5n+4} = +\infty.$$

Lemma 2.4. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_1 < m_{A,l} \le M_{A,l} < B_0,$$

then

$$\lim_{n\to\infty} x_{5n+2} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{5n+1} = \lim_{n\to\infty} x_{5n+3} = +\infty.$$

Lemma 2.5. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_5 < m_{A,l} \le M_{A,l} < B_4,$$

then

$$\lim_{n\to\infty} x_{5n+1} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{5n} = \lim_{n\to\infty} x_{5n+2} = +\infty.$$

Combining the above Lemmas 2.1–2.5, one can derive the following results.

Lemma 2.6. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that, for some* $i \in N(0,4)$*,*

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i.$$

Then

$$\lim_{n\to\infty} x_{5n+i+2} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{5n+i+1} = \lim_{n\to\infty} x_{5n++i+3} = +\infty,$$

which means $\{x_n\}_{n=-3}^{\infty}$ *is unbounded.*

Proof. The proof follows from Lemmas 2.1–2.5 and will be omitted. □

Remark 2.1. Note that if (2.1) does not hold for all $i \in N(0,4)$, then every positive solution of (1.1) is bounded and eventually becomes periodic.

3. $l \in N(1)$ and $m = 10$

In this section one assumes that $\{B_n\}_{n=0}^{\infty}$ is a positive periodic sequence with minimal period 10 and that, for some $i \in N(0,4)$, one of the following conditions holds:

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i \ \text{ and } \ B_{i+6} < m_{A,l} \le M_{A,l} < B_{i+5}, \tag{3.1}$$

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i \ \text{ and } \ B_{i+6} \le m_{A,l} \le M_{A,l} \le B_{i+5}, \tag{3.2}$$

$$B_{i+1} \le m_{A,l} \le M_{A,l} \le B_i \ \text{ and } \ B_{i+6} < m_{A,l} \le M_{A,l} < B_{i+5}. \tag{3.3}$$

It will be shown that every positive solution of (1.1) is unbounded provided that either (3.1), (3.2), or (3.3) is true. Let's first establish some useful lemmas, which will show that every positive solution of (1.1) is unbounded in each case where the solution will consist of subsequences, some of which converge to 0 and some of which diverge to infinity.

Lemma 3.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_4 < m_{A,l} \le M_{A,l} < B_3 \ \ \text{and} \ \ B_9 < m_{A,l} \le M_{A,l} < B_8,$$

then

$$\lim_{n\to\infty} x_{5n} = 0 \ \ \text{and} \ \ \lim_{n\to\infty} x_{5n+1} = \lim_{n\to\infty} x_{5n+4} = +\infty.$$

Proof. In view of (1.1) and the rule of iteration and properties of maximum and minimum for function, one obtains

$$\begin{aligned}
x_{10n+5} &= \max\left\{\frac{A_{10n+4}}{x_{10n+4}}, \frac{B_{10n+4}}{x_{10n+1}}\right\} \\
&= \max\left\{\frac{A_{10n+4}}{\max\left\{\frac{A_{10n+3}}{x_{10n+3}}, \frac{B_{10n+3}}{x_{10n}}\right\}}, \frac{B_{10n+4}}{\max\left\{\frac{A_{10n}}{x_{10n}}, \frac{B_{10n}}{x_{10n-3}}\right\}}\right\} \\
&= \max\left\{\min\left\{\frac{A_{10n+4}x_{10n+3}}{A_{10n+3}}, \frac{A_{10n+4}x_{10n}}{B_{10n+3}}\right\}, \min\left\{\frac{B_{10n+4}x_{10n}}{A_{10n}}, \frac{B_{10n+4}x_{10n-3}}{B_{10n}}\right\}\right\}.
\end{aligned}$$

As in Lemma 2.1, consider the following four cases.

$$\textit{Case 1: } \min\left\{\frac{A_{10n+4}x_{10n+3}}{A_{10n+3}}, \frac{A_{10n+4}x_{10n}}{B_{10n+3}}\right\} = \frac{A_{10n+4}x_{10n+3}}{A_{10n+3}} \le \frac{A_{10n+4}x_{10n}}{B_{10n+3}} \le \frac{M_{A,l}}{B_3}x_{10n}.$$

$$\textit{Case 2: } \min\left\{\frac{A_{10n+4}x_{10n+3}}{A_{10n+3}}, \frac{A_{10n+4}x_{10n}}{B_{10n+3}}\right\} = \frac{A_{10n+4}x_{10n}}{B_{10n+3}} \le \frac{M_{A,l}}{B_3}x_{10n}.$$

$$\textit{Case 3: } \min\left\{\frac{B_{10n+4}x_{10n}}{A_{10n}}, \frac{B_{10n+4}x_{10n-3}}{B_{10n}}\right\} = \frac{B_{10n+4}x_{10n}}{A_{10n}} \le \frac{B_4}{m_{A,l}}x_{10n}.$$

$$\textit{Case 4: } \min\left\{\frac{B_{10n+4}x_{10n}}{A_{10n}}, \frac{B_{10n+4}x_{10n-3}}{B_{10n}}\right\} = \frac{B_{10n+4}x_{10n-3}}{B_{10n}} \le \frac{B_{10n+4}x_{10n}}{A_{10n}} \le \frac{B_4}{m_{A,l}}x_{10n}.$$

Also notice that

$$\begin{aligned}
x_{10n+10} &= \max\left\{\frac{A_{10n+9}}{x_{10n+9}}, \frac{B_{10n+9}}{x_{10n+6}}\right\} \\
&= \max\left\{\frac{A_{10n+9}}{\max\left\{\frac{A_{10n+8}}{x_{10n+8}}, \frac{B_{10n+8}}{x_{10n+5}}\right\}}, \frac{B_{10n+9}}{\max\left\{\frac{A_{10n+5}}{x_{10n+5}}, \frac{B_{10n+5}}{x_{10n+2}}\right\}}\right\} \\
&= \max\left\{\min\left\{\frac{A_{10n+9}x_{10n+8}}{A_{10n+8}}, \frac{A_{10n+9}x_{10n+5}}{B_{10n+8}}\right\}, \min\left\{\frac{B_{10n+9}x_{10n+5}}{A_{10n+5}}, \frac{B_{10n+9}x_{10n+2}}{B_{10n+5}}\right\}\right\}.
\end{aligned}$$

As in Lemma 2.1, further consider the following four cases.

$$\textit{Case 5: } \min\left\{\frac{A_{10n+9}x_{10n+8}}{A_{10n+8}}, \frac{A_{10n+9}x_{10n+5}}{B_{10n+8}}\right\} = \frac{A_{10n+9}x_{10n+8}}{A_{10n+8}} \le \frac{A_{10n+9}x_{10n+5}}{B_{10n+8}} \le \frac{M_{A,l}}{B_8}x_{10n+5}.$$

$$\textit{Case } 6\text{: } \min\left\{\frac{A_{10n+9}x_{10n+8}}{A_{10n+8}}, \frac{A_{10n+9}x_{10n+5}}{B_{10n+8}}\right\} = \frac{A_{10n+9}x_{10n+5}}{B_{10n+8}} \le \frac{M_{A,l}}{B_8}x_{10n+5}.$$

$$\textit{Case } 7\text{: } \min\left\{\frac{B_{10n+9}x_{10n+5}}{A_{10n+5}}, \frac{B_{10n+9}x_{10n+2}}{B_{10n+5}}\right\} = \frac{B_{10n+9}x_{10n+5}}{A_{10n+5}} \le \frac{B_9}{m_{A,l}}x_{10n+5}.$$

$$\textit{Case } 8\text{: } \min\left\{\frac{B_{10n+9}x_{10n+5}}{A_{10n+5}}, \frac{B_{10n+9}x_{10n+2}}{B_{10n+5}}\right\} = \frac{B_{10n+9}x_{10n+2}}{B_{10n+5}} \le \frac{B_{10n+9}x_{10n+5}}{A_{10n+5}} \le \frac{B_9}{m_{A,l}}x_{10n+5}.$$

Let

$$M = \max\left\{\frac{M_{A,l}}{B_3}, \frac{B_4}{m_{A,l}}, \frac{M_{A,l}}{B_8}, \frac{B_9}{m_{A,l}}\right\}.$$

Then, by the known assumption, one can see $M < 1$. Combining the above 8 cases, one has

$$x_{10n+5} \le Mx_{10n}$$

and

$$x_{10n+10} \le Mx_{10n+5}.$$

Hence

$$x_{10(n+1)+5} \le Mx_{10(n+1)} \le M^2x_{10n+5} \le \cdots \le M^{2(n+1)}x_5$$

and

$$x_{10(n+1)} \le Mx_{10n+5} \le M^2x_{10n} \le \cdots \le M^{2(n+1)}x_0.$$

In view of $0 < x_{10(n+1)+5} \le M^{2(n+1)}x_5 \to 0$ and $0 < x_{10n+10} \le M^{2(n+1)}x_0 \to 0$, one has

$$\lim_{n\to\infty} x_{10n} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{10n+5} = 0,$$

which indicates

$$\lim_{n\to\infty} x_{5n} = 0.$$

Again, for all $n \ge 0$,

$$x_{5n+1} = \max\left\{\frac{A_{5n}}{x_{5n}}, \frac{B_{5n}}{x_{5n-3}}\right\} \ge \frac{A_{5n}}{x_{5n}} \to +\infty.$$

Thus

$$\lim_{n\to\infty} x_{5n+1} = +\infty.$$

In addition, noticing that

$$x_{5n+4} = \max\left\{\frac{A_{5n+3}}{x_{5n+3}}, \frac{B_{5n+3}}{x_{5n}}\right\} \ge \frac{B_{5n+3}}{x_{5n}} \to +\infty,$$

one has,

$$\lim_{n\to\infty} x_{5n+4} = +\infty.$$

Similar to the proof of Lemma 3.1, one can derive the following Lemmas 3.2–3.5. □

Lemma 3.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of* (1.1)*. Suppose that*

$$B_3 < m_{A,l} \le M_{A,l} < B_2 \quad \textit{and} \quad B_8 < m_{A,l} \le M_{A,l} < B_7,$$

then

$$\lim_{n\to\infty} x_{5n+4} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n} = \lim_{n\to\infty} x_{5n+3} = +\infty.$$

Lemma 3.3. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of* (1.1)*. Assume that*

$$B_2 < m_{A,l} \le M_{A,l} < B_1 \quad \textit{and} \quad B_7 < m_{A,l} \le M_{A,l} < B_6.$$

Then

$$\lim_{n\to\infty} x_{5n+3} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n+4} = \lim_{n\to\infty} x_{5n+3} = +\infty.$$

Lemma 3.4. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of* (1.1)*. Then the conditions*

$$B_1 < m_{A,l} \le M_{A,l} < B_0 \quad \textit{and} \quad B_6 < m_{A,l} \le M_{A,l} < B_5$$

imply

$$\lim_{n\to\infty} x_{5n+2} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n+3} = \lim_{n\to\infty} x_{5n+1} = +\infty.$$

Lemma 3.5. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of* (1.1)*. Then the conditions*

$$B_5 < m_{A,l} \le M_{A,l} < B_4 \quad \textit{and} \quad B_{10} < m_{A,l} \le M_{A,l} < B_9$$

indicate

$$\lim_{n\to\infty} x_{5n+1} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n+2} = \lim_{n\to\infty} x_{5n} = +\infty.$$

Combining the above Lemmas 3.1–3.5, the following first main consequence in this section may be immediately derived.

Theorem 3.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of* (1.1)*. Suppose that for some $i \in N(0,4)$,*

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i \quad \textit{and} \quad B_{i+6} m_{A,l} \le M_{A,l} < B_{i+5},$$

then

$$\lim_{n\to\infty} x_{5n+i+2} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n+i+3} = \lim_{n\to\infty} x_{5n+i+6} = +\infty.$$

In the next five lemmas one will assume that, for some $i \in N(0,4)$, one of the following two conditions is valid:
namely, either

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i \quad \text{and} \quad B_{i+6} \le m_{A,l} \le M_{A,l} \le B_{i+5},$$

or

$$B_{i+1} \le m_{A,l} \le M_{A,l} \le B_i \quad \text{and} \quad B_{i+6} < m_{A,l} \le M_{A,l} < B_{i+5}.$$

Lemma 3.6. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that either*

$$B_4 < m_{A,l} \le M_{A,l} < B_3 \quad and \quad B_9 \le m_{A,l} \le M_{A,l} \le B_8 \tag{3.4}$$

or

$$B_4 \le m_{A,l} \le M_{A,l} \le B_3 \quad and \quad B_9 < m_{A,l} \le M_{A,l} < B_8. \tag{3.5}$$

Then

$$\lim_{n\to\infty} x_{10n} = \lim_{n\to\infty} x_{10n+5} = 0$$

and

$$\lim_{n\to\infty} x_{10n+1} = \lim_{n\to\infty} x_{10n+4} = \lim_{n\to\infty} x_{10n+6} = \lim_{n\to\infty} x_{10n+9} = +\infty.$$

Moreover, for $\prod_{i=1}^{p} K_i > 1(<1)$*, where,* $K_n = \frac{A_{10n+2}A_{10n+7}}{B_1 B_6}$*,* $\lim_{n\to\infty} x_{10n+8} = +\infty(0)$*,*

$$\lim_{n\to\infty} x_{10n+2} = \lim_{n\to\infty} x_{10n+7} = 0(+\infty), \quad and \quad \lim_{n\to\infty} x_{10n+3} = \lim_{n\to\infty} x_{10n+8} = +\infty(0).$$

For $\prod_{i=1}^{p} K_i = 1$*,* $\{x_{10n+8}\}_{n=0}^{\infty}$*,* $\{x_{10n+2}\}_{n=0}^{\infty}$*,* $\{x_{10n+3}\}_{n=0}^{\infty}$ *and* $\{x_{10n+7}\}_{n=0}^{\infty}$ *are all p-periodic sequences.*

Proof. It suffices to consider the case where (3.4) is true. The proof for the case (3.5) is similar and will be omitted.

As in Lemma 2.1 and 3.1, Put $M = \max\left\{\frac{B_4}{m_{A,l}}, \frac{M_{A,l}}{B_3}\right\}$. According to the assumption (3.4), $M < 1$. From the inequalities above and the periodicity of $\{B_n\}_{n=0}^{\infty}$ with period 10, it follows by induction that, for all $n \ge 0$,

$$x_{10n+10} \le M x_{10n+5} \le M^2 x_{10n} \le \cdots \le M^{2(n+1)} x_0 \to 0. \tag{3.6}$$

Therefore,

$$\lim_{n\to\infty} x_{10n} = \lim_{n\to\infty} x_{10n+10} = 0.$$

Accordingly, still by (3.6), one has

$$\lim_{n\to\infty} x_{10n+5} = 0.$$

Because of

$$x_{10n+11} = \max\left\{\frac{A_{10n+10}}{x_{10n+10}}, \frac{B_{10n+10}}{x_{10n+7}}\right\} \ge \frac{A_{10n+10}}{x_{10n+10}} \to +\infty,$$

$$\lim_{n\to\infty} x_{10n+1} = \lim_{n\to\infty} x_{10n+11} = +\infty.$$

In addition, noticing that

$$x_{10n+14} = \max\left\{\frac{A_{10n+13}}{x_{10n+13}}, \frac{B_{10n+13}}{x_{10n+10}}\right\} \ge \frac{B_{10n+13}}{x_{10n+10}} \to +\infty$$

and

$$x_{10n+6} = \max\left\{\frac{A_{10n+5}}{x_{10n+5}}, \frac{B_{10n+5}}{x_{10n+2}}\right\} \geq \frac{A_{10n+5}}{x_{10n+5}} \to +\infty,$$

one has

$$\lim_{n\to\infty} x_{10n+4} = \lim_{n\to\infty} x_{10n+14} = +\infty \quad \text{and} \quad \lim_{n\to\infty} x_{10n+6} = +\infty.$$

Also, note that

$$x_{10n+9} = \max\left\{\frac{A_{10n+8}}{x_{10n+8}}, \frac{B_{10n+8}}{x_{10n+5}}\right\} \geq \frac{B_{10n+8}}{x_{10n+5}} \to +\infty.$$

So,

$$\lim_{n\to\infty} x_{10n+9} = +\infty.$$

As for $\{x_{10n+2}\}_{n=0}^{\infty}$, $\{x_{10n+3}\}_{n=0}^{\infty}$, $\{x_{10n+7}\}_{n=0}^{\infty}$ and $\{x_{10n+8}\}_{n=0}^{\infty}$, noticing for all $n \geq 0$,

$$x_{10n+2} = \max\left\{\frac{A_{10n+1}}{x_{10n+1}}, \frac{B_{10n+1}}{x_{10n-2}}\right\}, \quad x_{10n+3} = \max\left\{\frac{A_{10n+2}}{x_{10n+2}}, \frac{B_{10n+2}}{x_{10n-1}}\right\},$$

$$x_{10n+7} = \max\left\{\frac{A_{10n+6}}{x_{10n+6}}, \frac{B_{10n+6}}{x_{10n+3}}\right\}, \quad x_{10n+8} = \max\left\{\frac{A_{10n+7}}{x_{10n+7}}, \frac{B_{10n+7}}{x_{10n+4}}\right\},$$

One has eventually

$$x_{10n+2} = \frac{B_{10n+1}}{x_{10n-2}} = \frac{B_1}{x_{10n-2}}, \quad x_{10n+3} = \frac{A_{10n+2}}{x_{10n+2}} = \frac{A_{10n+2}x_{10n-2}}{B_1},$$

$$x_{10n+7} = \frac{B_{10n+6}}{x_{10n+3}} = \frac{B_1B_6}{A_{10n+2}x_{10n-2}}, \quad x_{10n+8} = \frac{A_{10n+7}}{x_{10n+7}} = \frac{A_{10n+2}A_{10n+7}}{B_1B_6}x_{10n-2}.$$

Denote $K_n = \frac{A_{10n+2}A_{10n+7}}{B_1B_6}$. Then $\{K_n\}$ is a periodic sequence with period p and

$$x_{10n+8} = K_n x_{10(n-1)+8} = \cdots = \left(\prod_{i=1}^{n} K_i\right)x_8.$$

So, for $\prod_{i=1}^{p} K_i > 1(<1)$, $\lim_{n\to\infty} x_{10n+8} = +\infty(0)$. Correspondingly,

$$\lim_{n\to\infty} x_{10n+2} = \lim_{n\to\infty} x_{10n+7} = 0(+\infty)$$

and

$$\lim_{n\to\infty} x_{10n+3} = \lim_{n\to\infty} x_{10n+8} = +\infty(0).$$

For $\prod_{i=1}^{p} K_i = 1$,

$$x_{10(n+p)+8} = \left(\prod_{i=1}^{n+p} K_i\right)x_8 = \left(\prod_{i=1}^{n} K_i\right)x_8 = x_{10n+8}.$$

Namely, $\{x_{10n+8}\}_{n=0}^{\infty}$ is a p-period sequence. Thereout, so are $\{x_{10n+2}\}_{n=0}^{\infty}$, $\{x_{10n+3}\}_{n=0}^{\infty}$ and $\{x_{10n+7}\}_{n=0}^{\infty}$. □

Analogously, one can obtain the following results.

Lemma 3.7. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1). *Suppose that either*

$$B_3 < m_{A,l} \le M_{A,l} < B_2 \quad and \quad B_8 \le m_{A,l} \le M_{A,l} \le B_7$$

or

$$B_3 \le m_{A,l} \le M_{A,l} \le B_2 \quad and \quad B_8 < m_{A,l} \le M_{A,l} < B_7.$$

Then

$$\lim_{n\to\infty} x_{10n+9} = \lim_{n\to\infty} x_{10n+4} = 0$$

and

$$\lim_{n\to\infty} x_{10n} = \lim_{n\to\infty} x_{10n+3} = \lim_{n\to\infty} x_{10n+5} = \lim_{n\to\infty} x_{10n+8} = +\infty.$$

Lemma 3.8. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1). *Assume that either*

$$B_2 < m_{A,l} \le M_{A,l} < B_1 \quad and \quad B_7 \le m_{A,l} \le M_{A,l} \le B_6$$

or

$$B_2 \le m_{A,l} \le M_{A,l} \le B_1 \quad and \quad B_7 < m_{A,l} \le M_{A,l} < B_6$$

Then

$$\lim_{n\to\infty} x_{10n+8} = \lim_{n\to\infty} x_{10n+3} = 0$$

and

$$\lim_{n\to\infty} x_{10n+9} = \lim_{n\to\infty} x_{10n+2} = \lim_{n\to\infty} x_{10n+4} = \lim_{n\to\infty} x_{10n+7} = +\infty.$$

Lemma 3.9. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1). *Suppose that either*

$$B_1 < m_{A,l} \le M_{A,l} < B_0 \quad and \quad B_6 \le m_{A,l} \le M_{A,l} \le B_5$$

or

$$B_1 \le m_{A,l} \le M_{A,l} \le B_0 \quad and \quad B_6 < m_{A,l} \le M_{A,l} < B_5,$$

then

$$\lim_{n\to\infty} x_{10n+7} = \lim_{n\to\infty} x_{10n+2} = 0$$

and

$$\lim_{n\to\infty} x_{10n+8} = \lim_{n\to\infty} x_{10n+1} = \lim_{n\to\infty} x_{10n+3} = \lim_{n\to\infty} x_{10n+6} = +\infty.$$

Lemma 3.10. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. The assumption that either*

$$B_5 < m_{A,l} \le M_{A,l} < B_4 \quad and \quad B_{10} \le m_{A,l} \le M_{A,l} \le B_9$$

or

$$B_4 \le m_{A,l} \le M_{A,l} \le B_3 \quad and \quad B_9 < m_{A,l} \le M_{A,l} < B_8$$

ensures

$$\lim_{n\to\infty} x_{10n+6} = \lim_{n\to\infty} x_{10n+1} = 0$$

and

$$\lim_{n\to\infty} x_{10n+2} = \lim_{n\to\infty} x_{10n+5} = \lim_{n\to\infty} x_{10n+7} = \lim_{n\to\infty} x_{10n} = +\infty.$$

Synthesizing the above Lemmas 3.6–3.10, one gets the second main result in this section.

Theorem 3.2. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that for some* $i \in N(0,4)$*, either*

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i \quad and \quad B_{i+6} \le m_{A,l} \le M_{A,l} \le B_{i+5}$$

or

$$B_{i+1} \le m_{A,l} \le M_{A,l} \le B_i \quad and \quad B_{i+6} < m_{A,l} \le M_{A,l} < B_{i+5}.$$

Then

$$\lim_{n\to\infty} x_{10n+i+7} = \lim_{n\to\infty} x_{10n+i+2} = 0$$

and

$$\lim_{n\to\infty} x_{10n+i+8} = \lim_{n\to\infty} x_{10n+i+1} = \lim_{n\to\infty} x_{10n+i+3} = \lim_{n\to\infty} x_{10n+i+6} = +\infty.$$

Proof. The proof follows from Lemmas 3.6–3.10 and will be omitted. □

The following boundedness conclusion is then direct.

Theorem 3.3. *If* (3.1), (3.2) *and* (3.3) *are not true for all* $i \in N(0,4)$*, then every positive solution of* (1.1) *is bounded and becomes eventually periodic.*

4. $l \in N(1)$ and $m = 5k$ for $k = 3,4,5,\cdots$

In this section one assumes that $\{B_n\}_{n=0}^{\infty}$ is a positive periodic sequence with minimal period $5k$ for $k = 3,4,5,\cdots$ and that for some $i \in N(0,4)$, one of the following two conditions holds:

(i) For all $j \in \{0,5,10,\cdots,5k-5\}$, $B_{i+1+j} < m_{A,l} \le M_{A,l} < B_{i+j}$.

(ii) There exists a $j \in \{0,5,10,\cdots,5k-5\}$ such that $B_{i+1+j} < m_{A,l} \le M_{A,l} < B_{i+j}$ and for all $t \in \{0,5,10,\cdots,5k-5\}$ with $t \ne j$, $B_{t+1+i} \le m_{A,l} \le M_{A,l} \le B_{t+i}$.

It will be verified that every positive solution of (1.1) is unbounded provided that either (i) or (ii) holds. First formulate the following Lemma 4.1.

Lemma 4.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1). *Suppose that*

$$B_5 < m_{A,l} \le M_{A,l} \le B_4, B_{10} < m_{A,l} \le M_{A,l} < B_9, \cdots, B_{5+5(k-1)} < m_{A,l} \le M_{A,l} < B_{5(k-1)+4}.$$

Then

$$\lim_{n\to\infty} x_{5kn+6} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5kn+7} = \lim_{n\to\infty} x_{5kn+5} = +\infty.$$

Proof. As in Lemma 2.1, set

$$M = \max\left\{\frac{M_{A,l}}{B_4}, \frac{M_{A,l}}{B_9}, \cdots, \frac{M_{A,l}}{5(k-1)+4}, \frac{B_5}{m_{A,l}}, \frac{B_{10}}{m_{A,l}}, \cdots, \frac{B_{5(k-1)+5}}{m_{A,l}}\right\}.$$

Then, $M < 1$. For any $r \in \{0, 1, \cdots, k-1\}$,

$$\begin{aligned}
x_{5kn+5(r+1)+1} &= \max\left\{\frac{A_{5kn+5(r+1)}}{x_{5kn+5(r+1)}}, \frac{B_{5kn+5(r+1)}}{x_{5kn+5(r+1)-3}}\right\} \\
&= \max\left\{\frac{A_{5kn+5(r+1)}}{\max\left\{\frac{A_{5kn+5(r+1)-1}}{x_{5kn+5(r+1)-1}}, \frac{B_{5kn+5(r+1)-1}}{x_{15kn+5(r+1)-4}}\right\}}, \frac{B_{5kn+5(r+1)}}{\max\left\{\frac{A_{5kn+5(r+1)-4}}{x_{5kn+5(r+1)-4}}, \frac{B_{5kn+5(r+1)-4}}{x_{5kn+5(r+1)-7}}\right\}}\right\} \\
&= \max\left\{\min\left\{\frac{A_{5kn+5(r+1)}x_{5kn+5(r+1)-1}}{A_{5kn+5(r+1)-1}}, \frac{A_{5kn+5(r+1)}x_{5kn+5(r+1)-4}}{B_{5kn+5(r+1)-1}}\right\},\right. \\
&\qquad \left.\min\left\{\frac{B_{5kn+5(r+1)}x_{5kn+5(r+1)-4}}{A_{5kn+5(r+1)-4}}, \frac{B_{5kn+5(r+1)}x_{5kn+5(r+1)-7}}{B_{5kn+5(r+1)-4}}\right\}\right\}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&\min\left\{\frac{A_{5kn+5(r+1)}x_{5kn+5(r+1)-1}}{A_{5kn+5(r+1)-1}}, \frac{A_{5kn+5(r+1)}x_{5kn+5(r+1)-4}}{B_{5kn+5(r+1)-1}}\right\} \\
&\quad \le \frac{A_{5kn+5(r+1)}x_{5kn+5(r+1)-4}}{B_{5kn+5(r+1)-1}} \le \frac{M_{A,l}}{B_{5kn+5r+4}} x_{5kn+5(r+1)-4} \le M x_{5kn+5(r+1)-4}
\end{aligned}$$

and

$$\begin{aligned}
&\min\left\{\frac{B_{5kn+5(r+1)}x_{5kn+5(r+1)-4}}{A_{5kn+5(r+1)-4}}, \frac{B_{5kn+5(r+1)}x_{5kn+5(r+1)-7}}{B_{5kn+5(r+1)-4}}\right\} \\
&\quad \le \frac{B_{5kn+5r+5}x_{5kn+5(r+1)-4}}{A_{5kn+5(r+1)-4}} \le \frac{B_{5kn+5r+5}x_{5kn+5(r+1)-4}}{m_p} \le M x_{5kn+5(r+1)-4}.
\end{aligned}$$

So

$$x_{5kn+5(r+1)+1} \le M x_{5kn+5r+1},$$

which reads

$$x_{5kn+5(r+m+1)+1} \le M x_{5kn+5(r+m)+1} \le \cdots \le M^m x_{5kn+5r+1} \to 0 (m \to +\infty).$$

Thereout,

$$\lim_{n\to\infty} x_{5n+1} = \lim_{n\to\infty} x_{5(n+1)+1} = \lim_{n\to\infty} x_{5n+6} = 0.$$

Also, for all $n \geq 0$,

$$x_{5n+7} = \max\left\{\frac{A_{5n+6}}{x_{5n+6}}, \frac{B_{5n+6}}{x_{5n+3}}\right\} \geq \frac{A_{5n+6}}{x_{5n+6}} \to +\infty,$$

which implies

$$\lim_{n\to\infty} x_{5n+2} = \lim_{n\to\infty} x_{5n+7} = +\infty.$$

From the relation

$$x_{5n+10} = \max\left\{\frac{A_{5n+9}}{x_{5n+9}}, \frac{B_{5n+9}}{x_{5n+6}}\right\} \geq \frac{B_{5n+9}}{x_{5n+6}} \to +\infty,$$

one can see that

$$\lim_{n\to\infty} x_{5n} = \lim_{n\to\infty} x_{5n+10} = +\infty.$$ □

Similar to the proof of Lemma 4.1, the following lemmas may be then derived.

Lemma 4.2. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1) *and*

$$B_4 < m_{A,l} \leq M_{A,l} \leq B_3, B_9 < m_{A,l} \leq M_{A,l} < B_8, \cdots, B_{5(k-1)+4} < m_{A,l} \leq M_{A,l} < B_{5(k-1)+3}.$$

Then

$$\lim_{n\to\infty} x_{5kn+5} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5kn+6} = \lim_{n\to\infty} x_{5kn+4} = +\infty.$$

Lemma 4.3. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Assume that*

$$B_3 < m_{A,l} \leq M_{A,l} \leq B_2, B_8 < m_{A,l} \leq M_{A,l} < B_7, \cdots, B_{5(k-1)+3} < m_{A,l} \leq M_{A,l} < B_{5(k-1)+2}.$$

Then

$$\lim_{n\to\infty} x_{5kn+4} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5kn+5} = \lim_{n\to\infty} x_{5kn+3} = +\infty.$$

Lemma 4.4. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_2 < m_{A,l} \leq M_{A,l} < B_1, B_7 < m_{A,l} \leq M_{A,l} < B_6, \cdots, B_{5(k-1)+2} < m_{A,l} \leq M_{A,l} < B_{5(k-1)+1}.$$

Then

$$\lim_{n\to\infty} x_{5kn+3} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5kn+4} = \lim_{n\to\infty} x_{5kn+2} = +\infty.$$

Lemma 4.5. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_1 < m_{A,l} \leq M_{A,l} \leq B_0, B_6 < m_{A,l} \leq M_{A,l} < B_5, \cdots, B_{1+5(k-1)} < m_{A,l} \leq M_{A,l} < B_{5(k-1)}.$$

Then

$$\lim_{n\to\infty} x_{5kn+2} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5kn+3} = \lim_{n\to\infty} x_{5kn+1} = +\infty.$$

The first main result in this section may be obtained.

Theorem 4.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1). *Suppose that for some* $i \in \{0,1,2,3,4\}$,

$$B_{i+1} < m_{A,l} \le M_{A,l} < B_i, \quad B_{i+1+5} < m_{A,l} \le M_{A,l} < B_{i+5}, \cdots,$$

and

$$B_{i+1+5(k-1)} < m_{A,l} \le M_{A,l} < B_{5(k-1)+i}.$$

Then

$$\lim_{n\to\infty} x_{5n+i+2} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{5n+i+3} = \lim_{n\to\infty} x_{5n+i+1} = +\infty.$$

Proof. The proof follows from Lemmas 4.1–4.5 and will be omitted. □

In the next five lemmas one will assume that, for some $i \in N(0,4)$, there exists a $j = 0,5,10,\cdots,5k-5$ such that $B_{i+1+j} < m_{A,l} \le M_{A,l} < B_{i+j}$ and for all $t = 0,5,10,\cdots,5k-5$, where $t \ne j$, $B_{t+1+i} \le m_{A,l} \le M_{A,l} \le B_{t+i}$.

Lemma 4.6. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1). *Suppose that*

$$B_5 < m_{A,l} \le M_{A,l} < B_4, B_{10} \le m_{A,l} \le M_{A,l} \le B_9, \cdots, B_{5+5(k-1)} \le m_{A,l} \le M_{A,l} \le B_{5(k-1)+4}.$$

Then, for all $r = 0,1,\cdots,k-1$, $\lim_{n\to\infty} x_{5k(n+1)+5r+1} = 0$ *and*

$$\lim_{n\to\infty} x_{5k(n+1)+5r+2} = \lim_{n\to\infty} x_{5k(n+1)+5r+5} = +\infty.$$

Proof. Similar to as in Lemma 3.1, let

$$M = \max\left\{\frac{M_{A,l}}{B_4}, \frac{B_5}{M_{A,l}}\right\} < 1.$$

Then,

$$\begin{aligned}
x_{5kn+6} &= \max\left\{\frac{A_{5kn+5}}{x_{5kn+5}}, \frac{B_{5kn+5}}{x_{5kn+2}}\right\} \\
&= \max\left\{\frac{A_{5kn+5}}{\max\left\{\frac{A_{5kn+4}}{x_{5kn+4}}, \frac{B_{5kn+4}}{x_{5kn+1}}\right\}}, \frac{B_{5kn+5}}{\max\left\{\frac{A_{5kn+1}}{x_{5kn+1}}, \frac{B_{5kn+1}}{x_{5kn-2}}\right\}}\right\} \\
&= \max\left\{\min\left\{\frac{A_{5kn+5}x_{5kn+4}}{A_{5kn+4}}, \frac{A_{5kn+5}x_{5kn+1}}{B_{5kn+4}}\right\}, \min\left\{\frac{B_{5kn+5}x_{5kn+1}}{A_{5kn+1}}, \frac{B_{5kn+5}x_{5kn-2}}{B_{5kn+1}}\right\}\right\}.
\end{aligned}$$

Notice that

$$\min\left\{\frac{A_{5kn+5}x_{5kn+4}}{A_{5kn+4}}, \frac{A_{5kn+5}x_{5kn+1}}{B_{5kn+4}}\right\} \le \frac{A_{5kn+5}x_{5kn+1}}{B_{5kn+4}} \le \frac{M_{A,l}}{B_{5kn+4}} x_{5kn+1} \le M x_{5kn+1}$$

and

$$\min\left\{\frac{B_{5kn+5}x_{5kn+1}}{A_{5kn+1}}, \frac{B_{5kn+5}x_{5kn-2}}{B_{5kn+1}}\right\} \le \frac{B_{5kn+5}x_{5kn+1}}{A_{5kn+1}} \le \frac{B_{5kn+5}}{M_{A,l}} x_{5kn+1} \le M x_{5kn+1}.$$

So,

$$x_{5kn+6} \le Mx_{5kn+1}.$$

For $q \in \{1,\cdots,k-1\}$, it follows from (1.1) that

$$\begin{aligned}
x_{5kn+5q+6} &= \max\left\{\frac{A_{5kn+5q+5}}{x_{5kn+5q+5}}, \frac{B_{5kn+5q+5}}{x_{5kn+5q+2}}\right\} \\
&= \max\left\{\frac{A_{5kn+5q+5}}{\max\left\{\frac{A_{5kn+5q+4}}{x_{5kn+5q+4}}, \frac{B_{5kn+5q+4}}{x_{5kn+5q+1}}\right\}}, \frac{B_{5kn+5q+5}}{\max\left\{\frac{A_{5kn+5q+1}}{x_{5kn+5q+1}}, \frac{B_{5kn+5q+1}}{x_{5kn+5q-2}}\right\}}\right\} \\
&= \max\left\{\min\left\{\frac{A_{5kn+5q+5}x_{5kn+5q+4}}{A_{5kn+5q+4}}, \frac{A_{5kn+5q+5}x_{5kn+5q+1}}{B_{5kn+5q+4}}\right\},\right. \\
&\qquad \left.\min\left\{\frac{B_{5kn+5q+5}x_{5kn+5q+1}}{A_{5kn+5q+1}}, \frac{B_{5kn+5q+5}x_{5kn+5q-2}}{B_{5kn+5q+1}}\right\}\right\}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\min\left\{\frac{A_{5kn+5q+5}x_{5kn+5q+4}}{A_{5kn+5q+4}}, \frac{A_{5kn+5q+5}x_{5kn+5q+1}}{B_{5kn+5q+4}}\right\} &\le \frac{A_{5kn+5q+5}x_{5kn+5q+1}}{B_{5kn+5q+4}} \\
&\le \frac{M_{A,l}}{B_{5kn+5q+4}}x_{5kn+5q+1} \\
&\le Mx_{5kn+5q+1}
\end{aligned}$$

and

$$\begin{aligned}
\min\left\{\frac{B_{5kn+5q+5}x_{5kn+5q+1}}{A_{5kn+5q+1}}, \frac{B_{5kn+5q+5}x_{5kn+5q-2}}{B_{5kn+5q+1}}\right\} &\le \frac{B_{5kn+5q+5}x_{5kn+5q+1}}{A_{5kn+5q+1}} \\
&\le \frac{B_{5kn+5q+5}}{M_{A,l}}x_{5kn+5q+1} \\
&\le Mx_{5kn+5q+1},
\end{aligned}$$

which reads

$$x_{5kn+5q+6} \le Mx_{5kn+5q+1}, \quad \text{for all } q \in \{1,\cdots,k-1\}.$$

Thereout, it produces

$$x_{5k(n+1)+1} \le x_{5k(n+1)-4} \le x_{5k(n+1)-9} \le \cdots \le x_{5kn+6} \le Mx_{5kn+1}.$$

It follows by induction that for all $n \ge 0$,

$$x_{5k(n+1)+1} \le M^{n+1}x_1.$$

Accordingly,

$$\lim_{n\to\infty} x_{5kn+1} = \lim_{n\to\infty} x_{5k(n+1)+1} = 0.$$

In addition, noting that

$$x_{5k(n+1)+2} = \max\left\{\frac{A_{5k(n+1)+1}}{x_{5k(n+1)+1}}, \frac{B_{5k(n+1)+1}}{x_{5k(n+1)-2}}\right\} \geq \frac{A_{5k(n+1)+1}}{x_{5k(n+1)+1}} \to +\infty,$$

one has

$$\lim_{n\to\infty} x_{5kn+2} = \lim_{n\to\infty} x_{5k(n+1)+2} = +\infty. \tag{4.1}$$

Also, owing to

$$x_{5k(n+1)+5} = \max\left\{\frac{A_{5k(n+1)+4}}{x_{5k(n+1)+4}}, \frac{B_{5k(n+1)+4}}{x_{5k(n+1)+1}}\right\} \geq \frac{B_{5k(n+1)+4}}{x_{5k(n+1)+1}} \to +\infty,$$

$$\lim_{n\to\infty} x_{5kn+5} = \lim_{n\to\infty} x_{5k(n+1)+5} = +\infty. \tag{4.2}$$

Furthermore,

$$x_{5k(n+1)+6} = \max\left\{\frac{A_{5k(n+1)+5}}{x_{5k(n+1)+5}}, \frac{B_{5k(n+1)+5}}{x_{5k(n+1)+2}}\right\}.$$

Hence, it follows from (4.1) and (4.2) that

$$\lim_{n\to\infty} x_{5k(n+1)+6} = \max\left\{\lim_{n\to\infty}\frac{A_{5k(n+1)+5}}{x_{5k(n+1)+5}}, \lim_{n\to\infty}\frac{B_{5k(n+1)+5}}{x_{5k(n+1)+2}}\right\} = 0.$$

In view of the relation

$$x_{5k(n+1)+7} = \max\left\{\frac{A_{5k(n+1)+6}}{x_{5k(n+1)+6}}, \frac{B_{5k(n+1)+6}}{x_{5k(n+1)+3}}\right\} \geq \frac{A_{5k(n+1)+6}}{x_{5k(n+1)+6}} \to \infty,$$

one can see that

$$\lim_{n\to\infty} x_{5kn+7} = \lim_{n\to\infty} x_{5k(n+1)+7} = +\infty.$$

Because of

$$x_{5k(n+1)+10} = \max\left\{\frac{A_{5k(n+1)+9}}{x_{5k(n+1)+9}}, \frac{B_{5k(n+1)+9}}{x_{5k(n+1)+6}}\right\} \geq \frac{B_{5k(n+1)+9}}{x_{5k(n+1)+6}} \to \infty,$$

$$\lim_{n\to\infty} x_{5kn+10} = \lim_{n\to\infty} x_{5k(n+1)+10} = +\infty.$$

Also from the observation

$$x_{5k(n+1)+11} = \max\left\{\frac{A_{5k(n+1)+10}}{x_{5k(n+1)+10}}, \frac{B_{5k(n+1)+10}}{x_{5k(n+1)+7}}\right\},$$

one obtains

$$\lim_{n\to\infty} x_{5k(n+1)+11} = 0.$$

Similarly continuing this process, one has the following results that for all $r = 0, 1, \cdots, k-1$,

$$\lim_{n\to\infty} x_{5k(n+1)+5r+1} = 0$$

and

$$\lim_{n\to\infty} x_{5k(n+1)+5r+2} = \lim_{n\to\infty} x_{5k(n+1)+5r+5} = +\infty. \qquad \square$$

The following lemmas may be analogously obtained.

Lemma 4.7. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_4 < m_{A,l} \le M_{A,l} \le B_3, B_9 \le m_{A,l} \le M_{A,l} < B_8, \cdots, B_{4+5(k-1)} \le m_{A,l} \le M_{A,l} < B_{5(k-1)+3}.$$

Then, for all $r = 0, 1, \cdots, k-1$, $\lim_{n\to\infty} x_{5k(n+1)+5r} = 0$ *and*

$$\lim_{n\to\infty} x_{5k(n+1)+5r+1} = \lim_{n\to\infty} x_{5k(n+1)+5r+4} = +\infty.$$

Lemma 4.8. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_3 < m_{A,l} \le M_{A,l} \le B_2, B_8 \le m_{A,l} \le M_{A,l} < B_7, \cdots, B_{3+5(k-1)} \le m_{A,l} \le M_{A,l} < B_{5(k-1)+2}.$$

Then, for all $r = 0, 1, \cdots, k-1$, $\lim_{n\to\infty} x_{5k(n+1)+5r-1} = 0$ *and*

$$\lim_{n\to\infty} x_{5k(n+1)+5r} = \lim_{n\to\infty} x_{5k(n+1)+3} = +\infty.$$

Lemma 4.9. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_2 < m_{A,l} \le M_{A,l} \le B_1, B_7 \le m_{A,l} \le M_{A,l} < B_6, \cdots, B_{2+5(k-1)} \le m_{A,l} \le M_{A,l} < B_{5(k-1)-2}.$$

Then, for all $r = 0, 1, \cdots, k-1$, $\lim_{n\to\infty} x_{5k(n+1)+5r-2} = 0$ *and*

$$\lim_{n\to\infty} x_{5k(n+1)+5r-1} = \lim_{n\to\infty} x_{5k(n+1)+5r+2} = +\infty.$$

Lemma 4.10. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that*

$$B_1 < m_{A,l} \le M_{A,l} \le B_0, B_6 \le m_{A,l} \le M_{A,l} < B_5, \cdots, B_{1+5(k-1)} \le m_{A,l} \le M_{A,l} < B_{5(k-1)}.$$

Then, for all $r = 0, 1, \cdots, k-1$, $\lim_{n\to\infty} x_{5k(n+1)+5r-3} = 0$ *and*

$$\lim_{n\to\infty} x_{5k(n+1)+5r-2} = \lim_{n\to\infty} x_{5k(n+1)+5r+1} = +\infty.$$

One is now in a position to formulate the second main results in this section.

Theorem 4.2. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of* (1.1)*. Suppose that for some* $i \in N(0,4)$ *the following two conditions are valid:*

(i) *There exists a* $j = 0, 5, 10, \cdots, 5k-5$ *such that* $B_{i+1+j} < m_{A,l} \le M_{A,l} < B_{i+j}$.

(ii) *For all* $t = 0, 5, 10, \cdots, 5k-5$*, where* $t \neq j$, $B_{i+1+t} \le m_{A,l} \le M_{A,l} \le B_{i+t}$.

Then, for all $r = 0, 1, \cdots, k-1$, $\lim_{n\to\infty} x_{5k(n+1)+5r+i-3} = 0$ *and*

$$\lim_{n\to\infty} x_{5k(n+1)+5r+i-2} = \lim_{n\to\infty} x_{5k(n+1)+5r+i+1} = +\infty.$$

Proof. For the case $j = 0$, the proof follows from Lemmas 4.6–4.10. The proofs for the other cases where $j = 5, 10, \cdots, 5k-5$ are similar and will be omitted. □

From the above conditions for unbounded solutions, one can easily draw a conclusion for conditions for bounded solutions.

Theorem 4.3. *Assume that the conditions* (i) *and* (ii) *do not hold for all* $i \in N(0,4)$*. Then every positive solution of* (1.1) *is bounded and becomes eventually periodic.*

5. Special case of $l = 1$

When $l = 1$, (1.1) reduces to the following form

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B_n}{x_{n-3}}\right\}, \quad n = 0, 1, \cdots, \tag{5.1}$$

where A is a positive constant. Without loss of generality, one may assumes $A = 1$. The previous Theorem **??** reduces to the following form.

Theorem 5.1. *Assume that $\{B_n\}_{n=0}^{\infty}$ is a positive periodic sequence with prime period 5. Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of (5.1). Suppose that, for some $i \in N(0,4)$, $B_{i+1} < 1 < B_i$. Then*

$$\lim_{n\to\infty} x_{5n+i+2} = 0 \quad \textit{and} \quad \lim_{n\to\infty} x_{5n+i+1} = \lim_{n\to\infty} x_{5n++i+3} = +\infty,$$

which means $\{x_n\}_{n=-3}^{\infty}$ is unbounded.

The other theorems mentioned previously in this paper, such as Theorems 3.1–4.3, have also the corresponding reduction forms and omitted here.

6. Conclusion

In this paper one considers the boundedness nature for positive solutions of a nonlinear max-type fourth order difference equation with periodic coefficients and derive a series of sufficient conditions ensuring the existence of bounded and unbounded solutions to this equation. Our interest in the future will be to consider more general max-type difference equation

$$x_{n+1} = \max\left\{\frac{A_n}{x_{n-l}}, \frac{B_n}{x_{n-m}}\right\}, \quad n = 0, 1, \cdots, \tag{6.1}$$

where $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are two periodic sequences of positive real numbers, $l, m \in \{1, 2, \cdots\}$ with $l < m$ and the initial values $x_{-m}, \cdots, x_{-1}$ and x_0 are arbitrary positive real numbers.

By this detailed (1.1), one attempts for (6.1) to discover the rule that how the delays and the periodicity of coefficients affect the boundedness property of solutions.

We hope, this work of this paper will shed some light to final revealing the rule.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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