# On the Area of the Symmetry Orbits of the Einstein-Vlasov-Scalar Field System with Plane and Hyperbolic Symmetry

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**Abstract** We prove in the case of cosmological models for the Einstein-Vlasov-scalar field system, that the area radius of compact hypersurfaces tends to a constant value as the past boundary of the maximal Cauchy development is approached. In other case, there is at least one Cauchy hypersurface of constant areal time coordinate in plane and hyperbolic symmetric spacetimes. Moreover, we show that the areal time coordinate $R = t$ which covers these spacetimes runs from zero at infinity with the singularity occuring at $R = 0$. The sources of the equations are generated by a distribution function and a massless scalar field, subject to the Vlasov and wave equations respectively.

## 1. Introduction

Consider the Einstein-Vlasov-scalar field system with spherical, plane and hyperbolic symmetries. For more information on this system which describes the evolution of self-gravitating collisionless matter and scalar waves within the context of general relativity, see [6], [7], [8], where the system has been studied in areal coordinates and global existences results were obtained directly. Three types of time coordinates which have been studied in the inhomogeneous Einstein-Vlasov system case are constant mean curvature, areal and conformal coordinates. A constant mean curvature time coordinate $t$ is one where each hypersurface of constant time has constant mean curvature and on each hypersurface of this kind the value of $t$ is the mean curvature of that slice. In the case of areal coordinates the time coordinate is a function of the area of the surface of symmetry. In the case of conformal coordinates the metric is conformally flat on the manifold $Q$

which is the quotient of spacetime by the symmetry group. $Q$ is a two-dimensional Lorentzian manifold.

Now consider the past maximal globally hyperbolic development of data on an initial hypersurface, see [8]. Using conformal coordinates, we prove that along any past inextendible timelike curve, the time coordinate $R = t$ which is the area of the symmetry orbits tends to a constant value $R_0 = 0$, independent of which curve is chosen. This was proved in the case of $T^2$-symmetric spacetimes with Vlasov matter by [9] with results generalized later in some direction by [5].

Let us recall the formulation of the Einstein-Vlasov-scalar field system. We consider a four-dimensional spacetime manifold $M$, with local coordinates $(x^\alpha) = (t, x^i)$ on which $x^0 = t$ denotes the time and $(x^i)$ the space coordinates. Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On $M$, a lorentzian metric $g$ is given with signature $(-, +, +, +)$. We consider a self-gravitating collisionless gas and restrict ourselves to the case where all particles have the same rest mass, normalized to 1, and move forward in time. We denote by $(p^\alpha)$ the momenta of the particles. The conservation of the quantity $g_{\alpha\beta}p^\alpha p^\beta$ requires that the phase space of the particle is the seven-dimensional submanifold

$$PM = \{g_{\alpha\beta}p^\alpha p^\beta = -1; \; p^0 > 0\}$$

of $TM$ which is coordinatized by $(t, x^i, p^i)$. If the coordinates are such that the components $g_{0i}$ vanish then the component $p^0$ is expressed by other coordinates via

$$p^0 = \sqrt{-g^{00}}\sqrt{1 + g_{ij}p^i p^j}$$

The distribution function of the particles is a non-negative real-valued function denoted by $f$, that is defined on $PM$. In addition we consider a massless scalar field $\phi$ which is a real-valued function on $M$. The Einstein-Vlasov-scalar field system now reads:

$$\partial_t f + \frac{p^i}{p^0}\partial_{x^i}f - \frac{1}{p^0}\Gamma^i_{\beta\gamma}p^\beta p^\gamma \partial_{p^i}f = 0,$$

$$\nabla^\alpha \nabla_\alpha \phi = 0,$$

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

$$T_{\alpha\beta} = -\int_{\mathbb{R}^3} f p_\alpha p_\beta |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p_0} + \left(\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2}g_{\alpha\beta}\nabla_\nu \phi \nabla^\nu \phi\right)$$

where $p_\alpha = g_{\alpha\beta}p^\beta$, $|g|$ denotes the modulus of determinant of the metric $g_{\alpha\beta}$, $\Gamma^\lambda_{\alpha\beta}$ the Christoffel symbols, $G_{\alpha\beta}$ the Einstein tensor, and $T_{\alpha\beta}$ the energy-momentum tensor.

Note that since the contribution of $f$ to the energy-momentum tensor is divergence-free [3], the form of the contribution of the scalar field to the energy-momentum tensor determines the field equation for $\phi$.

We refer to [4] for the notion of spherical, plane and hyperbolic symmetry. We now consider a solution of the Einstein-Vlasov-scalar field system where all unknowns are invariant under one of these symmetries. We write the system in conformal coordinates. The circumstances under which coordinates of this type exist are discussed in [2]. In such coordinates the metric $g$ takes the form

$$ds^2 = e^{2\mu(t,r)}(-dt^2 + dr^2) + R(t,r)^2(d\theta^2 + \sin_k^2\theta\, d\varphi^2) \tag{1.1}$$

where

$$\sin_k\theta = \begin{cases} \sin\theta & \text{for } k = 1 \text{ (spherical symmetry)}; \\ 1 & \text{for } k = 0 \text{ (plane symmetry)}; \\ \sinh\theta & \text{for } k = -1 \text{ (hyperbolic symmetry)} \end{cases}$$

Here the timelike coordinate $t$ locally labels spatial hypersurfaces of the spacetime, and each such hypersurface consists of all of the group orbits which have area $R$. $r \in \mathbb{R}$ and $(\theta, \varphi)$ range in the domains $[0, \pi] \times [0, 2\pi]$, $[0, 2\pi] \times [0, 2\pi]$, $[0, \infty[ \times [0, 2\pi]$ respectively, and stand for angular coordinates. The functions $\mu$ and $R$ are periodic in $r$ with period 1 and independent of $\theta$ and $\varphi$. It has been shown in [2] that due to the symmetry, $f$ can be written as a function of

$$t, r, w := e^\mu p^1 \text{ and } F := R^4\left[(p^2)^2 + \sin_k^2\theta(p^3)^2\right],$$

i.e. $f = f(t, r, w, F)$ and $F$ is conserved quantity along particle orbits. In these variables, we have $p^0 = e^{-\mu}\sqrt{1 + w^2 + F/R^2} =: e^{-\mu}\langle p\rangle$. The scalar field is a function of $t$ and $r$ which is periodic in $r$ with period 1.

We denote by a dot and by a prime the derivatives of the metric components and of the scalar field with respect to $t$ and $r$ respectively. Using the results of [2], the complete Einstein-Vlasov-scalar field system can be written in the following form:

$$\partial_t f + \frac{w}{\langle p\rangle}\partial_r f + \left[-\dot{\mu}w - \mu'\left(\langle p\rangle + \frac{w^2}{\langle p\rangle}\right) + e^{-2\mu}R'\frac{F}{R^3\langle p\rangle}\right]\partial_w f = 0, \tag{1.2}$$

$$-R'' + \dot{\mu}\dot{R} + \mu'R' + \frac{1}{2R}[\dot{R}^2 - R'^2 + ke^{2\mu}] = 4\pi Re^{2\mu}\rho, \tag{1.3}$$

$$\dot{R}' - \dot{\mu}R' - \mu'\dot{R} = 4\pi Re^{2\mu}j, \tag{1.4}$$

$$\ddot{R} - R'' + \frac{1}{R}[\dot{R}^2 - R'^2 + ke^{2\mu}] = 4\pi Re^{2\mu}(\rho - p), \tag{1.5}$$

$$\ddot{\mu} - \mu'' - \frac{1}{R^2}[\dot{R}^2 - R'^2 + ke^{2\mu}] = 4\pi Re^{2\mu}(p - \rho - q), \tag{1.6}$$

$$\ddot{\phi} - \phi'' + 2\frac{\dot{R}}{R}\dot{\phi} - 2\frac{R'}{R}\phi' = 0 \tag{1.7}$$

where (1.7) is the wave equation in $\phi$ and

$$\rho(t,r) = e^{-2\mu}T_{00}(t,r)$$

$$= \frac{\pi}{R^2}\int_{-\infty}^{+\infty}\int_0^{+\infty}\langle p\rangle f(t,r,w,F)dFdw + \frac{1}{2}e^{-2\mu}(\dot{\phi}^2 + \phi'^2), \quad (1.8)$$

$$p(t,r) = e^{-2\mu}T_{11}(t,r)$$

$$= \frac{\pi}{R^2}\int_{-\infty}^{+\infty}\int_0^{+\infty}\frac{w^2}{\langle p\rangle}f(t,r,w,F)dFdw + \frac{1}{2}e^{-2\mu}(\dot{\phi}^2 + \phi'^2), \quad (1.9)$$

$$j(t,r) = -e^{-2\mu}T_{01}(t,r)$$

$$= \frac{\pi}{R^2}\int_{-\infty}^{+\infty}\int_0^{+\infty} wf(t,r,w,F)dFdw - e^{-2\mu}\dot{\phi}\phi', \quad (1.10)$$

$$q(t,r) = \frac{2}{R^2}T_{22}(t,r)$$

$$= \frac{2}{R^2\sin_k^2\theta}T_{33}(t,r,\theta)$$

$$= \frac{\pi}{R^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{F}{\langle p\rangle}f(t,r,w,F)dFdw + e^{-2\mu}(\dot{\phi}^2 - \phi'^2). \quad (1.11)$$

We prescribe initial data at time $t = t_0$:

$$f(t_0,r,w,F) = \mathring{f}(r,w,F), \quad \mu(t_0,r) = \mathring{\mu}(r), \quad \dot{\mu} = \mu_1(r)$$

$$R(t_0,r) = \mathring{R}(r), \quad \phi(t_0,r) = \mathring{\phi}(r), \quad \dot{\phi}(t_0,r) = \psi(r), \quad \dot{R}(t_0,r) = R_1(r)$$

The equations (1.3), (1.4) are constraints and (1.5), (1.6) are the evolution equations.

In [8], it is proved using areal time coordinates that $]0, t_0]$ ($t_0 > 0$) is the past maximal existence interval of solutions of this system with the singularity occurring at $t = 0$. In the case $k < 0$, some restriction on the initial data needed. In [7], the evolution of the spacetime in the expanding direction is analysed. A global existence theorem in areal time coordinates is proved for all $t \in [t_0, +\infty[$, where $t$ denotes the area radius of the surfaces of symmetry of the induced spacetime. In [2] (theorem 6.1), it is proved in the case of hyperbolic symmetry with Vlasov matter that $(M, g)$ can be covered by symmetric compact hypersurfaces of constant area radius. And the area radius of these hypersurfaces takes all values in $]t_0, +\infty[$ where $t_0 > 0$. Following these, [9] and [5], we prove (without any restriction in the initial data) in the case of plane and hyperbolic symmetric spacetimes with Vlasov-scalar field matter that $t_0 = 0$, i.e. the area of the group orbits goes to zero at the end of the maximal Cauchy development in the contracting direction. The strategy will be to show that, given a $C^\infty$ solution to the Einstein-Vlasov-scalar field system with metric (1.1) on $]0, t_0]\times S$, where $S$ is a 2-torus, or an hyperbolic plane,

in the case of plane or hyperbolic symmetry respectively. Then for any $t_1 \in \, ]0, t_0]$ such that $t_1 > 0$, the spacetime extends to $t_1$. It will be convenient to fix $t_1 \in \, ]0, t_0]$ and to fix some arbitrary $t_2 \in [t_1, t_0]$. If the metric, the matter content functions and all their derivatives are continuously bounded, then it follows from [9] and [5] (by a long chain of geometrical arguments), that the spacetime further extends to the areal time interval $(0, t_0)$. The existence of conformal symmetries has the effect of simplifying the resulting system of partial differential equations and thus making the integration process simpler.

## 2. $C^\infty$-bounds of Quantities

Let a smooth solution of the system (1.2)-(1.11) be given on some time interval $(t_1, t_0]$. We want to show that if this interval is bounded and if $R$ is bounded away from zero on this interval then $f$, $R$, $\mu$, $\phi$ and all their derivatives are bounded as well with bounds depending on the data and the lower bound on $R$.

**Lemma 2.1.** *Let $D^+ = \partial_t + \partial_r$; $D^- = \partial_t - \partial_r$ ; $X = R(\dot{\phi} - \phi')$; $Y = R(\dot{\phi} + \phi')$; $a = \frac{-\dot{R} + R'}{R}$; $b = -\frac{\dot{R} + R'}{R}$.*
*Then as a consequence of the wave equation (1.7), $X$ and $Y$ satisfy the system*

$$D^+ X = aY \tag{2.1}$$
$$D^- Y = bX \tag{2.2}$$

**Lemma 2.2.** *Let $D^+$, $D^-$, $a$, $b$, $X$ and $Y$ be defined as in the previous lemma. Set $a_1 = \partial_r a$, $b_1 = \partial_r b$, $X_1 = \partial_r X$ and $Y_1 = \partial_r Y$.*
*Then using Lemma 2.1 and the field equations (1.3), (1.4), $X_1$ and $Y_1$ satisfy the system*

$$D^+ X_1 = aY_1 + a_1 Y \tag{2.3}$$
$$D^- Y_1 = bX_1 + b_1 X_1 \tag{2.4}$$

*with*

$$a_1 = (\mu' - \dot{\mu})a + \frac{ab - R'a}{R} - \frac{e^{2\mu}}{2R^2} - 4\pi e^{2\mu}(\rho + j);$$

$$b_1 = -\frac{(\mu' + \dot{\mu})b + R'b}{R} + \frac{ab}{2R} - \frac{e^{2\mu}}{2R} + 4\pi R e^{2\mu}(j - \rho).$$

*Proof.* $\partial_r a = \frac{R'' - \dot{R}' - R'a}{R}$. Add equations (1.3) and (1.4) to replace the term $R'' - \dot{R}'$ and obtain $a_1$.

Analogously, $\partial_r b = -\frac{R'' + \dot{R}' + R'b}{R}$. Subtract equations (1.3), (1.4) to replace the term $R'' + \dot{R}'$ and obtain $b_1$. A direct calculation gives (2.3) and (2.4). $\qquad\square$

Now, we define an auxiliary variables $\tau := \frac{1}{\sqrt{2}}(t - r)$, $\xi := \frac{1}{\sqrt{2}}(t + r)$. Then

$$\partial_\tau := \frac{1}{\sqrt{2}}(\partial_t - \partial_r) \quad \text{and} \quad \partial_\xi := \frac{1}{\sqrt{2}}(\partial_t + \partial_r).$$

The analysis which follows is modeled on the one in [2].

**Step 1:** *Uniformly $C^1$-bounds on R.*

Using (1.3) and (1.4), we obtain after a short calculation

$$\partial_r(\partial_\xi R) = \frac{1}{\sqrt{2}}(\dot{R}' + R'')$$

$$= \left(\sqrt{2}\partial_\xi\mu + \frac{1}{\sqrt{2}}\partial_\tau R\right)\partial_\xi R - \frac{1}{2\sqrt{2}R}e^{2\mu} - 2\sqrt{2}\pi Re^{2\mu}(\rho - j)$$

$$< \left(\sqrt{2}\partial_\xi\mu + \frac{1}{\sqrt{2}}\partial_\tau R\right)\partial_\xi R \qquad (\text{since } |j| < \rho).$$

If $R_\xi(t,r) = 0$ for some $t \in\, ]t_1, t_0]$ and $r \in \mathbb{R}$, then by the periodicity of $R$ with respect to $r$,

$$0 = \partial_\xi R(t, r+1) < \partial_\xi R(t, r)\exp\left(\int_r^{r+1}\left(\sqrt{2}\partial_\xi\mu + \frac{1}{\sqrt{2}}\partial_\tau R\right)ds\right) = 0,$$

a contradiction. Thus $\partial_\xi R \neq 0$ on $]t_1, t_0] \times S$. Similarly

$$\partial_r(\partial_\tau R) = \frac{1}{\sqrt{2}}(\dot{R}' - R'')$$

$$= -\left(\sqrt{2}\partial_\tau\mu + \frac{1}{\sqrt{2}R}\partial_\xi R\right)\partial_\tau R + \frac{1}{2\sqrt{2}R}e^{2\mu} + 2\sqrt{2}\pi Re^{2\mu}(\rho + j)$$

$$> -\left(\sqrt{2}\partial_\tau\mu + \frac{1}{\sqrt{2}}\partial_\xi R\right)\partial_\tau R \quad (\text{since } \rho + j > 0).$$

This yields the same assertion for $\partial_\tau R$. This implies that the quantity

$$g^{\alpha\beta}\partial_{x^\alpha}R\partial_{x^\beta}R = e^{-2\mu}(R'^2 - \dot{R}^2) = -2e^{-2\mu}\partial_\xi R\partial_\tau R$$

does not change sign. Since $R$ is periodic and continuous in $r$, there must exist points where $R' = 0$, hence the quantity above is negative everywhere, and by our choice of time direction,

$$\dot{R} > 0, \ |R'| < \dot{R} \text{ on } (t_1, t_0] \times S. \tag{2.5}$$

By (1.5) and the fact that $\rho \geq p$,

$$\partial_\tau\partial_\xi R = -\frac{1}{2R}\partial_\tau R\partial_\xi R + e^{2\mu}\left(\frac{1}{R} + 4\pi R(\rho - p)\right) > -\frac{1}{2R}\partial_\tau R\partial_\xi R.$$

We fix some $(t, r) \in (t_1, t_0] \times \mathbb{R}$. Then for $s \in [t_1, t)$,

$$\frac{d}{ds}\partial_\xi R(s, r+t-s) = \sqrt{2}\partial_\tau R\partial_\xi R(s, r+t-s)$$

$$> -\frac{1}{2R(s, r+t-s)}\partial_\tau R(s, r+t-s)\partial_\xi R(s, r+t-s)$$

$$> -\frac{1}{\sqrt{2}}\frac{d}{ds}[\ln R(s, r+t-s)]\partial_\xi R(s, r+t-s).$$

Integrating this differential inequality yields

$$\partial_\xi R(t,r) < \frac{R(t_0, r+t-t_0)}{R(t,r)} \partial_\xi R(t_0, r+t-t_0).$$

Similarly,

$$\partial_\tau R(t,r) < \frac{R(t_0, r-t+t_0)}{R(t,r)} \partial_\tau R(t_0, r-t+t_0).$$

Both estimates together imply that $\dot{R}$ is bounded from above on $(t_1, t_0] \times S$ with a bound of the desired sort. This provides from (2.5), bounds for $R'$.

**Step 2:** $C^1$-*bounds on* $\phi$.

**Proposition 2.3.** *Let a, b, X and Y be defined as in Lemma* 2.1 *and*

$$m(t) = \sup\{(|a|, |b|)(t,r); \ r \in \mathbb{R}\}$$

$$K(t) = \sup\{(|X| + |Y|)(t,r); \ r \in \mathbb{R}\}.$$

*If* $(X, Y)$ *is a solution of* (2.1) *and* (2.2) *with*

$$X(t_0) = \mathring{R}(r)(\psi(r) - \mathring{\phi}'(r))$$

*and*

$$Y(t_0) = \mathring{R}(r)(\psi(r) + \mathring{\phi}'(r))$$

*then for any* $t \in (t_1, t_0]$, *we have*

$$K(t) \le K(t_0) + \int_t^{t_0} m(s)K(s)ds \tag{2.6}$$

*Proof.* It is similar to c.f. ([6, Proposition 2.3, p. 697]). The characteristic curves $(t, \gamma_i(t))$, $i = 1, 2$ of the second order partial differential equation (1.7) satisfy the differential equation $\dot{\gamma}_i = \pm 1$. On these characteristic curves, we have from Lemma 2.1, $D^+ = D^- = \frac{d}{dt}$. Then (2.1)-(2.2) become

$$\begin{cases} \dfrac{d}{dt} X(t, \gamma_1(t)) = aY(t, \gamma_1(t)) \\ \dfrac{d}{dt} Y(t, \gamma_2(t)) = bX(t, \gamma_2(t)) \end{cases}$$

Integrate this system on $[t, t_0]$, take the absolute value in each equation, add the two inequalities and take the supremum (in space) of each term to obtain (2.6). □

Since $a$ and $b$ are bounded by Step 1, we deduced from (2.6) and the Gronwall lemma that $K(t)$ is bounded. Then $X$ and $Y$ are bounded. Since $R$ is bounded away from 0, we conclude that $\dot{\phi}$ and $\phi'$ are bounded. Then $\phi$ is bounded.

**Step 3:** $C^1$-*bounds on* $\mu$.

From (1.5)-(1.6) we find

$$\partial_{\tau\xi}\mu = -\frac{1}{R}\partial_{\tau\xi}R - 2\pi e^{2\mu}q.$$

We fix some $(t,r) \in (t_1, t_0[ \times \mathbb{R}$. Then for $s \in [t_1, t)$,

$$\frac{d}{ds}\partial_\xi\mu(s, r+t-s) = \partial_t\partial_\xi\mu - \partial_\theta\partial_\xi\mu$$

$$= \sqrt{2}\partial_{\tau\xi}\mu$$

$$= -\frac{\sqrt{2}}{R}\frac{d}{ds}\partial_\xi R(s, r+t-s) - 2\sqrt{2}\pi e^{2\mu}q(s, r+t-s).$$

Integrating this and integrating by parts the term containing $\partial_\xi R$ yields

$$\partial_\xi\mu(t,r) = \partial_\xi\mu(t_0, r+t-t_0) + \sqrt{2}\frac{\partial_\xi R(t_0, r+t-t_0)}{R(t_0, r+t-t_0)} - \sqrt{2}\frac{\partial_\xi R(t,r)}{R(t,r)}$$

$$+ \sqrt{2}\int_t^{t_0}\left(2\pi e^{2\mu}q - \frac{\partial_\tau R\partial_\xi R}{R^2}\right)(s, r+t-s)ds. \tag{2.7}$$

We know that

$$\int_t^{t_0}\partial_\xi\partial_\tau R(s, r+t-s)ds = \frac{\sqrt{2}}{2}(\partial_\xi R(t_0, r+t-t_0) - \partial_\xi R(t,r)). \tag{2.8}$$

The right hand side of the previous relation is bounded.

Note that $\partial_\xi\partial_\tau R = \frac{1}{2}(\ddot{R} - R'')$. Then (1.5) gives

$$\partial_\xi\partial_\tau R = \frac{1}{2}(\ddot{R} - R'') = \frac{1}{2R}[-(\partial_t R)^2 + (\partial_r R)^2 - ke^{2\mu}] + 4\pi Re^{2\mu}(\rho - p);$$

and the left hand side of (2.8) can be written as

$$\int_t^{t_0}\left\{\frac{1}{2R}[-(\partial_t R)^2 + (\partial_r R)^2] - \frac{k}{2R}e^{2\mu} + 4\pi Re^{2\mu}(\rho - p)\right\}(s, r+t-s)ds.$$

Here the first term is bounded by Step 1. The second and third terms are non-negative ($k = 0$ or $k = -1$) and therefore are bounded by (2.8). Thus by (1.11), (1.8), (1.9) and Step 2,

$$\int_t^{t_0}e^{2\mu}q(s, r+t-s)ds \le \int_t^{t_0}\left[\frac{1}{R^2}e^{2\mu}(\rho - p) + \dot{\phi}^2 - \phi'^2\right](s, r+t-s)ds$$

is bounded as well since $R$ is bounded away from zero. Thus (2.7) implies that $\partial_\xi\mu$ is bounded. Analogously to (2.7), we have

$$\partial_\tau\mu(t,r) = \partial_\tau\mu(t_0, r-t+t_0) + \sqrt{2}\frac{\partial_\tau R(t_0, r-t+t_0)}{R(t_0, r-t+t_0)} - \sqrt{2}\frac{\partial_\tau R(t,r)}{R(t,r)}$$

$$+ \sqrt{2}\int_t^{t_0}\left(2\pi e^{2\mu}q - \frac{\partial_\tau R\partial_\xi R}{R^2}\right)(s, r-t+s)ds \tag{2.9}$$

from which we can conclude that $\partial_\tau \mu$ is bounded. Therefore $\dot{\mu}$, $\mu'$ and $\mu$ are bounded.

**Step 4:** *bounds on matter quantities $\rho$, $p$, $j$, $q$.*
We have for any $t \in \,]t_1, t_0]$, $r \in \mathbb{R}$, $w \in \mathbb{R}$, $F > 0$,

$$f(t, r, w, F) = f(t_0, \Theta(t_0, t, r, w, F), W(t_0, t, r, w, F), F)$$

where $(\Theta(\cdot, t, r, w, F), W(\cdot, t, r, w, F))$ is the solution of the characteristic system

$$\begin{cases} \dot{r} = \dfrac{w}{\langle p \rangle} \\ \dot{w} = -\dot{\mu} w - \mu'(\langle p \rangle + \dfrac{w^2}{\langle p \rangle}) + e^{-2\mu} R' \dfrac{F}{R^3 \langle p \rangle} \end{cases}$$

of the Vlasov equation with $\Theta(t, t, r, w, F) = r$, $W(t, t, r, w, F) = w$. This representation of $f$ implies that $f$ is non-negative and bounded by its maximum at $t = t_0$. By Steps 1 and 3 the right hand side of the second equation in the characteristic system is linearly bounded in $w$. If the $w$-support of $f$ is compact i.e. if

$$\sup\{|w|/(r, w, F) \in \operatorname{supp} f(t), t \in \,]t_1, t_0]\} < \infty,$$

then by Step 2, $\rho$, $p$, $j$ and $q$ are bounded.

**Step 5:** *bounds on second order derivatives of $\phi$.*

**Proposition 2.4.** *Let $a_1$, $b_1$, $X_1$ and $Y_1$ be defined as in Lemma 2.2 and $m(t)$, $K(t)$ be defined as in Proposition 2.3. Set*

$$u_1(t) = \sup\{(|a_1|, |b_1|)(t, r); r \in \mathbb{R}\}$$

$$A_1(t) = \sup\{(|X_1| + |Y_1|)(t, r); r \in \mathbb{R}\}.$$

*If in addition to the assumptions of Proposition 2.3 the quantities $X_1$ and $Y_1$ satisfy (2.3)-(2.4) and agree with $\partial_r X$ and $\partial_r Y$ respectively for $t = t_0$ then:*
*If $t \in \,]t_1, t_0]$, we have*

$$A_1(t) \leq A_1(t_0) + \int_t^{t_0} u_1(s) K(s) ds + \int_t^{t_0} m(s) A_1(s) ds \tag{2.10}$$

*Proof.* Analogous to the proof of Proposition 2.4 using this time Lemma 2.2. Since $m(t)$, $u_1(t)$ and $K(t)$ are bounded by previous steps, we deduced from (2.10) and the Gronwall lemma that $A_1(t)$ is bounded. Then $X_1$ and $Y_1$ are bounded. Since $R$ is bounded away from 0, we conclude using the $C^1$-bounds of $R$ and $\phi$ that $\dot{\phi}'$ and $\phi''$ are bounded. The bounds of $\ddot{\phi}$ is deduced from the wave equation (1.7).

**Step 6:** *bounds on second order derivatives of $R$ and $\mu$.*
Steps 1, 2, 3 and 4 together with (1.2), (1.3) and (1.4) imply respectively that $R''$, $\dot{R}'$ and $\ddot{R}$ are bounded on $(t_1, t_0] \times S$. Now subtract equations (2.7)-(2.9) to obtain a formula for $\mu'$. When this formula is differentiated with respect to $r$ there results

a number of terms which are bounded by the previous steps and the terms

$$2\pi \int_t^{t_0} e^{2\mu} \frac{\pi}{R^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F}{\langle p \rangle} \partial_r f(s, r+t-s, w, F) dF dw ds \tag{2.11}$$

$$2\pi \int_t^{t_0} e^{2\mu} \frac{\pi}{R^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F}{\langle p \rangle} \partial_r f(s, r-t+s, w, F) dF dw ds \tag{2.12}$$

We introduce the differential operators $W = \sqrt{2}\partial_\tau = \partial_t - \partial_r$, $S = \partial_t + \frac{w}{\langle p \rangle}\partial_r$. We have $\partial_r = \frac{\langle p \rangle}{\langle p \rangle + w}(S - W)$. By the Vlasov equation,

$$Sf(s, r+t-s, w, F)$$

$$= \left[ \dot{\mu} w + \mu'(\langle p \rangle + \frac{w^2}{\langle p \rangle}) - e^{-2\mu} R' \frac{F}{R^3 \langle p \rangle} \right] \partial_w f(s, r+t-s, w, F).$$

When this is substituted into (2.11), the resulting term can be integrated by parts with respect to $w$, and all the terms which then appear are bounded by the previous steps. Next we have

$$(Wf)(s, r+t-s, w, F) = \frac{d}{ds}[f(s, r+t-s, w, F)],$$

so that the corresponding term in (2.11) can be integrated by parts with respect to $s$ which again results in bounded terms. In order to deal with (2.12), we replace $w$ by $-w$ and the rest of argument should then be obvious, and $\mu''$ is seen to be bounded on $(t_1, t_2) \times S$. By (1.6), $\ddot{\mu}$ is bounded. Adding equations (2.7) and (2.9) gives the formula for $\dot{\mu}$. Therefore $\dot{\mu}'$ can be dealt with like $\mu''$.

**Step 7:** *Higher order derivatives.*
Using the characteristic system of Step 4, $C^2$-bounds on $R$ and $\mu$ give bounds on the first order derivatives of $\Theta(\cdot, t, r, w, F)$ and $W(\cdot, t, r, w, F)$ with respect to $r$, $w$, $F$. This yields corresponding $C^1$-bounds first on $f$ and then as in Step 4 and using Step 5, on $\rho$, $p$, $j$, $q$. These in turn imply $C^3$-bounds on $R$. Iterating process in Step 5 deals $C^3$-bounds on $\phi$. The third order derivatives of $\mu$ then have to be dealt with by repeating the argument of Step 6. This process can be iterated to bound any desired derivative on $(t_1, t_0] \times S$ in terms of the data at $t = t_0$ and the positive lower bound on $R$.

Later will require a slight generalization of these results in order to show that the arguments of section 5 of ([2]) generalize to cover the case of Vlasov-scalar field system. Once it has been established in Step 1 that the gradient of $R$ is timelike on a region which is covered by Cauchy surface of constant conformal time, the estimates in the later steps hold for any future subset $Z$ of the half-plane $t \leq t_0$ provided $Z$ is a future set. This means by definition that any future directed causal curve in the region $t \leq t_0$ starting at a point of $Z$ remains in $Z$. Thus if $R$ is bounded away from zero on $Z$ and $t$ is bounded on $Z$ then all the unknowns and their derivatives can be controlled on $Z$.

With the presence of the scalar field, the maximal Cauchy development of initial data on a hypersurface of constant conformal time is not a union of hypersurfaces of constant conformal time. Its past boundary is in general of the form $t_1 = h(r)$ where $h$ is a non constant function. We suppose that $h$ has a lower bound.

Consider a special choice of $Z$ defined by

$$Z = \left\{ (t,r) \in \mathbb{R}^2; \ t_1 \le t \le t_0, \ r_1 + t_0 - t < r < r_2 - t_0 + t; \right.$$

$$\left. r_1 < r_2, \ t_0 - \frac{r_2 - r_1}{2} < t_1 \right\}.$$

Suppose a solution of the system in conformal coordinates defined on $Z$ is such that $R$ is bounded away from zero. Then the solution extend smoothly to the boundary of $Z$ at $t = t_1$. They define smooth Cauchy data for the system. Applying the standard local existence theorem (without symmetry) allows the solution to be extended through that boundary. Repeating the construction of the conformal coordinates then shows that we get an extension of the solution written in conformal coordinates through that boundary.

**Theorem 2.5.** *Let $(M, g, f, \phi)$ be the maximal globally hyperbolic development of initial data for the Einstein-Vlasov-scalar field system with surface symmetry. Then M can be covered by symmetric compact hypersurfaces of constant area radius. The area radius of these hypersurfaces takes all values in the range $]0, +\infty[$.*

***Proof.*** Suppose that $t_1 = t_0$. From the bound on the support of the distribution function and the $C^\infty$ bounds obtained previously, it follows that there is a $C^\infty$ extension of the metric functions and the matter distribution-scalar field functions to $t_1$ satisfying the system. Therefore $\{t_1\} \times S$ is contained in the Cauchy development of any Cauchy surface in $]t_1, \infty[ \times S$. This contradicts the fact that the Cauchy development does not extend to $\{t_0\} \times S$. Since $t_1$ was allowed to be any positive number in $[t_0, \infty[$, it must be the case that $t_0 = 0$. $\square$

## 3. Conclusion

Since all metric functions, the Vlasov field, the scalar field and all their derivatives have been shown to be uniformly bounded, the maximal Cauchy development cannot have $t_1 > 0$. Therefore, $(M, g)$ admits a global foliation by areal coordinates with the time coordinate $t$ taking all values in $(0, t_0)$, i.e. $t_1 = 0$, see [9], [5]. This result was obtained in the hyperbolic case without any restriction on the initial data.

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