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Research Article

Resonance Stability of Oblate Infinitesimal in the Neighbourhood of Triangular Equilibrium Points for Triaxial Primaries in the Elliptic Restricted Three Body Problem

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Abstract. The present study aims to investigates the existence of resonance and linear stability of oblate infinitesimal in the neighbourhood of triangular equilibrium points of the elliptical restricted three body problem, considering effect of triaxial primaries in circular and elliptical cases. For this the Hamiltonian function, convergent in nature and describing the motion of the infinitesimal body in the neighbourhood of the triangular equilibrium solutions is derived. Also, the Hamiltonian for the system is expanded in powers of the generalized components of momenta. Further, canonical transformation has also been used to study the stability of the triangular equilibrium points. The study primarily focuses on establishing the relation for determining the range of stability at and near the resonance frequency at $\omega_2 = \frac{1}{2}$. It is observed that the parametric resonance is only possible at the resonance frequency $\omega_2 = \frac{1}{2}$ in circular and elliptical cases.

Keywords. Elliptical restricted three body problem, Lagrangian points, Stability, Parametric resonance

Mathematics Subject Classification (2020). 70F07, 70F15, 70H03

1. Introduction

The present study aims to investigate the condition of existence of resonance and their stability of oblate infinitesimal in the neighbourhood of triangular equilibrium points, when both the primaries are triaxial in the *Elliptical Restricted Three Body Problem* (ER3BP). For the long term evolution of dynamical system resonance plays a very important role. We have adopted the method given by Markeev [10] in which the Hamiltonian function pertaining to the problem is made independent of time by using several canonical transformation. The existence of resonance and their stability of oblate infinitesimal at and near the resonance frequency $\omega_2 = \frac{1}{2}$ has been analysed using simulation technique by drawing the region of stability in circular as well as in Elliptical case. The study of infinitesimal motion around the triangular point in ER3BP has been described in detail by Danby [3] and the problem on characteristics exponent of the equilibrium solution was studied by Bennett [1] and many others.

The ER3BP has been described in considerable details by Danby [3], Szebehely, Rabe. Markeev, Halan and Rana and Khasan. The influence of the eccentricity of the orbits of the primaries with or without radiation pressure on the existence and stability of the equilibrium points are studied by on the non-linear stability of the triangular liberation points of the R3BP in presence of resonance was studied by Chandra and Kumar [2]. Linear and non-linear stability of the triangular libration point for the photo gravitational ER3BP was studied by Kumar and Choudhary [8], Markellos [11], and Markeev [9].

The stability of infinitesimal mass around the equilibrium points of the elliptical restricted three body have been studied by above mentioned authors considering the various perturbation forces. The authors have investigated the different aspects of the elliptic problem. The existence of the liberation points and their stability in the radiation elliptic restricted three body problem has been investigated. The stability of the motion of infinitesimal around the triangular equilibrium points are depending on μ and e. The different aspects of the same problem in details have been investigated. The existence of libration points and their stability in the photo gravitational elliptical restricted three body problem have been studied. The analytical investigation concerning the structure of asymptotic pertubative approximation for small amplitude motions of the third point mass in the neighbourhood of a Lagrangian equilateral libration positions in the planar, elliptical restricted three bodies have been investigated. After a sequence of canonical transformations, they formulated the Hamiltonian governing the motion of the negligible mass body using the eccentric anomaly of the primaries elliptical Keplarian orbits as the independent variable. They studied the Liberalized system of differential equation of motion obtained from expanding the Hamiltonian around a Lagrangian solution. The approximated integrated of the elliptical restricted three body problem by means of perturbation technique based on Lie series development, which led to an approximated solution of the differential system of canonical equation of motion derived from the chosen Hamiltonian function have been discussed.

The present study aims to investigate the condition of existence of resonance and to study the linear stability of oblate infinitesimal body around the triangular equilibrium points in the model of elliptical restricted three body problem, when both the primaries are triaxial in elliptical as well as circular case. The method given by Markeev is used in which the Hamiltonian function pertaining to the problem is made independent of time by using several canonical transformations. $L_{4,5}$ in the elliptic restricted three body problem in presence of parametric resonance. In this model, both the primaries are triaxial. This is achieved by the method given in Markeev [10] in which the Hamiltonian is made independent of time using canonical transformation. This model can easily be applied in the astrophysical applications for the study of many stellar systems.

The present paper is organized in five sections. Section 1 describes introduction; Section 2 describes the equation of motion of the problem; In Section 3 the stability of the system in Circular case $e = 0$ has been investigated; Section 4 describes the stability of the system in elliptical case $e \neq 0$ has been investigated; Section 5 present the discussion and conclusion.

2. Equation of Motion

The differential equation of the motion of the oblate infinitesimal mass in elliptical restricted three body problem under triaxial primaries in the barycentric, pulsating and rotating, non-dimensional coordinates are given by Narayan *et al*. [12]

$$x'' - 2y' = \frac{1}{(1+e\cos v)}\left(\frac{\partial \sigma}{\partial x}\right), \quad y'' + 2x' = \frac{1}{(1+e\cos v)}\left(\frac{\partial \sigma}{\partial y}\right), \tag{2.1}$$

where ($'$) denotes differentiation with respect to v,

$$\Omega = \left(\frac{x^2+y^2}{2}\right) + \frac{1}{n^2}\Big\{\frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)[(2\sigma_1-\sigma_2)+A_4]}{2r_1^3} - \frac{3(1-\mu)[(\sigma_1-\sigma_2)y^2+A_4]}{2r_1^5}$$
$$+ \frac{\mu[(2\sigma_1'-\sigma_2')+A_4]}{2r_2^3}\frac{3\mu[(\sigma_1'-\sigma_2')y^2+A_4]}{2r_2^5}\Big\} \tag{2.2}$$

where

$$n^2 = 1 + \frac{3}{2}e^2 + \frac{3}{2}(2\sigma_1-\sigma_2) + \frac{3}{2}(2\sigma_1'-\sigma_2'),$$
$$r_1^2 = (x+\mu)^2 + y^2, \quad r_2^2 = (x-1+\mu)^2 + y^2, \quad \mu = \frac{m_2}{m_1+m_2}, \tag{2.3}$$

where m_1 and are masses of the primaries σ_1, σ_2, σ_1', σ_2' and A_4 are oblateness triaxial parameters while e and v are the eccentricity of orbital and true anamoly of the primaries, respectively. The coordinates of the triangular equilibrium points are:

$$x = \Big\{\frac{1}{2} - \frac{e^2}{2(1-\mu)} - \frac{\mu e^2}{2(1-\mu)} + \frac{e^2}{2} - \frac{11}{8}\sigma_1 - \frac{1-3\mu}{2\mu}\sigma_1 + \frac{11}{8}\sigma_2 + \frac{1-2\mu}{2\mu}\sigma_2 + \frac{11}{8}\sigma_1'$$
$$+ \frac{5\mu}{2(1-\mu)}\sigma_1' - \frac{7}{8}\sigma_2' - \frac{3\mu}{2(1-\mu)}\sigma_2' + 2A_4 + \frac{5\mu}{2(1-\mu)}A_4\Big\},$$

$$y=\frac{\sqrt{3}}{2}\left[1+\frac{2}{3}\left\{-\frac{e^2}{3(1-\mu)}+\frac{\mu e^2}{3(1-\mu)}-\frac{e^2}{3}-\frac{11}{12}\sigma_1+\frac{2}{3}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{11}{12}\sigma_2-\frac{2}{3}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2\right.\right.$$
$$\left.\left.-\frac{11}{12}\sigma_1'-\frac{5\mu}{2(1-\mu)}\sigma_1'+\frac{7}{12}\sigma_2'-\frac{1}{3}\left(\frac{3\mu}{(1-\mu)}\right)\sigma_2'-\frac{4}{3}A_4+\frac{2}{6}\left(\frac{5\mu}{(1-\mu)}\right)A_4\right\}\right] \tag{2.4}$$

Thus, the coordinates of the triangular equilibrium points has been obtained up to first order terms in the parameter σ_1, σ_2, σ_1', σ_2' and A_4 which is represented by (2.4). The system (2.1) described the motion of dynamical system with lagrangian, which is represented as

$$L=\frac{1}{2}(\dot{x}^2+\dot{y}^2)+(\dot{y}x-\dot{x}y)+\frac{1}{1+e\cos v}+\left[\left(\frac{x^2+y^2}{2}\right)+\frac{1}{n^2}\left\{\frac{(1-\mu)}{r_1}+\frac{\mu}{r_2}\right.\right.$$
$$+\frac{(1-\mu)[(2\sigma_1-\sigma_2)+A_4]}{2r_1^3}-\frac{3(1-\mu)[(\sigma_1-\sigma_2)y^2+A_4]}{2r_1^5}$$
$$\left.\left.+\frac{\mu[(2\sigma_1'-\sigma_2')+A_4]}{2r_2^3}-\frac{3\mu[(\sigma_1'-\sigma_2')y^2+A_4]}{2r_2^5}\right\}\right]. \tag{2.5}$$

The Hamiltonian of the problem is given by:

$$H=-L+p_x\dot{x}+p_y\dot{y}, \tag{2.6}$$

where

$$p_x=\frac{\partial L}{\partial \dot{x}}=\dot{x}-y$$

and

$$p_y=\frac{\partial L}{\partial \dot{y}}=\dot{y}+x,$$

where p_x and p_y are the generalized component of momenta. Thus, using equation (2.5) and equation (2.6), the perturbed Hamiltonian is given by:

$$H=\frac{1}{2}(p_x^2+p_y^2)+(yp_x-xp_y)+\frac{e\cos v}{2(1+e\cos v)}(x^2+y^2)$$
$$-\frac{1}{1+e\cos v}\left[\frac{1}{n^2}\left\{\frac{(1-\mu)}{r_1}+\frac{\mu}{r_2}+\frac{(1-\mu)[(2\sigma_1-\sigma_2)+A_4]}{2r_1^3}-\frac{3(1-\mu)[(\sigma_1-\sigma_2)y^2+A_4]}{2r_1^5}\right.\right.$$
$$\left.\left.+\frac{\mu[(2\sigma_1'-\sigma_2')+A_4]}{2r_2^3}-\frac{3\mu[(\sigma_1'-\sigma_2')y^2+A_4]}{2r_2^5}\right\}\right]. \tag{2.7}$$

Since the two triangular equilibrium points are symmetrical, the nature of the oscillation ofinfinitesimal near two points will be the same. Hence, in further calculation the motion near theequilibrium point L_4 will be considered. So, shifting the origin to L_4 by the change of variables given by:

$$\left.\begin{aligned} x&=\xi+q_1,\\ y&=\eta+q_2,\\ p_x&=p_\xi+p_1,\\ p_y&=p_\eta+p_2,\end{aligned}\right\} \tag{2.8}$$

where the displacement of infinitesimal at and near the equilibrium point L_4 is represented as follows:

$$\xi = \left\{\frac{1}{2} - \frac{e^2}{2(1-\mu)} - \frac{\mu e^2}{2(1-\mu)} + \frac{e^2}{2} - \frac{11}{8}\sigma_1 - \frac{1-3\mu}{2\mu}\sigma_1 + \frac{11}{8}\sigma_2 + \frac{1-2\mu}{2\mu}\sigma_2 + \frac{11}{8}\sigma_1' + \frac{5\mu}{2(1-\mu)}\sigma_1' - \frac{7}{8}\sigma_2' - \frac{3\mu}{2(1-\mu)}\sigma_2' + 2A_4 + \frac{5\mu}{2(1-\mu)}A_4\right\},$$

$$\eta = \frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}\left\{-\frac{e^2}{3(1-\mu)} + \frac{\mu e^2}{3(1-\mu)} - \frac{e^2}{3} - \frac{11}{12}\sigma_1 + \frac{2}{3}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1 + \frac{11}{12}\sigma_2 - \frac{2}{3}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2 - \frac{11}{12}\sigma_1' - \frac{5\mu}{2(1-\mu)}\sigma_1' + \frac{7}{12}\sigma_2' - \frac{1}{3}\left(\frac{3\mu}{(1-\mu)}\right)\sigma_2' - \frac{4}{3}A_4 + \frac{2}{6}\left(\frac{5\mu}{(1-\mu)}\right)A_4\right\}\right],$$

$$p_\xi = -\frac{\sqrt{3}}{2}\left[1 + \frac{2}{3}\left\{-\frac{e^2}{3(1-\mu)} + \frac{\mu e^2}{3(1-\mu)} - \frac{e^2}{3} - \frac{11}{12}\sigma_1 + \frac{2}{3}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1 + \frac{11}{12}\sigma_2 - \frac{2}{3}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2 - \frac{11}{12}\sigma_1' - \frac{5\mu}{2(1-\mu)}\sigma_1' + \frac{7}{12}\sigma_2' - \frac{1}{3}\left(\frac{3\mu}{(1-\mu)}\right)\sigma_2' - \frac{4}{3}A_4 + \frac{2}{6}\left(\frac{5\mu}{(1-\mu)}\right)A_4\right\}\right],$$

$$p_\eta = \left\{\frac{1}{2} - \frac{e^2}{2(1-\mu)} - \frac{\mu e^2}{2(1-\mu)} + \frac{e^2}{2} - \frac{11}{8}\sigma_1 - \frac{1-3\mu}{2\mu}\sigma_1 + \frac{11}{8}\sigma_2 + \frac{1-2\mu}{2\mu}\sigma_2 + \frac{11}{8}\sigma_1' + \frac{5\mu}{2(1-\mu)}\sigma_1' - \frac{7}{8}\sigma_2' - \frac{3\mu}{2(1-\mu)}\sigma_2' + 2A_4 + \frac{5\mu}{2(1-\mu)}A_4\right\}. \tag{2.9}$$

The solution of equation (2.9) in the new variable are given by $q_1 = q_2 = p_1 = p_2 = 0$ which is the equilibrium position. Now, the Hamiltonian H can be written in the form of H_k as the sum of terms of the kth degree homogeneous in the variable q_1, q_2, p_1, p_2 as:

$$H = \sum_0^\infty H_k\,,$$

$$H_0 + H_1 + H_2 + H_3 + \dots, \tag{2.10}$$

where

$$H_0 = H(\xi, \eta, p_\xi, p_\eta) = \text{constant} \tag{2.11}$$

and

$$H_1 = 0. \tag{2.12}$$

The Hamiltonian can also be written as:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + (yp_x - xp_y) + \frac{e\cos v}{2(1+e\cos v)}(x^2 + y^2)$$

$$-\frac{1}{1+e\cos v}\left[\frac{1}{n^2}\left\{(1-\mu)r_1^{-1}+\mu r_2^{-1}+\frac{(1-\mu)[(2\sigma_1-\sigma_2)+A_4]}{2}r_1^{-3}\right.\right.$$
$$-\frac{3(1-\mu)[(\sigma_1-\sigma_2)y^2+A_4]}{2}r_1^{-5}+\frac{\mu[(2\sigma_1'-\sigma_2')+A_4]}{2}r_2^{-3}$$
$$\left.\left.-\frac{3\mu[(\sigma_1'-\sigma_2')y^2+A_4]}{2}r_2^{-5}\right\}\right]. \tag{2.13}$$

Now, expressing the coordinates in the form of general coordinates, we have

$$\begin{aligned}
r_1^{-1}&=f(q_1,q_2),\\
r_2^{-1}&=g(q_1,q_2),\\
r_1^{-3}&=\alpha(q_1,q_2),\\
r_2^{-3}&=\beta(q_1,q_2),\\
r_1^{-5}&=a(q_1,q_2),\\
r_2^{-5}&=b(q_1,q_2).
\end{aligned} \tag{2.14}$$

Now, for analyzing the linear stability expanding each function of equation (2.14) by Taylor's theoremupto second order terms, we have:

$$\begin{aligned}
f(q_1,q_2)&=f(0,0)+[q_1f_1(0,0)+q_2f_2(0,0)]+\frac{1}{2}[q_1^2f_{11}(0,0)+2q_1q_2f_{12}(0,0)+q_2^2f_{22}(0,0)]\\
&\quad+\frac{1}{6}[q_1^3f_{111}(0,0)+3q_1^2q_2f_{112}(0,0)+3q_1q_2^2f_{122}(0,0)+q_2^3f_{222}(0,0)]\\
&\quad+\frac{1}{24}[q_1^4f_{1111}(0,0)+4q_1^3q_2f_{1112}(0,0)+6q_1^2q_2^2f_{1122}(0,0)+4q_1q_2^3f_{1222}(0,0)\\
&\quad+q_2^4f_{2222}(0,0)],
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
g(q_1,q_2)&=g(0,0)+[q_1g_1(0,0)+q_2g_2(0,0)]+\frac{1}{2}[q_1^2g_{11}(0,0)+2q_1q_2g_{12}(0,0)+q_2^2g_{22}(0,0)]\\
&\quad+\frac{1}{6}[q_1^3g_{111}(0,0)+3q_1^2q_2g_{112}(0,0)+3q_1q_2^2g_{122}(0,0)+q_2^3g_{222}(0,0)]\\
&\quad+\frac{1}{24}[q_1^4g_{1111}(0,0)+4q_1^3q_2g_{1112}(0,0)+6q_1^2q_2^2g_{1122}(0,0)+4q_1q_2^3g_{1222}(0,0)\\
&\quad+q_2^4g_{2222}(0,0)],
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\alpha(q_1,q_2)&=\alpha(0,0)+[q_1\alpha_1(0,0)+q_2\alpha_2(0,0)]+\frac{1}{2}[q_1^2\alpha_{11}(0,0)+2q_1q_2\alpha_{12}(0,0)+q_2^2\alpha_{22}(0,0)]\\
&\quad+\frac{1}{6}[q_1^3\alpha_{111}(0,0)+3q_1^2q\alpha_{112}(0,0)+3q_1q_2^2\alpha_{122}(0,0)+q_2^3\alpha_{222}(0,0)]\\
&\quad+\frac{1}{24}[q_1^4\alpha_{1111}(0,0)+4q_1^3q_2\alpha_{1112}(0,0)+6q_1^2q_2^2\alpha_{1122}(0,0)+4q_1q_2^3\alpha_{1222}(0,0)\\
&\quad+q_2^4\alpha_{2222}(0,0)],
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
\beta(q_1,q_2)&=\beta(0,0)+[q_1\beta_1(0,0)+q_2\beta_2(0,0)]\\
&\quad+\frac{1}{2}[q_1^2\beta_{11}(0,0)+2q_1q_2\beta_{12}(0,0)+q_2^2\beta_{22}(0,0)]
\end{aligned}$$

$$+\frac{1}{6}[q_1^3\beta_{111}(0,0)+3q_1^2q_2\beta_{112}(0,0)+3q_1q_2^2\beta_{122}(0,0)+q_2^3\beta_{222}(0,0)]$$
$$+\frac{1}{24}[q_1^4\beta_{1111}(0,0)+4q_1^3q_2\beta_{1112}(0,0)+6q_1^2q_2^2\beta_{1122}(0,0)+4q_1q_2^3\beta_{1222}(0,0)$$
$$+q_2^4\beta_{2222}(0,0)], \tag{2.18}$$

$$a(q_1,q_2)=a(0,0)+[q_1a_1(0,0)+q_2a_2(0,0)]+\frac{1}{2}[q_1^2a_{11}(0,0)+2q_1q_2a_{12}(0,0)+q_2^2a_{22}(0,0)]$$
$$+\frac{1}{6}[q_1^3a_{111}(0,0)+3q_1^2q_2a_{112}(0,0)+3q_1q_2^2a_{122}(0,0)+q_2^3a_{222}(0,0)]$$
$$+\frac{1}{24}[q_1^4a_{1111}(0,0)+4q_1^3q_2a_{1112}(0,0)+6q_1^2q_2^2a_{1122}(0,0)+4q_1q_2^3a_{1222}(0,0)$$
$$+q_2^4a_{2222}(0,0)] \tag{2.19}$$

and

$$b(q_1,q_2)=b(0,0)+[q_1b_1(0,0)+q_2b_2(0,0)]+\frac{1}{2}[q_1^2b_{11}(0,0)+2q_1q_2b_{12}(0,0)+q_2^2b_{22}(0,0)]$$
$$+\frac{1}{6}[q_1^3b_{111}(0,0)+3q_1^2q_2b_{112}(0,0)+3q_1q_2^2b_{122}(0,0)+q_2^3b_{222}(0,0)]$$
$$+\frac{1}{24}[q_1^4b_{1111}(0,0)+4q_1^3q_2b_{1112}(0,0)+6q_1^2q_2^2b_{1122}(0,0)+4q_1q_2^3b_{1222}(0,0)$$
$$+q_2^4b_{2222}(0,0)] \tag{2.20}$$

and evaluating values at the equilibrium position.

The values of $f(q_1,q_2)$, at the equilibrium points are evaluated as:

$$f(0,0)=\left\{1+\frac{5}{12}\left(\frac{e^2}{1-\mu}\right)+\frac{1}{12}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{1}{12}e^2+\frac{55}{48}\sigma_1+\frac{1}{6}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{55}{48}\sigma_2-\frac{1}{6}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2\right.$$
$$\left.-\frac{11}{48}\sigma_1'-\frac{1}{12}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{7}{48}\sigma_2'+\frac{5}{4}\left(\frac{\mu}{1-\mu}\right)\sigma_2'-\frac{1}{3}A_4-\frac{5}{12}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_1(0,0)=\left\{-\frac{1}{2}+\frac{3}{8}\left(\frac{e^2}{1-\mu}\right)+\frac{3}{8}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{3}{8}e^2-\frac{11}{32}\sigma_1+\frac{3}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{11}{32}\sigma_2-\frac{3}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2\right.$$
$$\left.+\frac{11}{32}\sigma_1'-\frac{3}{8}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{21}{32}\sigma_2'-\frac{3}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{3}{2}A_4-\frac{2}{9}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_2(0,0)=-\frac{\sqrt{3}}{2}\left\{1+\frac{37}{36}\left(\frac{e^2}{1-\mu}\right)+\frac{17}{36}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{17}{36}e^2+\frac{407}{144}\sigma_1+\frac{17}{18}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{407}{144}\sigma_2\right.$$
$$-\frac{17}{18}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{187}{144}\sigma_1'-\frac{17}{36}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{119}{144}\sigma_2'-\frac{53}{12}\left(\frac{\mu}{1-\mu}\right)\sigma_2'-\frac{17}{9}A_4$$
$$\left.-\frac{37}{36}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{11}(0,0)=\left\{-\frac{1}{4}-\frac{19}{16}\left(\frac{e^2}{1-\mu}\right)-\frac{23}{16}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{23}{16}e^2-\frac{209}{64}\sigma_1-\frac{23}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{209}{64}\sigma_2\right.$$
$$\left.+\frac{23}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{253}{64}\sigma_1'+\frac{33}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{161}{64}\sigma_2'-\frac{19}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{23}{4}A_4-\frac{3}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{12}(0,0)=\sqrt{3}\left\{\frac{3}{4}+\frac{31}{52}\left(\frac{e^2}{1-\mu}\right)-\frac{13}{52}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{13}{52}e^2+\frac{341}{192}\sigma_1-\frac{3}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{341}{192}\sigma_2\right.$$
$$+\frac{13}{24}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{143}{192}\sigma_1'+\frac{13}{48}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{91}{192}\sigma_2'+\frac{31}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{13}{12}A_4$$
$$\left.-\frac{31}{24}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{22}(0,0)=\left\{\frac{5}{4}+\frac{73}{64}\left(\frac{e^2}{1-\mu}\right)+\frac{53}{36}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{41}{8}e^2+\frac{429}{64}\sigma_1+\frac{27}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{429}{64}\sigma_2\right.$$
$$-\frac{27}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{297}{64}\sigma_1'-\frac{27}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{189}{64}\sigma_2'+\frac{73}{36}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{27}{4}A_4$$
$$\left.-\frac{23}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{111}(0,0)=\left\{\frac{21}{8}+\frac{171}{32}\left(\frac{e^2}{1-\mu}\right)-\frac{83}{32}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{61}{32}e^2+\frac{571}{128}\sigma_1+\frac{723}{48}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{571}{128}\sigma_2\right.$$
$$-\frac{723}{48}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{521}{128}\sigma_1'-\frac{61}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{427}{128}\sigma_2'-\frac{309}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{61}{8}A_4$$
$$\left.+\frac{53}{48}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{112}(0,0)=\sqrt{3}\left\{-\frac{3}{8}+\frac{1419}{288}\left(\frac{e^2}{1-\mu}\right)-\frac{609}{288}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{1419}{288}e^2+\frac{15609}{1152}\sigma_1-\frac{609}{144}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$-\frac{15609}{1152}\sigma_2+\frac{609}{144}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{6699}{1152}\sigma_1'+\frac{609}{288}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{4263}{1152}\sigma_2'$$
$$\left.+\frac{1419}{96}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{609}{72}A_4+\frac{2901}{144}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{122}(0,0)=\left\{-\frac{33}{8}-\frac{233}{32}\left(\frac{e^2}{1-\mu}\right)-\frac{33}{32}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{33}{32}e^2-\frac{1463}{128}\sigma_1-\frac{33}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{1463}{128}\sigma_2\right.$$
$$+\frac{33}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{803}{128}\sigma_1'+\frac{33}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1643}{128}\sigma_2'-\frac{651}{32}\left(\frac{\mu}{1-\mu}\right)\sigma_2'+\frac{33}{8}A_4$$
$$\left.+\frac{213}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{222}(0,0)=\sqrt{3}\left\{-\frac{9}{8}-\frac{137}{32}\left(\frac{e^2}{1-\mu}\right)-\frac{133}{32}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{133}{32}e^2-\frac{1507}{128}\sigma_1-\frac{133}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{1507}{128}\sigma_2+\frac{133}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1463}{128}\sigma_1'+\frac{133}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1099}{128}\sigma_2'$$
$$\left.-\frac{411}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{141}{8}A_4+\frac{137}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$f_{1111}(0,0)=\left\{-\frac{111}{16}-\frac{5825}{64}\left(\frac{e^2}{1-\mu}\right)+\frac{915}{64}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{915}{64}e^2-\frac{2475}{256}\sigma_1+\frac{915}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{2475}{256}\sigma_2-\frac{915}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{3135}{256}\sigma_1'-\frac{915}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{11235}{256}\sigma_2'$$

$$-\frac{675}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{915}{16}A_4+\frac{225}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\Bigg\},$$

$$f_{1112}(0,0)=\sqrt{3}\Bigg\{-\frac{75}{16}-\frac{2995}{192}\left(\frac{e^2}{1-\mu}\right)-\frac{2135}{192}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{2135}{192}e^2-\frac{7745}{768}\sigma_1-\frac{2135}{96}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{7745}{768}\sigma_2+\frac{2135}{96}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1715}{768}\sigma_1'+\frac{2135}{128}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{14945}{768}\sigma_2'$$
$$-\frac{5515}{64}\left(\frac{\mu}{1-\mu}\right)\sigma_2'+\frac{2135}{48}A_4+\frac{2995}{192}\left(\frac{5\mu}{1-\mu}\right)A_4\Bigg\},$$

$$f_{1122}(0,0)=\Bigg\{\frac{123}{16}+\frac{45}{64}\left(\frac{e^2}{1-\mu}\right)-\frac{1691}{64}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{1691}{64}e^2+\frac{495}{256}\sigma_1-\frac{1445}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{495}{256}\sigma_2+\frac{1445}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{14685}{256}\sigma_1'+\frac{1445}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{7665}{256}\sigma_2'$$
$$-\frac{45}{2}\left(\frac{\mu}{1-\mu}\right)\sigma_2'+\frac{1335}{16}A_4-\frac{4055}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\Bigg\},$$

$$f_{1222}(0,0)=\frac{\sqrt{3}}{2}\Bigg\{\frac{135}{8}+\frac{1405}{32}\left(\frac{e^2}{1-\mu}\right)+\frac{665}{32}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{665}{32}e^2+\frac{15455}{128}\sigma_1+\frac{185}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{15455}{128}\sigma_2-\frac{665}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{16555}{128}\sigma_1'-\frac{165}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{4655}{128}\sigma_2'$$
$$+\frac{2455}{32}\left(\frac{\mu}{1-\mu}\right)\sigma_2'-\frac{665}{8}A_4-\frac{2285}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\Bigg\},$$

$$f_{2222}(0,0)=\Bigg\{\frac{9}{16}+\frac{1335}{64}\left(\frac{e^2}{1-\mu}\right)-\frac{435}{64}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{435}{64}e^2+\frac{14685}{256}\sigma_1+\frac{525}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{14685}{256}\sigma_2-\frac{525}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{11385}{256}\sigma_1'-\frac{75}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{13965}{256}\sigma_2'$$
$$+\frac{4005}{64}\left(\frac{\mu}{1-\mu}\right)\sigma_2'-\frac{1995}{16}A_4-\frac{1335}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\Bigg\},$$

The values of $g(q_1,q_2)$, at the equilibrium points are evaluated as:

$$g(0,0)=\Bigg\{1-\frac{1}{12}\left(\frac{e^2}{1-\mu}\right)-\frac{5}{12}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{5}{12}e^2-\frac{11}{48}\sigma_1-\frac{5}{6}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{11}{48}\sigma_2$$
$$+\frac{5}{6}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{55}{48}\sigma_1'+\frac{5}{12}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{35}{48}\sigma_2'-\frac{1}{12}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{5}{3}A_4$$
$$+\frac{1}{12}\left(\frac{5\mu}{1-\mu}\right)A_4\Bigg\},$$

$$g_1(0,0)=\Bigg\{\frac{1}{2}+\frac{3}{8}\left(\frac{e^2}{1-\mu}\right)-\frac{1}{8}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{1}{8}e^2+\frac{33}{32}\sigma_1-\frac{1}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{33}{32}\sigma_2+\frac{1}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2$$
$$+\frac{11}{32}\sigma_1'+\frac{1}{8}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{7}{32}\sigma_2'+\frac{3}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{1}{2}A_4-\frac{3}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\Bigg\},$$

$$g_2(0,0)=-\frac{\sqrt{3}}{2}\Bigg\{1-\frac{17}{36}\left(\frac{e^2}{1-\mu}\right)-\frac{37}{36}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{37}{36}e^2-\frac{187}{144}\sigma_1-\frac{37}{18}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$

$$+\frac{187}{144}\sigma_2+\frac{37}{18}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{407}{144}\sigma_1'+\frac{37}{36}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{259}{144}\sigma_2'-\frac{17}{36}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$+\frac{37}{9}A_4+\frac{17}{36}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{11}(0,0)=\bigg\{-\frac{1}{4}+\frac{23}{16}\left(\frac{e^2}{1-\mu}\right)+\frac{19}{16}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{19}{16}e^2+\frac{253}{64}\sigma_1+\frac{19}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{253}{64}\sigma_2$$
$$-\frac{19}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{209}{64}\sigma_1'-\frac{19}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{133}{64}\sigma_2'+\frac{23}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{21}{4}A_4$$
$$-\frac{23}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{12}(0,0)=\frac{3\sqrt{3}}{2}\bigg\{-\frac{1}{2}-\frac{13}{72}\left(\frac{e^2}{1-\mu}\right)+\frac{31}{72}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{31}{72}e^2-\frac{143}{288}\sigma_1+\frac{31}{36}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{143}{288}\sigma_2-\frac{31}{36}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{341}{288}\sigma_1'-\frac{31}{72}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{217}{288}\sigma_2'-\frac{41}{72}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$-\frac{31}{18}A_4+\frac{13}{72}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{22}(0,0)=\bigg\{\frac{5}{4}-\frac{27}{16}\left(\frac{e^2}{1-\mu}\right)-\frac{39}{16}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{39}{16}e^2-\frac{1133}{256}\sigma_1-\frac{39}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{1133}{256}\sigma_2$$
$$+\frac{39}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1716}{256}\sigma_1'+\frac{39}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1092}{256}\sigma_2'-\frac{29}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{39}{4}A_4$$
$$+\frac{27}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{111}(0,0)=\bigg\{-\frac{21}{8}-\frac{59}{32}\left(\frac{e^2}{1-\mu}\right)+\frac{161}{32}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{161}{32}e^2+\frac{1991}{128}\sigma_1+\frac{161}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{1991}{128}\sigma_2-\frac{161}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{1771}{128}\sigma_1'-\frac{161}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{1127}{128}\sigma_2'$$
$$+\frac{241}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{161}{8}A_4-\frac{61}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{112}(0,0)=\frac{3\sqrt{3}}{2}\bigg\{-\frac{1}{4}-\frac{667}{144}\left(\frac{e^2}{1-\mu}\right)-\frac{143}{144}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{143}{144}e^2-\frac{3377}{576}\sigma_1-\frac{143}{72}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{3377}{576}\sigma_2+\frac{143}{72}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1573}{576}\sigma_1'+\frac{143}{144}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1001}{576}\sigma_2'$$
$$-\frac{307}{144}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{143}{36}A_4+\frac{307}{144}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{122}(0,0)=\bigg\{\frac{33}{8}-\frac{33}{32}\left(\frac{e^2}{1-\mu}\right)-\frac{213}{32}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{213}{32}e^2-\frac{363}{128}\sigma_1-\frac{213}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{363}{128}\sigma_2$$
$$+\frac{213}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{2343}{128}\sigma_1'+\frac{213}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1491}{128}\sigma_2'-\frac{33}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$+\frac{213}{8}A_4+\frac{33}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{222}(0,0)=\frac{\sqrt{3}}{2}\bigg\{-\frac{9}{4}+\frac{133}{16}\left(\frac{e^2}{1-\mu}\right)+\frac{377}{16}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{377}{16}e^2+\frac{1463}{64}\sigma_1+\frac{137}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{1463}{64}\sigma_2-\frac{137}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{1507}{64}\sigma_1'-\frac{137}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{2639}{64}\sigma_2'$$
$$+\frac{733}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{137}{4}A_4-\frac{73}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{1111}(0,0)=\bigg\{-\frac{111}{16}-\frac{915}{64}\left(\frac{e^2}{1-\mu}\right)+\frac{225}{64}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{225}{64}e^2-\frac{10065}{256}\sigma_1+\frac{225}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{10065}{256}\sigma_2-\frac{225}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{10175}{256}\sigma_1'-\frac{225}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{1575}{256}\sigma_2'$$
$$-\frac{75}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{225}{16}A_4+\frac{915}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{1112}(0,0)=\frac{3\sqrt{3}}{2}\bigg\{\frac{25}{8}-\frac{2135}{288}\left(\frac{e^2}{1-\mu}\right)-\frac{2995}{288}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{2995}{288}e^2-\frac{2915}{128}\sigma_1+\frac{2995}{144}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{2915}{128}\sigma_2-\frac{2995}{144}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{32945}{1152}\sigma_1'-\frac{2995}{288}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{20965}{1152}\sigma_2'$$
$$-\frac{5915}{288}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{2995}{72}A_4+\frac{2135}{288}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{1122}(0,0)=\bigg\{\frac{123}{16}+\frac{935}{64}\left(\frac{e^2}{1-\mu}\right)-\frac{45}{64}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{45}{64}e^2+\frac{16665}{256}\sigma_1-\frac{45}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{16665}{256}\sigma_2+\frac{45}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{2475}{256}\sigma_1'+\frac{45}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1575}{256}\sigma_2'$$
$$+\frac{1335}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{45}{16}A_4-\frac{1335}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{1222}(0,0)=\frac{\sqrt{3}}{2}\bigg\{-\frac{135}{8}+\frac{665}{32}\left(\frac{e^2}{1-\mu}\right)+\frac{1405}{32}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{1405}{32}e^2-\frac{7315}{128}\sigma_1$$
$$+\frac{1405}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{7315}{128}\sigma_2-\frac{1405}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{15455}{128}\sigma_1'-\frac{1085}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'$$
$$+\frac{7355}{128}\sigma_2'+\frac{245}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{1405}{8}A_4-\frac{245}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$g_{2222}(0,0)=\bigg\{\frac{9}{16}-\frac{1995}{64}\left(\frac{e^2}{1-\mu}\right)-\frac{1335}{64}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{1335}{64}e^2-\frac{66345}{256}\sigma_1-\frac{1335}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{66345}{256}\sigma_2+\frac{1335}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{14685}{256}\sigma_1'+\frac{1335}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{9345}{256}\sigma_2'$$
$$-\frac{1995}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{1335}{16}A_4+\frac{1995}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\}.$$

The values of $\alpha(q_1,q_2)$, at the equilibrium points are evaluated as:

$$\alpha(0,0)=\bigg\{1+\frac{5}{4}\left(\frac{e^2}{1-\mu}\right)+\frac{1}{4}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{1}{8}e^2+\frac{55}{16}\sigma_1+\frac{1}{2}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{55}{16}\sigma_2-\frac{1}{2}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2$$

$$-\frac{11}{16}\sigma_1'-\frac{1}{4}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{7}{16}\sigma_2'+\frac{5}{4}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-A_4-\frac{5}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_1(0,0)=\bigg\{-\frac{3}{2}-\frac{13}{8}\frac{e^2}{1-\mu}+\frac{7}{8}\frac{\mu e^2}{1-\mu}-\frac{7}{8}e^2-\frac{143}{32}\sigma_1+\frac{7}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{143}{32}\sigma_2-\frac{7}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2$$
$$-\frac{77}{32}\sigma_1'-\frac{7}{8}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{49}{32}\sigma_2'-\frac{13}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{7}{2}A_4+\frac{13}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_2(0,0)=\frac{\sqrt{3}}{2}\bigg\{-3-\frac{21}{4}\frac{e^2}{1-\mu}-\frac{9}{4}\frac{\mu e^2}{1-\mu}+\frac{9}{4}e^2-\frac{231}{16}\sigma_1-\frac{9}{2}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{231}{16}\sigma_2$$
$$+\frac{9}{2}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{99}{16}\sigma_1'+\frac{9}{4}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{63}{16}\sigma_2'-\frac{21}{4}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+9A_4+\frac{21}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_{11}(0,0)=\bigg\{\frac{3}{4}-\frac{45}{16}\frac{e^2}{1-\mu}-\frac{105}{16}\frac{\mu e^2}{1-\mu}+\frac{105}{16}e^2-\frac{495}{64}\sigma_1-\frac{105}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{495}{64}\sigma_2$$
$$+\frac{105}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1155}{64}\sigma_1'+\frac{105}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{735}{64}\sigma_2'-\frac{245}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$+\frac{105}{4}A_4+\frac{45}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_{12}(0,0)=\frac{\sqrt{3}}{2}\bigg\{\frac{15}{2}+\frac{95}{8}\frac{e^2}{1-\mu}-\frac{5}{8}\frac{\mu e^2}{1-\mu}+\frac{5}{8}e^2+\frac{1045}{32}\sigma_1-\frac{5}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{1045}{32}\sigma_2$$
$$+\frac{5}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{55}{32}\sigma_1'-\frac{15}{8}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{315}{32}\sigma_2'+\frac{95}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{5}{2}A_4-\frac{95}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_{22}(0,0)=\bigg\{\frac{33}{4}+\frac{345}{16}\frac{e^2}{1-\mu}+\frac{165}{16}\frac{\mu e^2}{1-\mu}-\frac{165}{16}e^2+\frac{3795}{64}\sigma_1+\frac{165}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{3795}{64}\sigma_2$$
$$-\frac{165}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{1815}{64}\sigma_1'-\frac{125}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{1155}{64}\sigma_2'+\frac{345}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{165}{4}A_4$$
$$-\frac{505}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_{111}(0,0)=\bigg\{\frac{75}{8}+\frac{1065}{32}\frac{e^2}{1-\mu}+\frac{645}{32}\frac{\mu e^2}{1-\mu}-\frac{645}{32}e^2+\frac{3315}{128}\sigma_1+\frac{645}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{3315}{128}\sigma_2$$
$$-\frac{645}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{5505}{128}\sigma_1'-\frac{645}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{4515}{128}\sigma_2'+\frac{2115}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{645}{8}A_4$$
$$-\frac{1065}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_{112}(0,0)=\frac{\sqrt{3}}{2}\bigg\{-\frac{45}{4}+\frac{25}{16}\frac{e^2}{1-\mu}+\frac{605}{16}\frac{\mu e^2}{1-\mu}-\frac{605}{16}e^2-\frac{6725}{64}\sigma_1+\frac{605}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{6725}{64}\sigma_2$$
$$-\frac{605}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{6655}{64}\sigma_1'-\frac{605}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{4235}{64}\sigma_2'+\frac{25}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{605}{4}A_4$$
$$-\frac{25}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$\alpha_{122}(0,0)=\left\{-\frac{225}{8}-315\frac{e^2}{1-\mu}-\frac{105}{2}\frac{\mu e^2}{1-\mu}+\frac{105}{2}e^2-\frac{3465}{4}\sigma_1-105\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{3465}{4}\sigma_2\right.$$
$$+105\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1155}{8}\sigma_1'+\frac{295}{4}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{735}{8}\sigma_2'-315\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+210A_4$$
$$\left.+350\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$\alpha_{222}(0,0)=\frac{\sqrt{3}}{2}\left\{-\frac{135}{4}-\frac{2025}{16}\frac{e^2}{1-\mu}-\frac{1125}{16}\frac{\mu e^2}{1-\mu}+\frac{1125}{16}e^2-\frac{82225}{192}\sigma_1-\frac{1125}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{82225}{192}\sigma_2+\frac{1125}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{12375}{64}\sigma_1'+\frac{1965}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{7875}{64}\sigma_2'$$
$$\left.-\frac{5595}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{1125}{4}A_4+\frac{2025}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$\alpha_{1111}(0,0)=\left\{-\frac{855}{16}-\frac{7035}{64}\frac{e^2}{1-\mu}+\frac{2625}{64}\frac{\mu e^2}{1-\mu}-\frac{2625}{64}e^2-\frac{77385}{256}\sigma_1+\frac{2625}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{77385}{256}\sigma_2-\frac{2625}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{28875}{256}\sigma_1'-\frac{2625}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{18375}{256}\sigma_2'$$
$$\left.-\frac{31185}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{2625}{16}A_4+\frac{7035}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$\alpha_{1112}(0,0)=\frac{\sqrt{3}}{2}\left\{-\frac{315}{8}-\frac{7455}{32}\frac{e^2}{1-\mu}-\frac{7035}{32}\frac{\mu e^2}{1-\mu}+\frac{7035}{32}e^2-\frac{1988385}{128}\sigma_1-\frac{80835}{128}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{1988385}{128}\sigma_2+\frac{80835}{128}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1785}{128}\sigma_1'+\frac{42315}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{49245}{128}\sigma_2'$$
$$\left.-\frac{15645}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{1365}{8}A_4+\frac{53655}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$\alpha_{1122}(0,0)=\left\{\frac{1395}{16}+\frac{7455}{64}\frac{e^2}{1-\mu}-\frac{10605}{64}\frac{\mu e^2}{1-\mu}+\frac{10605}{64}e^2+\frac{8085}{256}\sigma_1-\frac{10605}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$-\frac{8085}{256}\sigma_2+\frac{10605}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{116655}{256}\sigma_1'+\frac{12005}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{74235}{256}\sigma_2'$$
$$\left.+\frac{66605}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{10605}{16}A_4-\frac{15855}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$\alpha_{1222}(0,0)=\frac{\sqrt{3}}{2}\left\{\frac{1575}{8}+\frac{20895}{32}\frac{e^2}{1-\mu}+\frac{6195}{32}\frac{\mu e^2}{1-\mu}-\frac{6195}{32}e^2-\frac{689535}{384}\sigma_1+\frac{6195}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{689535}{384}\sigma_2-\frac{6195}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{105945}{128}\sigma_1'-\frac{6195}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{130095}{384}\sigma_2'$$
$$\left.+\frac{87045}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{6195}{8}A_4-\frac{20895}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$\alpha_{2222}(0,0)=\left\{\frac{1665}{16}+\frac{34125}{64}\frac{e^2}{1-\mu}+\frac{26985}{64}\frac{\mu e^2}{1-\mu}-\frac{26985}{64}e^2+\frac{1126125}{768}\sigma_1+\frac{26985}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$

$$-\frac{1126125}{768}\sigma_2-\frac{26985}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{890505}{768}\sigma_1'-\frac{33705}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{56685}{768}\sigma_2'$$
$$+\frac{145635}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{26985}{16}A_4-\frac{7245}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\}.$$

The values of $\beta(q_1,q_2)$, at the equilibrium points are evaluated as:

$$\beta(0,0)=\Big\{1-2\left(\frac{e^2}{1-\mu}\right)+\left(\frac{\mu e^2}{1-\mu}\right)-e^2+\frac{11}{2}\sigma_1+2\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{11}{2}\sigma_2-2\left(\frac{1-2\mu}{2\mu}\right)\sigma_2$$
$$-\frac{11}{2}\sigma_1'-\frac{5}{8}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{7}{4}\sigma_2'+2\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-4A_4-\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_1(0,0)=\Big\{\frac{3}{2}+\frac{13}{2}\left(\frac{e^2}{1-\mu}\right)+4\left(\frac{\mu e^2}{1-\mu}\right)-4e^2+\frac{143}{8}\sigma_1+8\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{143}{8}\sigma_2-8\left(\frac{1-2\mu}{2\mu}\right)\sigma_2$$
$$-11\sigma_1'-\frac{37}{8}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+7\sigma_2'+\frac{13}{2}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-16A_4-4\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_2(0,0)=\sqrt{3}\Big\{-\frac{3}{2}-\frac{14}{3}\left(\frac{e^2}{1-\mu}\right)-\frac{17}{6}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{17}{6}e^2-\frac{77}{6}\sigma_1-\frac{17}{3}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{77}{6}\sigma_2$$
$$+\frac{17}{3}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{187}{24}\sigma_1'+\frac{83}{24}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{119}{24}\sigma_2'-\frac{14}{3}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{14}{3}A_4$$
$$+\frac{17}{6}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{11}(0,0)=\Big\{\frac{3}{4}-\frac{15}{2}\left(\frac{e^2}{1-\mu}\right)-\frac{45}{4}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{45}{4}e^2-\frac{165}{8}\sigma_1-\frac{125}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{165}{8}\sigma_2$$
$$+\frac{125}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{407}{16}\sigma_1'+\frac{165}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{315}{16}\sigma_2'-\frac{65}{4}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+45A_4$$
$$+\frac{45}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{12}(0,0)=\frac{15\sqrt{3}}{2}\Big\{-\frac{1}{2}-\frac{49}{18}\left(\frac{e^2}{1-\mu}\right)-\frac{16}{9}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{16}{9}e^2-\frac{385}{72}\sigma_1-\frac{32}{9}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{385}{72}\sigma_2$$
$$+\frac{32}{9}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{44}{9}\sigma_1'+\frac{149}{72}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{28}{9}\sigma_2'-\frac{49}{18}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{64}{9}A_4$$
$$+\frac{32}{18}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{22}(0,0)=\Big\{\frac{33}{4}+\frac{75}{2}\left(\frac{e^2}{1-\mu}\right)+\frac{105}{4}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{105}{4}e^2+\frac{825}{8}\sigma_1+\frac{105}{2}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{825}{8}\sigma_2$$
$$-\frac{105}{2}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{1155}{16}\sigma_1'-\frac{505}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{735}{16}\sigma_2'+\frac{75}{2}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-105A_4$$
$$-\frac{125}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{111}(0,0)=\Big\{-\frac{75}{8}-\frac{705}{8}\left(\frac{e^2}{1-\mu}\right)-75\left(\frac{\mu e^2}{1-\mu}\right)+75e^2-\frac{7755}{32}\sigma_1-150\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{7755}{32}\sigma_2$$

$$+150\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{825}{4}\sigma_1'+\frac{4485}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{525}{4}\sigma_2'-\frac{705}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+265A_4$$
$$+\frac{565}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{112}(0,0)=\frac{\sqrt{3}}{2}\Big\{-\frac{45}{4}+20\left(\frac{e^2}{1-\mu}\right)+\frac{235}{4}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{235}{4}e^2+55\sigma_1+\frac{235}{2}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-55\sigma_2$$
$$-\frac{235}{2}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{2585}{16}\sigma_1'-\frac{925}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{1645}{16}\sigma_2'+20\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{35}{3}A_4$$
$$-\frac{275}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{122}(0,0)=\Big\{\frac{225}{8}+\frac{915}{4}\left(\frac{e^2}{1-\mu}\right)+\frac{1305}{8}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{1305}{8}e^2+\frac{10065}{16}\sigma_1+\frac{1305}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{10065}{16}\sigma_2+\frac{1305}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{15755}{32}\sigma_1'-\frac{5745}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{9135}{32}\sigma_2'$$
$$+220\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{2305}{6}A_4-\frac{1305}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{222}(0,0)=\frac{\sqrt{3}}{2}\Big\{-\frac{135}{4}-220\left(\frac{e^2}{1-\mu}\right)-\frac{695}{4}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{695}{4}e^2-605\sigma_1-695\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+605\sigma_2+695\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{7645}{16}\sigma_1'+\frac{3305}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{4865}{16}\sigma_2'-220\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$+345A_4+\frac{275}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{1111}(0,0)=\Big\{-\frac{855}{16}-\frac{1155}{8}\left(\frac{e^2}{1-\mu}\right)+\frac{105}{16}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{105}{16}e^2-\frac{12705}{32}\sigma_1+\frac{105}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{12705}{32}\sigma_2-\frac{105}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{1155}{64}\sigma_1'+\frac{32235}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{735}{64}\sigma_2'$$
$$-\frac{1155}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{1715}{4}A_4-\frac{105}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{1112}(0,0)=\frac{\sqrt{3}}{2}\Big\{\frac{315}{8}+\frac{4655}{8}\left(\frac{e^2}{1-\mu}\right)+560\left(\frac{\mu e^2}{1-\mu}\right)-560e^2+\frac{51205}{32}\sigma_1+\frac{71365}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{51205}{32}\sigma_2-\frac{71365}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-1540\sigma_1'-\frac{68215}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+980\sigma_2'$$
$$+\frac{4655}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{36785}{16}A_4-\frac{12655}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{1122}(0,0)=\Big\{\frac{1395}{16}+\frac{3247}{24}\left(\frac{e^2}{1-\mu}\right)-\frac{11219}{48}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{11219}{48}e^2+\frac{35717}{96}\sigma_1$$
$$-\frac{11219}{24}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{35717}{96}\sigma_2+\frac{11219}{24}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{21941}{24}\sigma_1'$$
$$+\frac{27131}{192}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{78533}{192}\sigma_2'+\frac{3247}{24}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{8103}{4}A_4+\frac{11203}{48}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{1222}(0,0)=\frac{\sqrt{3}}{2}\Big\{-\frac{1575}{8}-\frac{20475}{8}\left(\frac{e^2}{1-\mu}\right)-\frac{2205}{2}\left(\frac{\mu e^2}{1-\mu}\right)+\frac{2205}{2}e^2-\frac{225225}{32}\sigma_1$$
$$-2205\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{225225}{32}\sigma_2+2205\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{24255}{8}\sigma_1'+\frac{45675}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'$$
$$-\frac{16065}{32}\sigma_2'-\frac{20475}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+945A_4+1575\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$\beta_{2222}(0,0)=\Big\{\frac{1665}{16}+\frac{8085}{8}\left(\frac{e^2}{1-\mu}\right)+\frac{14385}{16}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{14385}{16}e^2+\frac{88935}{32}\sigma_1+\frac{14385}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{88935}{32}\sigma_2-\frac{14385}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{158235}{64}\sigma_1'-\frac{40845}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{100695}{64}\sigma_2'$$
$$+\frac{8085}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{7595}{4}A_4+\frac{735}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\}.$$

The values of $a(q_1,q_2)$, at the equilibrium points are evaluated as:

$$a(0,0)=\Big\{1+\frac{25}{12}\frac{e^2}{1-\mu}+\frac{5}{12}\frac{\mu e^2}{1-\mu}-\frac{5}{12}e^2+\frac{275}{48}\sigma_1+\frac{5}{6}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{275}{48}\sigma_2-\frac{5}{6}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2$$
$$-\frac{55}{48}\sigma_1'-\frac{5}{12}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{35}{48}\sigma_2'+\frac{25}{4}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{5}{3}A_4-\frac{25}{12}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_1(0,0)=\Big\{-\frac{5}{2}-\frac{115}{24}\frac{e^2}{1-\mu}+\frac{25}{24}\frac{\mu e^2}{1-\mu}-\frac{25}{24}e^2-\frac{1265}{96}\sigma_1-\frac{35}{12}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{1265}{96}\sigma_2$$
$$+\frac{35}{12}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{275}{96}\sigma_1'-\frac{25}{24}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{175}{96}\sigma_2'-\frac{155}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{25}{6}A_4$$
$$+\frac{115}{24}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_2(0,0)=-\frac{5\sqrt{3}}{2}\Big\{1+\frac{113}{36}\frac{e^2}{1-\mu}+\frac{29}{36}\frac{\mu e^2}{1-\mu}-\frac{29}{36}e^2+\frac{1067}{144}\sigma_1+\frac{29}{18}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{1067}{144}\sigma_2$$
$$-\frac{29}{18}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{143}{48}\sigma_1'-\frac{11}{12}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{91}{144}\sigma_2'+\frac{307}{36}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{29}{9}A_4$$
$$-\frac{97}{36}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_{11}(0,0)=\Big\{\frac{15}{4}+\frac{35}{48}\frac{e^2}{1-\mu}-\frac{665}{48}\frac{\mu e^2}{1-\mu}+\frac{665}{48}e^2+\frac{385}{192}\sigma_1-\frac{665}{24}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{385}{192}\sigma_2$$
$$+\frac{665}{24}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{7315}{192}\sigma_1'+\frac{665}{48}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{4655}{192}\sigma_2'+\frac{595}{16}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$+\frac{665}{12}A_4-\frac{35}{48}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_{12}(0,0)=\frac{35\sqrt{3}}{2}\Big\{\frac{1}{2}+\frac{29}{24}\frac{e^2}{1-\mu}+\frac{1}{24}\frac{\mu e^2}{1-\mu}-\frac{1}{24}e^2+\frac{319}{96}\sigma_1+\frac{1}{12}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{319}{96}\sigma_2$$
$$-\frac{1}{12}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{11}{96}\sigma_1'-\frac{37}{24}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{7}{96}\sigma_2'+\frac{119}{24}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{1}{6}A_4-\frac{29}{24}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_{22}(0,0)=\left\{\frac{85}{4}+\frac{1435}{16}\frac{e^2}{1-\mu}+\frac{511}{16}\frac{\mu e^2}{1-\mu}-\frac{511}{16}e^2+\frac{15785}{64}\sigma_1+\frac{511}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$-\frac{15785}{64}\sigma_2-\frac{511}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{5621}{64}\sigma_1'-\frac{1253}{48}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{3577}{64}\sigma_2'$$
$$\left.+\frac{14035}{48}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{511}{4}A_4-\frac{5425}{48}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$a_{111}(0,0)=\left\{\frac{105}{8}+\frac{2625}{32}\frac{e^2}{1-\mu}+\frac{3465}{32}\frac{\mu e^2}{1-\mu}-\frac{3465}{32}e^2+\frac{3675}{128}\sigma_1+\frac{2205}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$-\frac{3675}{128}\sigma_2-\frac{2205}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{945}{128}\sigma_1'-\frac{2205}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{15435}{128}\sigma_2'$$
$$\left.+\frac{3675}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{2205}{8}A_4-\frac{2625}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$a_{112}(0,0)=\frac{35\sqrt{3}}{2}\left\{-\frac{5}{4}-\frac{79}{48}\frac{e^2}{1-\mu}+\frac{133}{48}\frac{\mu e^2}{1-\mu}-\frac{133}{48}e^2-\frac{869}{192}\sigma_1+\frac{133}{24}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{869}{192}\sigma_2-\frac{133}{24}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{1463}{192}\sigma_1'-\frac{133}{48}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{931}{192}\sigma_2'-\frac{709}{48}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$\left.-\frac{133}{12}A_4+\frac{79}{48}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$a_{122}(0,0)=\left\{-\frac{805}{8}-\frac{10325}{32}\frac{e^2}{1-\mu}-\frac{1505}{32}\frac{\mu e^2}{1-\mu}+\frac{1505}{32}e^2-\frac{113575}{128}\sigma_1-\frac{1505}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{113575}{128}\sigma_2+\frac{1505}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{16555}{128}\sigma_1'+\frac{1505}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{10535}{128}\sigma_2'$$
$$\left.-\frac{42875}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{1505}{8}A_4+\frac{8645}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$a_{222}(0,0)=\frac{\sqrt{3}}{2}\left\{-\frac{525}{4}-\frac{875}{3}\frac{e^2}{1-\mu}-\frac{13055}{48}\frac{\mu e^2}{1-\mu}+\frac{13055}{48}e^2+\frac{292985}{192}\sigma_1-\frac{13055}{24}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$-\frac{292985}{192}\sigma_2+\frac{13055}{24}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{143605}{192}\sigma_1'+\frac{4165}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{107065}{192}\sigma_2'$$
$$\left.-\frac{92785}{48}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{13055}{12}A_4+\frac{26635}{48}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$a_{1111}(0,0)=\left\{-\frac{2415}{16}-\frac{33285}{64}\frac{e^2}{1-\mu}-\frac{2625}{64}\frac{\mu e^2}{1-\mu}+\frac{2625}{64}e^2-\frac{2740815}{768}\sigma_1-\frac{2625}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$
$$+\frac{2740815}{768}\sigma_2+\frac{2625}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{28875}{256}\sigma_1'+\frac{2625}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{82845}{768}\sigma_2'$$
$$\left.-\frac{109935}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{2625}{16}A_4+\frac{33285}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\right\},$$

$$a_{1112}(0,0)=\frac{35\sqrt{3}}{2}\left\{-\frac{9}{8}-\frac{579}{32}\frac{e^2}{1-\mu}-\frac{735}{32}\frac{\mu e^2}{1-\mu}+\frac{735}{32}e^2+\frac{1551}{128}\sigma_1-\frac{735}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1\right.$$

$$-\frac{1551}{128}\sigma_2+\frac{735}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{3795}{128}\sigma_1'+\frac{747}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{5145}{128}\sigma_2'$$
$$-\frac{579}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{735}{8}A_4+\frac{579}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_{1122}(0,0)=\Big\{\frac{5915}{16}+\frac{204371}{192}\frac{e^2}{1-\mu}-\frac{134441}{192}\frac{\mu e^2}{1-\mu}+\frac{134441}{192}e^2+\frac{2248093}{768}\sigma_1$$
$$-\frac{134441}{96}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{2248093}{768}\sigma_2+\frac{134441}{96}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{105061}{48}\sigma_1'$$
$$+\frac{134473}{192}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1031087}{768}\sigma_2'+\frac{204371}{192}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{134441}{48}A_4$$
$$-\frac{102251}{96}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_{1222}(0,0)=\frac{\sqrt{3}}{2}\Big\{\frac{6615}{8}+\frac{106785}{32}\frac{e^2}{1-\mu}+\frac{27405}{32}\frac{\mu e^2}{1-\mu}-\frac{27405}{32}e^2-\frac{3523905}{384}\sigma_1$$
$$+\frac{27405}{16}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{3523905}{384}\sigma_2-\frac{27405}{16}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{301455}{128}\sigma_1'$$
$$+\frac{28035}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{575505}{384}\sigma_2'+\frac{106785}{32}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{27405}{8}A_4-\frac{106785}{32}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$a_{2222}(0,0)=\Big\{\frac{10185}{16}+\frac{214515}{64}\frac{e^2}{1-\mu}+\frac{127575}{64}\frac{\mu e^2}{1-\mu}-\frac{127575}{64}e^2+\frac{2359665}{256}\sigma_1$$
$$+\frac{127575}{32}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{2359665}{256}\sigma_2-\frac{127575}{32}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{1569645}{256}\sigma_1'$$
$$-\frac{147735}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{1457505}{256}\sigma_2'+\frac{214515}{64}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{127575}{16}A_4-\frac{214515}{64}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\}.$$

The values of $b(q_1,q_2)$, at the equilibrium points are evaluated as:

$$b(0,0)=\Big\{1+\frac{10}{3}\frac{e^2}{1-\mu}+\frac{5}{3}\frac{\mu e^2}{1-\mu}-\frac{5}{3}e^2+\frac{55}{6}\sigma_1+\frac{10}{3}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{55}{6}\sigma_2-\frac{10}{3}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2$$
$$-\frac{55}{12}\sigma_1'-\frac{25}{12}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{35}{12}\sigma_2'+\frac{10}{3}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{20}{3}A_4+\frac{5}{3}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$b_1(0,0)=\Big\{\frac{5}{2}+\frac{85}{6}\frac{e^2}{1-\mu}+\frac{25}{3}\frac{\mu e^2}{1-\mu}-\frac{25}{3}e^2+\frac{935}{24}\sigma_1+\frac{50}{3}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{935}{24}\sigma_2$$
$$-\frac{50}{3}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{275}{12}\sigma_1'-\frac{295}{48}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{175}{12}\sigma_2'+\frac{85}{6}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{100}{3}A_4$$
$$+\frac{10}{3}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$b_2(0,0)=-\frac{5\sqrt{3}}{2}\Big\{1+\frac{40}{9}\frac{e^2}{1-\mu}+\frac{23}{9}\frac{\mu e^2}{1-\mu}-\frac{23}{9}e^2+\frac{110}{9}\sigma_1+\frac{46}{9}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{110}{9}\sigma_2$$
$$-\frac{46}{9}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{253}{36}\sigma_1'-\frac{121}{72}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{161}{36}\sigma_2'+\frac{40}{9}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{92}{9}A_4$$

$$+\frac{19}{9}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$b_{11}(0,0)=\bigg\{\frac{15}{4}-\frac{35}{6}\frac{e^2}{1-\mu}-\frac{245}{12}\frac{\mu e^2}{1-\mu}+\frac{245}{12}e^2-\frac{385}{24}\sigma_1+\frac{35}{3}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{385}{24}\sigma_2$$
$$-\frac{35}{3}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{2695}{48}\sigma_1'+\frac{805}{48}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{1715}{48}\sigma_2'-\frac{35}{6}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{1085}{9}A_4$$
$$+\frac{245}{12}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$b_{12}(0,0)=\frac{35\sqrt{3}}{2}\bigg\{-\frac{1}{2}-\frac{61}{18}\frac{e^2}{1-\mu}-\frac{19}{9}\frac{\mu e^2}{1-\mu}+\frac{19}{9}e^2-\frac{517}{72}\sigma_1-\frac{38}{9}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{517}{72}\sigma_2$$
$$+\frac{38}{9}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{209}{36}\sigma_1'+\frac{311}{72}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{133}{36}\sigma_2'-\frac{61}{18}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{76}{9}A_4$$
$$+\frac{19}{9}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$b_{22}(0,0)=\bigg\{\frac{85}{4}+70\frac{e^2}{1-\mu}+\frac{315}{4}\frac{\mu e^2}{1-\mu}-\frac{315}{4}e^2+\frac{2695}{8}\sigma_1+\frac{1015}{6}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1-\frac{2695}{8}\sigma_2$$
$$-\frac{1015}{6}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{3465}{16}\sigma_1'-\frac{1435}{16}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{2205}{16}\sigma_2'+\frac{245}{2}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'$$
$$-315A_4-\frac{1085}{12}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$b_{111}(0,0)=\bigg\{-\frac{105}{8}-\frac{1575}{8}\frac{e^2}{1-\mu}-\frac{735}{4}\frac{\mu e^2}{1-\mu}+\frac{735}{4}e^2-\frac{17325}{32}\sigma_1-\frac{735}{2}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{17325}{32}\sigma_2+\frac{735}{2}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{8085}{16}\sigma_1'+\frac{5985}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{5145}{16}\sigma_2'$$
$$-\frac{1015}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+700A_4+\frac{735}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$b_{112}(0,0)=\frac{\sqrt{3}}{2}\bigg\{-\frac{175}{4}-\frac{385}{9}\frac{e^2}{1-\mu}-\frac{3185}{36}\frac{\mu e^2}{1-\mu}+\frac{3185}{36}e^2-\frac{4235}{36}\sigma_1+\frac{4375}{18}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{4235}{36}\sigma_2-\frac{4375}{18}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{48125}{144}\sigma_1'-\frac{10885}{144}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{833}{48}\sigma_2'$$
$$-\frac{385}{48}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{8785}{9}A_4-\frac{4375}{36}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$b_{122}(0,0)=\bigg\{\frac{805}{8}+\frac{3395}{4}\frac{e^2}{1-\mu}+\frac{4585}{8}\frac{\mu e^2}{1-\mu}-\frac{4585}{8}e^2+\frac{37345}{16}\sigma_1+\frac{4585}{4}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{37345}{16}\sigma_2-\frac{4585}{4}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{50435}{32}\sigma_1'-\frac{20545}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{32095}{32}\sigma_2'$$
$$+\frac{1645}{2}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{2555}{2}A_4-\frac{4585}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\bigg\},$$

$$b_{222}(0,0)=\frac{\sqrt{3}}{2}\Big\{-\frac{525}{4}-\frac{2905}{3}\frac{e^2}{1-\mu}-\frac{8225}{12}\frac{\mu e^2}{1-\mu}+\frac{8225}{12}e^2-\frac{31955}{12}\sigma_1-\frac{8225}{6}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{31955}{12}\sigma_2+\frac{8225}{6}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{90475}{48}\sigma_1'+\frac{39515}{48}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{57575}{48}\sigma_2'$$
$$-\frac{1960}{3}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{3815}{3}A_4+\frac{4445}{12}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$b_{1111}(0,0)=\Big\{-\frac{2415}{16}-\frac{6405}{8}\frac{e^2}{1-\mu}-\frac{5145}{16}\frac{\mu e^2}{1-\mu}+\frac{5145}{16}e^2-\frac{70455}{32}\sigma_1-\frac{5145}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$+\frac{70455}{32}\sigma_2+\frac{5145}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{56595}{64}\sigma_1'+\frac{139125}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{36015}{64}\sigma_2'$$
$$-\frac{6405}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{35}{4}A_4+\frac{5145}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$b_{1112}(0,0)=\frac{\sqrt{3}}{2}\Big\{\frac{315}{8}+\frac{11795}{8}\frac{e^2}{1-\mu}+\frac{6545}{4}\frac{\mu e^2}{1-\mu}-\frac{6545}{4}e^2+\frac{129745}{32}\sigma_1+\frac{6545}{2}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{129745}{32}\sigma_2-\frac{6545}{2}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{71995}{16}\sigma_1'-\frac{89705}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{45815}{16}\sigma_2'$$
$$+\frac{11795}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{120505}{16}A_4-\frac{27335}{8}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$b_{1122}(0,0)=\Big\{\frac{5915}{16}+\frac{8505}{8}\frac{e^2}{1-\mu}-\frac{5355}{16}\frac{\mu e^2}{1-\mu}+\frac{5355}{16}e^2+\frac{93555}{32}\sigma_1-\frac{5355}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{93555}{32}\sigma_2+\frac{5355}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{58905}{64}\sigma_1'-\frac{945}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{37485}{64}\sigma_2'$$
$$+\frac{8505}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'+\frac{5355}{4}A_4+\frac{19215}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$b_{1222}(0,0)=\frac{\sqrt{3}}{2}\Big\{-\frac{6615}{8}-\frac{62895}{8}\frac{e^2}{1-\mu}-\frac{21945}{4}\frac{\mu e^2}{1-\mu}+\frac{21945}{4}e^2-\frac{691845}{32}\sigma_1$$
$$-\frac{21945}{2}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1+\frac{619845}{32}\sigma_2+\frac{21945}{2}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2+\frac{241395}{16}\sigma_1'$$
$$+\frac{151305}{32}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'-\frac{153615}{16}\sigma_2'-\frac{62895}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{6545}{2}A_4+\frac{1155}{4}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\},$$

$$b_{2222}(0,0)=\Big\{\frac{10185}{16}+\frac{50715}{8}\frac{e^2}{1-\mu}+\frac{69615}{16}\frac{\mu e^2}{1-\mu}-\frac{69615}{16}e^2+\frac{557865}{32}\sigma_1+\frac{69615}{8}\left(\frac{1-3\mu}{2\mu}\right)\sigma_1$$
$$-\frac{557865}{32}\sigma_2-\frac{69615}{8}\left(\frac{1-2\mu}{2\mu}\right)\sigma_2-\frac{765765}{64}\sigma_1'-\frac{335475}{64}\left(\frac{5\mu}{1-\mu}\right)\sigma_1'+\frac{487305}{64}\sigma_2'$$
$$+\frac{50715}{8}\left(\frac{3\mu}{1-\mu}\right)\sigma_2'-\frac{69615}{4}A_4-\frac{44415}{16}\left(\frac{5\mu}{1-\mu}\right)A_4\Big\}.$$

Equation (2.13) can be expressed in the following form:

$$H_2=\frac{1}{2}(p_1^2+p_2^2)+(p_1q_2-p_2q_1)+\frac{e\cos v}{2(1+e\cos v)}(q_1^2+q_2^2)-\frac{1}{1+e\cos v}(q_1^2A-q_1q_2B-q_2^2C), \tag{2.21}$$

where

$$A=\left\{\frac{1}{8}+\frac{19}{32}\left(\frac{e^2}{1-\mu}\right)-\frac{19}{32}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{29}{32}e^2+\frac{21}{16}\mu e^2+\frac{107}{128}\sigma_1+\frac{23}{32\mu}\sigma_1-\frac{171}{32}\mu\sigma_1-\frac{247}{128}\sigma_2-\frac{23}{32\mu}\sigma_2\right.$$
$$\left.+\frac{177\mu\sigma_2}{32}-\frac{301}{128}\sigma_1'+\frac{171}{32}\mu\sigma_1'+\frac{185}{128}\sigma_2'-\frac{135}{32}\mu\sigma_2'-\frac{1}{4}A_4-\frac{41}{16}\mu A_4\right\},$$
$$B=\sqrt{3}\left\{\frac{3}{4}+\frac{31}{52}\left(\frac{e^2}{1-\mu}\right)-\frac{2091}{1872}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{91}{104}e^2-\frac{2431}{624}\mu e^2-\frac{3}{2}\mu-\frac{1187}{192}\sigma_1-\frac{3}{16\mu}\sigma_1+\frac{1063}{96}\mu\sigma_1\right.$$
$$\left.-\frac{1759}{192}\sigma_2+\frac{13}{48\mu}\sigma_2-\frac{821\mu\sigma_2}{96}-\frac{289}{192}\sigma_1'+\frac{4159}{192}\mu\sigma_1'+\frac{125}{192}\sigma_2'-\frac{3923}{192}\mu\sigma_2'+\frac{37}{3}A_4+\frac{973}{24}\mu A_4\right\},$$
$$C=\left\{\frac{5}{8}+\frac{73}{72}\left(\frac{e^2}{1-\mu}\right)-\frac{323}{288}\left(\frac{\mu e^2}{1-\mu}\right)-\frac{241}{144}e^2+\frac{563}{288}\mu e^2-\frac{1401}{128}\sigma_1+\frac{27}{32\mu}\sigma_1+\frac{2215}{512}\mu\sigma_1+\frac{1305}{128}\sigma_2\right.$$
$$\left.-\frac{27}{32\mu}\sigma_2-\frac{2743\mu\sigma_2}{512}-\frac{537}{128}\sigma_1'-\frac{1104}{512}\mu\sigma_1'+\frac{309}{128}\sigma_2'+\frac{201}{32}\mu\sigma_2'-\frac{585}{32}A_4+\frac{297}{32}\mu A_4\right\}. \tag{2.22}$$

3. Stability of Triangular Equilibrium Point in Circular Case

The second order Hamiltonian of the problem is given by (2.21) becomes the Hamiltonian for the circular problem upto 2nd order terms given by:

$$H=\frac{1}{2}(p_1^2+p_2^2)+(p_1q_2-p_2q_1)+(q_1^2A-q_1q_2B-q_2^2C). \tag{3.1}$$

The variational equation for the circular problem can be written as :

$$\dot{p}_i=-\frac{\partial H_2}{\partial q_i}$$

and

$$\dot{q}_i=\frac{\partial H_2}{\partial p_i},$$

where H_2 is given by (3.1), where $(i=1,2)$. Hence, we have:

$$\ddot{q}_1-2\dot{q}_2=(1-2A)q_1+Bq_2,$$
$$\ddot{q}_2+2\dot{q}_1=Bq_1+(1+2C)q_2. \tag{3.2}$$

Now, for further investigation of motion, using the transformations,

$$q_1=Le^{\lambda t},$$
$$q_2=Me^{\lambda t},$$

where L, M and λ are parameters. Substituting, the above transformation in (3.2), the characteristics equation is obtained as:

$$\begin{vmatrix}\lambda^2-(1-2A) & -(2\lambda+B)\\ (2\lambda-B) & \lambda^2-(1+2C)\end{vmatrix}=0. \tag{3.3}$$

Solving, the above equation by substituting the values of A, B and C from (2.22), we have

$$\lambda^4+\left\{1-\frac{4951}{256}\mu\sigma_1+\frac{4159}{256}\mu\sigma_2-\frac{379}{16}\mu A_4+\frac{309}{32}\mu\sigma_1'-21\mu\sigma_2'\right\}\lambda^2$$

$$-\frac{27}{4}\mu(1-\mu)\left\{1+\frac{1063}{72}\sigma_1-\frac{821}{72}\sigma_2+\frac{973}{18}A_4-\frac{1967}{96}\sigma_1'-\frac{329}{32}\sigma_2'\right\}=0. \tag{3.4}$$

Let us consider,

$$\lambda_1 = i\omega_1,$$

$$\lambda_2 = i\omega_2.$$

The equation (3.4) can be written as:

$$\omega^4+\left\{1-\frac{4951}{256}\mu\sigma_1+\frac{4159}{256}\mu\sigma_2-\frac{379}{16}\mu A_4+\frac{309}{32}\mu\sigma_1'-21\mu\sigma_2'\right\}\omega^2$$
$$-\frac{27}{4}\mu(1-\mu)\left\{1+\frac{1063}{72}\sigma_1-\frac{821}{72}\sigma_2+\frac{973}{18}A_4-\frac{1967}{96}\sigma_1'-\frac{329}{32}\sigma_2'\right\}=0. \tag{3.5}$$

Thus,

$$\omega_{1,2}^2=\frac{1}{2}\Bigg[\left\{1-\frac{4951}{256}\mu\sigma_1+\frac{4159}{256}\mu\sigma_2-\frac{379}{16}\mu A_4+\frac{309}{32}\mu\sigma_1'-21\mu\sigma_2'\right\}\pm$$
$$\left\{1-\frac{4951}{128}\mu\sigma_1+\frac{4159}{128}\mu\sigma_2-\frac{379}{8}\mu A_4+\frac{309}{16}\mu\sigma_1'-42\mu\sigma_2'\right\}$$
$$-27\mu(1-\mu)\left\{1+\frac{1063}{72}\sigma_1-\frac{821}{72}\sigma_2+\frac{973}{18}A_4-\frac{1967}{96}\sigma_1'-\frac{329}{32}\sigma_2'\right\}\Bigg]^{\frac{1}{2}}$$

i.e.,

$$\omega_1=\Bigg\{\frac{1}{2}\Bigg[\left\{1-\frac{4951}{256}\mu\sigma_1+\frac{4159}{256}\mu\sigma_2-\frac{379}{16}\mu A_4+\frac{309}{32}\mu\sigma_1'-21\mu\sigma_2'\right\}$$
$$+\left\{1-\frac{4951}{128}\mu\sigma_1+\frac{4159}{128}\mu\sigma_2-\frac{379}{8}\mu A_4+\frac{309}{16}\mu\sigma_1'-42\mu\sigma_2'\right\}$$
$$-27\mu(1-\mu)\left\{1+\frac{1063}{72}\sigma_1-\frac{821}{72}\sigma_2+\frac{973}{18}A_4-\frac{1967}{96}\sigma_1'-\frac{329}{32}\sigma_2'\right\}\Bigg]^{\frac{1}{2}}\Bigg\}^{\frac{1}{2}}, \tag{3.6}$$

$$\omega_2=\Bigg\{\frac{1}{2}\Bigg[\left\{1-\frac{4951}{256}\mu\sigma_1+\frac{4159}{256}\mu\sigma_2-\frac{379}{16}\mu A_4+\frac{309}{32}\mu\sigma_1'-21\mu\sigma_2'\right\}$$
$$-\left\{1-\frac{4951}{128}\mu\sigma_1+\frac{4159}{128}\mu\sigma_2-\frac{379}{8}\mu A_4+\frac{309}{16}\mu\sigma_1'-42\mu\sigma_2'\right\}$$
$$-27\mu(1-\mu)\left\{1+\frac{1063}{72}\sigma_1-\frac{821}{72}\sigma_2+\frac{973}{18}A_4-\frac{1967}{96}\sigma_1'-\frac{329}{32}\sigma_2'\right\}\Bigg]^{\frac{1}{2}}\Bigg\}^{\frac{1}{2}}. \tag{3.7}$$

The equilibrium position is stable if $\omega_{1,2}$ are purely imaginary and hence, bifurcation occurs, Hence,

$$\Bigg[\left\{1-\frac{4951}{128}\mu\sigma_1+\frac{4159}{128}\mu\sigma_2-\frac{379}{8}\mu A_4+\frac{309}{16}\mu\sigma_1'-42\mu\sigma_2'\right\}$$
$$-27\mu(1-\mu)\left\{1+\frac{1063}{72}\sigma_1-\frac{821}{72}\sigma_2+\frac{973}{18}A_4-\frac{1967}{96}\sigma_1'-\frac{329}{32}\sigma_2'\right\}\Bigg]\geq 0. \tag{3.8}$$

If the equality relation holds, then we have :

$$\Bigg[\left\{1-\frac{4951}{128}\mu\sigma_1+\frac{4159}{128}\mu\sigma_2-\frac{379}{8}\mu A_4+\frac{309}{16}\mu\sigma_1'-42\mu\sigma_2'\right\}$$

$$-27\mu(1-\mu)\left\{1+\frac{1063}{72}\sigma_1-\frac{821}{72}\sigma_2+\frac{973}{18}A_4-\frac{1967}{96}\sigma_1'-\frac{329}{32}\sigma_2'\right\}\Bigg]=0. \tag{3.9}$$

Hence, the value of μ is given by:

$$\mu=\frac{1}{2}\Bigg[\left\{1+\frac{4951}{128}\mu\sigma_1-\frac{4159}{128}\mu\sigma_2+\frac{379}{8}\mu A_4-\frac{309}{16}\mu\sigma_1'+42\mu\sigma_2'\right\}$$
$$\pm\sqrt{\frac{23}{27}}\left\{1+\frac{2888973}{2944}\sigma_1-\frac{2240325}{2944}\sigma_2+\frac{617409}{184}A_4-\frac{121581}{92}\sigma_1'-\frac{221697}{368}\sigma_2'\right\}\Bigg]. \tag{3.10}$$

Since, $\mu \le \frac{1}{2}$, the positive sign is inadmissible. Hence, the region of stability in the first approximation can be written as:

$$0<\mu<\frac{1}{2}\Bigg[\left\{1+\frac{4951}{128}\mu\sigma_1-\frac{4159}{128}\mu\sigma_2+\frac{379}{8}\mu A_4-\frac{309}{16}\mu\sigma_1'+42\mu\sigma_2'\right\}$$
$$-\sqrt{\frac{23}{27}}\left\{1+\frac{2888973}{2944}\sigma_1-\frac{2240325}{2944}\sigma_2+\frac{617409}{184}A_4-\frac{121581}{92}\sigma_1'-\frac{221697}{368}\sigma_2'\right\}\Bigg]. \tag{3.11}$$

Thus, the value of μresponsible for stable equilibrium points is given by:

$$\mu_{\text{critical}}=0.038520-433.51364\sigma_1+334.9302\sigma_2-1524.798A_4+600.2034\sigma_1'+299.0123\sigma_2'. \tag{3.12}$$

Also, we have

$$\omega_1(\mu_c)=\omega_2(\mu_c)=\frac{1}{\sqrt{2}}$$

and

$$\omega_1(0)=1,$$
$$\omega_2(0)=0.$$

Thus, the parametric resonance is possible in the neighbourhood of the value of μ for whichthe frequencies ω_1 and ω_2 given in equation (3.6) and equation (3.7) satisfy at least one of the following relations:

$$\omega_1=\frac{N}{2},$$
$$\omega_2=\frac{N}{2},$$
$$\omega_1-\omega_2=N.$$

The dependence of ω_1 and ω_2 on μ is given in the graph. In the region equation (3.11), the only resonance $\omega_2=\frac{1}{2}$ is possible. The corresponding value of μ for $\omega_2=\frac{1}{2}$ is given by:

$$\mu_0=0.0285959+1.779061\sigma_1+0.2812080\sigma_2-1.51279A_4-1.1319865\sigma_1'+0.373684\sigma_2'. \tag{3.13}$$

The equation(3.13) gives the boundary of the stability region when the orbit is circular. The result is same as given in Kamel [7], when $\delta=0$, $\sigma_1=0$, $\sigma_2=0$. The region of stability and instability are plotted in the graphs with the help of MATLAB.

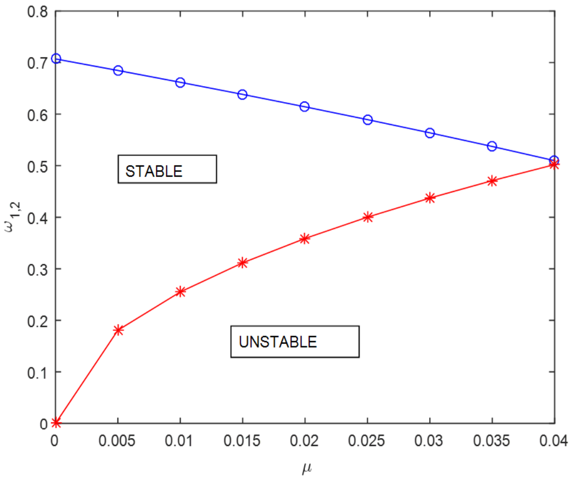

Figure 1. Correlation between μ and $\omega_{1,2}$ for $A_4 = 0.0$; $\sigma_1 = 0.0001$; $\sigma_2 = 0.0005$; $\sigma'_1 = 0.0001$; $\sigma'_2 = 0.0002$

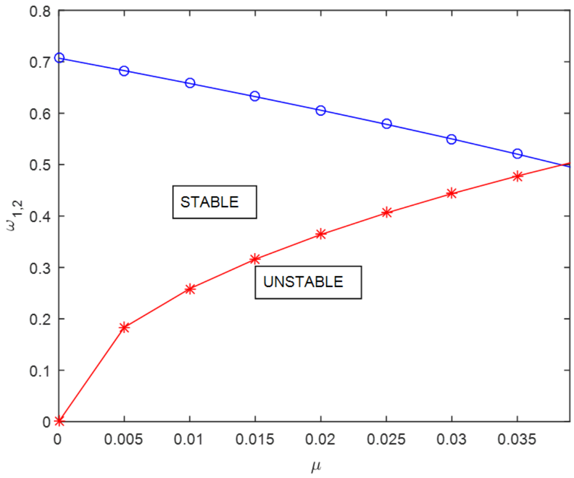

Figure 2. Correlation between μ and $\omega_{1,2}$ for $A_4 = 0.00001$; $\sigma_1 = 0.00003$; $\sigma_2 = 0.00001$; $\sigma'_1 = 0.00002$; $\sigma'_2 = 0.00001$

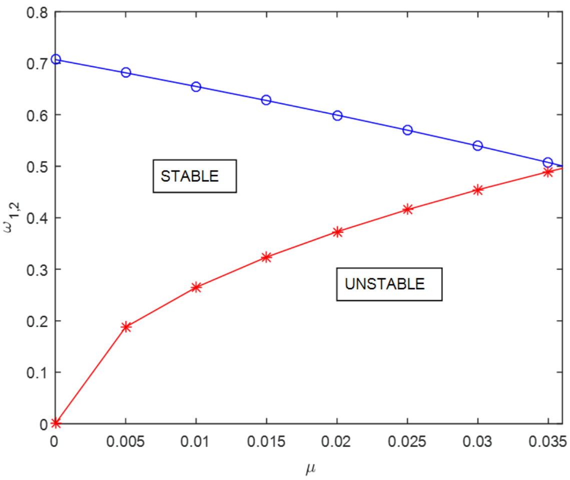

Figure 3. Correlation between μ and $\omega_{1,2}$ for $A_4 = 0.005$; $\sigma_1 = 0$; $\sigma_2 = 0$; $\sigma'_1 = 0.007$; $\sigma'_2 = 0.003$

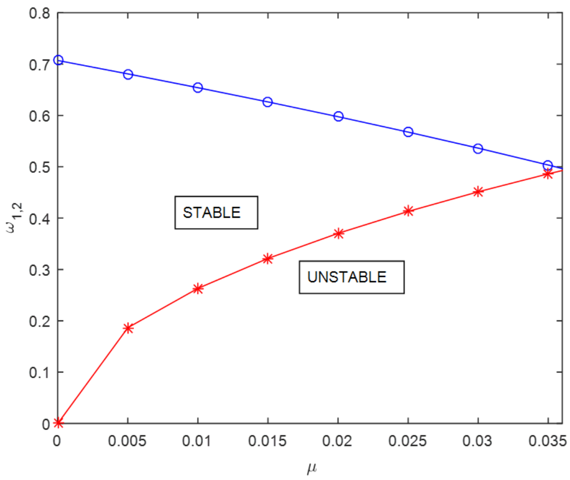

Figure 4. $A_4 = 0.0005$; $\sigma_1 = 0.0006$; $\sigma_2 = 0.0004$; $\sigma'_1 = 0.0$; $\sigma'_2 = 0.0$

4. Stability of Triangular Equilibrium Points in Elliptical Case

The Hamiltonian of the system is given by (2.21). For small values of e, expanding the function H_2in powers of e, up to first approximation, we have:

$$\begin{aligned} H_2 &= \frac{1}{2}(p_1^2+p_2^2)+(p_1q_2-p_2q_1) \\ &\quad + e\cos v\left\{\left(\frac{1}{2}-A\right)q_1^2+Bq_1q_2+\left(\frac{1}{2}+C\right)q_2^2\right\}+(q_1^2A-q_1q_2B-q_2^2C). \end{aligned} \tag{4.1}$$

Now using the Canonical transformation,

$$[q_1,q_2,p_1,p_2]=[q'_1,q'_2,p'_1,p'_2]N, \tag{4.2}$$

where

$$N = \begin{bmatrix} a_1 & a_1c_1 & -a_1c_1 & a_1(1-\omega_1^2 b_1) \\ a_2 & a_2c_2 & -a_2c_2 & a_2(1-\omega_2^2 b_2) \\ 0 & a_1b_1 & a_1(1-b_1) & a_1c_1 \\ 0 & -a_2b_2 & a_2(1-b_2) & -a_2c_2 \end{bmatrix} \tag{4.3}$$

and

$$a_i = \frac{1}{2}\sqrt{\frac{2l_i}{|\omega_i^2 - \frac{1}{2}|}},$$
$$b_i = \frac{2}{l_i},$$
$$c_i = \frac{-B}{l_i},$$
$$l_i = \omega_i^2 + 2C + 1, \tag{4.4}$$

where $i = 1,2$ and A, B, C are same as given in equation (2.22) and ω_i are given by equations (3.6) and (3.7).

Now rewriting equation (2.22) as:

$$H_2 = H_2^{(0)} + H_2^{(1)},$$

where $H_2^{(0)}$ is the Hamiltonian independent of eccentricity and $H_2^{(1)}$is the Hamiltonian containing first order approximation in e. Hence, we have:

$$H_2^{(0)} = \frac{1}{2}(p_1^2 + p_2^2) + (p_1q_2 - p_2q_1) + e\cos v\{Aq_1^2 - Bq_1q_2 - Cq_2^2\},$$
$$H_2^{(1)} = e\cos v\left\{\left(\frac{1}{2} - A\right)q_1^2 + Bq_1q_2 + \left(\frac{1}{2} + C\right)q_2^2\right\}.$$

Now using the transforming given in the equation (4.2), equation (4.3) and equation (4.4) in the above equation, we have:

$$H_2 = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) + (p_1q_2 - p_2q_1) + e\cos v\{aq_2^2 + bp_2^2 + cq_2^2p_2^2 + \ldots\}, \tag{4.5}$$

where

$$a = Aa_2^2 + Ba_2^2c_2 + Ca_2^2c_2;$$
$$b = Ca_2^2b_2;$$
$$c = -Ba_2^2b_2 - Ca_2^2b_2c_2. \tag{4.6}$$

Here, dots denote the second order terms in p_1' and q_1', which are not taken into consideration for further calculations. Now taking the transformations of variables:

$$q_i' = (-1)^{i+1}\frac{\sqrt{2\alpha_i}}{\omega_i}\sin\omega_i(v - (-1)^{i-1}\beta_i),$$
$$p_i' = \sqrt{2\alpha_i}\cos\omega_i(v - (-1)^{i-1}\beta_i) \tag{4.7}$$

and

$$\omega_2 = \frac{1}{2} + \varepsilon, \quad |\varepsilon| \ll 1. \tag{4.8}$$

The terms containing eccentricity are considered as perturbations in the Hamiltonian. So, using the above transformation of variables, the perturbation in the Hamiltonian is given by:

$$H_2^{(1)} = \left[e\cos v \frac{2a\alpha_2}{\omega_2^2} \sin^2 \omega_2(v-\beta_2) + 2b\alpha_2 \cos^2 \omega_2(v-\beta_2) - \frac{c\alpha_2}{\omega_2} \sin^2 \omega_2(v-\beta_2) \right].$$

Now, averaging the terms with finite frequencies between 0 to 2π, in the above equation, we have:

$$H_2^{(1)} = e\left[\left\{ \left(\frac{b-4a}{2} \right) \cos(2\varepsilon v - 2\omega_2\beta_2) + c\sin(2\varepsilon v - 2\omega_2\beta_2) \right\} \alpha_2 \right]. \tag{4.9}$$

Now, let,

$$U = \left(\frac{b-4a}{2} \right),$$

$$V = c.$$

Substituting, the above values and simplifying the equation (4.9), we have:

$$H_2^{(1)} = e[\{U\cos\omega_2(-2\varepsilon v + \beta_2) - V\sin\omega_2(-2\varepsilon v + \beta_2)\}\alpha_2]. \tag{4.10}$$

Now in order to eliminate v, from equation (4.10) using canonical transformations:

$$\bar{\alpha}_1 = \alpha_1, \quad \bar{\beta}_1 = \beta_1, \quad \bar{\alpha}_2 = \alpha_2, \quad \bar{\beta}_2 = \beta_2 - 2\varepsilon v. \tag{4.11}$$

Hence, the non-periodic part of the perturbation is given as:

$$\bar{H}_2 = e[\{U\cos 2\omega_2\bar{\beta}_2 - V\sin 2\omega_2\bar{\beta}_2\}\alpha_2 - 2\varepsilon\alpha_2]. \tag{4.12}$$

Now, the equation(4.12) is independent of v. Hence, its integral is:

$$\bar{H}_2 = h_1 = \text{constant} \tag{4.13}$$

So, using equation (4.13) in equation (4.12), we have:

$$e[\{U\cos 2\omega_2\bar{\beta}_2 - V\sin 2\omega_2\bar{\beta}_2\}\alpha_2 - 2\varepsilon\alpha_2] = h_1. \tag{4.14}$$

Now, taking,

$$\cos\theta = \frac{U}{(U^2+V^2)^{\frac{1}{2}}},$$

$$\sin\theta = \frac{V}{(U^2+V^2)^{\frac{1}{2}}}. \tag{4.15}$$

Then the above equation (4.14) can be simplified as:

$$\cos(2\omega_2\bar{\beta}_2 + \theta) = \frac{h + \frac{2\varepsilon}{e(U^2+V^2)^{\frac{1}{2}}}\bar{\alpha}_2}{\bar{\alpha}_2}, \tag{4.16}$$

where

$$h = \frac{h_1}{e(U^2+V^2)^{\frac{1}{2}}}.$$

Satisfying the condition:

$$\left|\frac{2\varepsilon}{e(U^2+V^2)^{\frac{1}{2}}}\right| < 1,$$

i.e.,

$$|\varepsilon| < \frac{1}{2}e(U^2+V^2)^{\frac{1}{2}}. \tag{4.17}$$

The inequality given by the equation (4.17) determines the region of parametric resonance in the $\mu - e$ plane in the neighbourhood of the point μ_0 corresponding to $\omega_2 = \frac{1}{2}$. The boundary of the region in the first approximation, accurate upto $O(e)^2$, is given by:

$$\mu_0 - eA_1 < \mu < \mu_0 + eA_1, \tag{4.18}$$

where

$$A_1 = \left.\frac{(U^2+V^2)^{\frac{1}{2}}}{2\frac{d\omega_2(\mu)}{d\mu}}\right|_{\mu=\mu_0}. \tag{4.19}$$

Now, substituting all the required values in the above equation, the boundary of the region given by the equation (4.18) of parametric resonance and local bifurcation about $\omega_2 = \frac{1}{2}$ in the first approximation in e has the form

$$\mu = 0.038520 - 433.51364\sigma_1 + 334.9302\sigma_2 - 1524.798A_4 + 600.2034\sigma_1' + 299.0123\sigma_2' \pm e. \tag{4.20}$$

The region of stability and instability are plotted in the graphs with the help of MATLAB. When the orbit is taken elliptical the boundary region is by the equation (4.20).

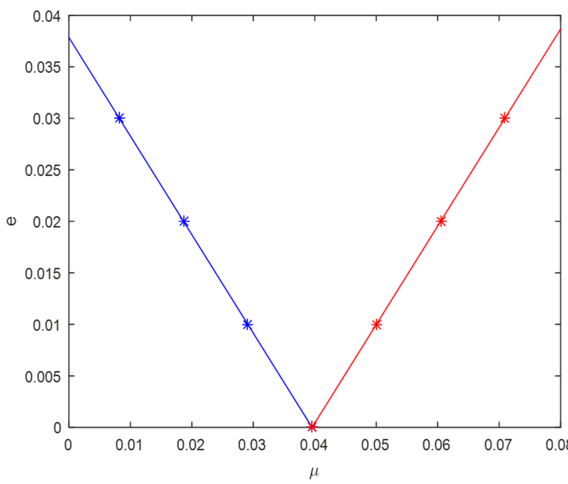

Figure 5. Correlation between μ and e for $A_4 = 0.0002$; $\sigma_1 = 0.0003$; $\sigma_2 = 0.0002$; $\sigma_1' = 0.0004$; $\sigma_2' = 0.0002$

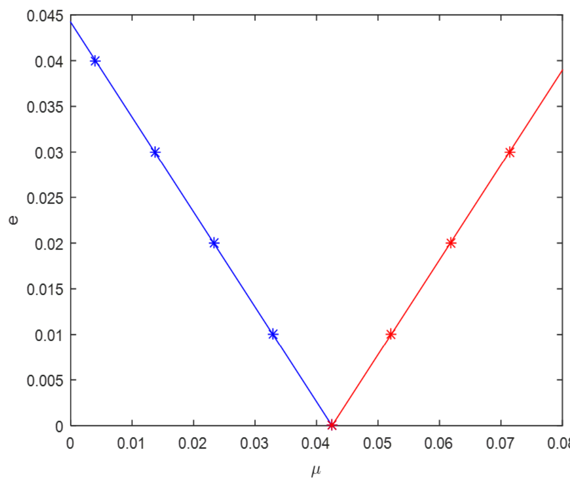

Figure 6. Correlation between μ and e for $A_4 = 0.0$; $\sigma_1 = 0.0003$; $\sigma_2 = 0.0002$; $\sigma_1' = 0.0004$; $\sigma_2' = 0.0002$

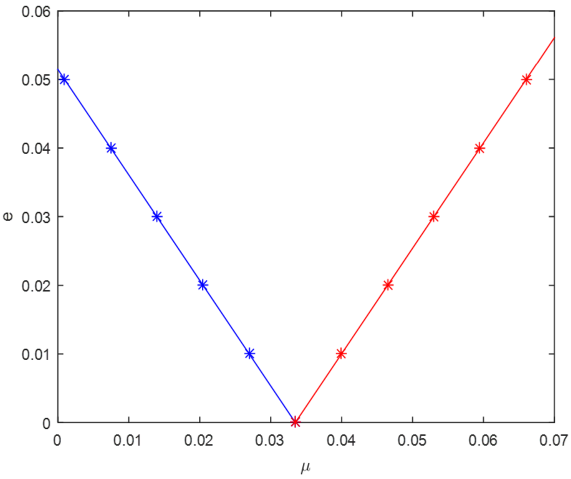

Figure 7. Correlation between μ and e for $A_4 = 0.0$; $\sigma_1 = 0$; $\sigma_2 = 0$; $\sigma'_1 = 0.0003$; $\sigma'_2 = 0.0001$

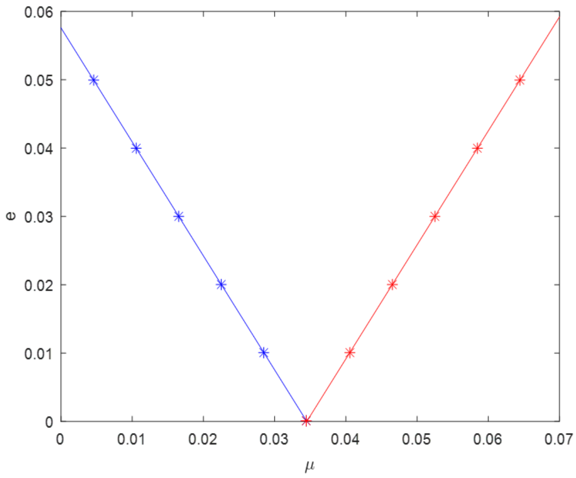

Figure 8. Correlation between μ and e for $A_4 = 0.0004$; $\sigma_1 = 0.0005$; $\sigma_2 = 0.0001$; $\sigma'_1 = 0$; $\sigma'_2 = 0$

5. Conclusion

The stability of oblateness infinitesimal moving around the triangular equilibrium points when both the primaries are triaxial under the Elliptical restricted three body problem has been discussed. We have constructed a suitable normalized convergent Hamiltonian function and investigated the stability of infinitesimal around the triangular equilibrium points and the perturbed system analytically and numerically due to Triaxiality of primaries in circular case up to the second order terms. The region of stability and instability of the linear problem for the value of eccentricity $e = 0$ has been analyzed using simulation technique of the problem. The region of stability of the linear problem in μ and $\omega_{1,2}$ plane has been clearly marked as shown in Figures 1-4. As observed from the figure the stable region decreases with increase in the value of triaxial factor. We conclude that the effect of the oblateness of infinitesimal and triaxiality of primaries affects the location and resonance stability of triangular equilibrium points of the elliptical restricted three-body problem in the particular case, at and near the resonance frequency

The method used by Khasan [6] has been adopted to investigate the stability of infinitesimal around the triangular equilibrium points. We have investigated the stability of infinitesimal mass in the model of Elliptical restricted three body problem at and near the resonance frequency $\omega_2 = 1/2$. The generalized component of moments for Hamiltonian function of the perturbed system is expanded up to second order. We have established a relation in $\mu - e$ plane for determining the range of stability using simulation techniques with the help of MATLAB-15. The stable and unstable region for stability of the linear problem $\mu - e$ has been plotted as shown in Figures 5-8.The region of stability decreases very slightly with the increase in the value of the parameters: oblateness and triaxiality.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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