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Mixed Type Second Order Symmetric Duality in Multiobjective Programming

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Abstract. A pair of mixed type multiobjective second-order symmetric dual program is formulated. Weak, strong and converse duality theorems are validated under bonvexity-boncavity and pseudobonvexity-pseudoboncavity of the Kernel function appearing in the primal and dual programs. Under additional conditions on the Kernel functional constituting the objective and constraint functions, these programs are shown to be self dual. This formulation of the programs not only generalizes mixed type first order symmetric multiobjective duality results but also unifies the pair of Wolfe and Mond-Weir type second order symmetric multiobjective programs.

1. **Introduction**

Duality theory has played an important role in development of optimization theory. Inception of duality theory in linear programming may be traced to classical minmax theory of Von Neumann [16] and was first explicitly given by Gale, Kuhn and Tucker [8] Duality results have proved to be useful in the growth of numerical algorithms for solving certain classes of optimization problems. For nonlinear programming problem, the existence of duality theory helps to develop numerical algorithms, as it provides suitable stopping rule for primal and dual problems. Applications of duality are prominent in physics, management science, economics and engineering.

Following Dorn [7], first order symmetric and self duality results in mathematical programming have been derived by a number of authors, notably, Dantzig *et al.* [6], Mond [15], and Bazaraa and Goode [1]. In these aforecited references, the authors have studied first order symmetric duality under the assumptions of convexity-concavity of the kernel function involved. Mond and Cottle [13] gave self duality by re-examining the programs of [15]. Later Mond and Weir [14] presented different pair of symmetric dual nonlinear program

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with a view to relax convexity-concavity of kernel functions to pseudoconvexity-pseudoconcavity.

Mangasarian [11] was the first to identify second order formulation the nonlinear programs under the assumptions that are complicated and somewhat difficult to verify. Mond [12] introduced the concept of second order convex functions named as bonvex functions by Bector and Chandra [2] and establish second order symmetric duality results under the assumptions of second order convexity on functions involved in the primal program. In [12] Mond has essentially remarked that the study of second order duality is important due to its computational advantage over first order duality as it provided tighter bound for the value of the objective function when approximations are used. Subsequently, Bector and Chandra [3] presented a pair of Mond-Weir type [14] second order symmetric and self dual programs in order to relax the bonvexity-boncavity conditions on the kernel function, considered in Mond [15] to pseudoconvexity-pseudoboncavity.

Chandra, Husain and Abha [5] presented a new symmetric dual formulation (called mixed symmetric dual formulation) for a class of nonlinear programming problem and derived various duality results. Their mixed formulation unifies the Wolfe [18] and Mond Weir type [14] symmetric dual formulations respectively, incorporated by Dantzig *et al.* [6] and Mond-Weir [14].

Recently Suneja *et al.* [17] studied Mond-Weir type second order symmetric duality in multiobjective programming by establishing usual duality theorems under η-bonvexity and η-boncavity assumptions. They also proved self duality theorems under skew symmetry of the kernel function that occur in the formulation of the problems. In [17] each component of the multiobjective dual models involves different auxiliary variables p_i and q_i, $i = 1, 2, \dots, k$, disagreeing with the formulation of second order dual model having single auxiliary variable p, presented by Mangasarian [11].

The purpose of this research is to present multiobjective version of the second order mixed symmetric and self duality in traditional mathematical programming with a single objective treated by Husain and Abha [9]. This formulation of the problems considers the same auxiliary variable p in the primal and the same auxiliary variable q in the dual, which is the conformity with the Mangasarian's [11] formulation. Obviously, our formulation unifies Wolfe and Mond-Weir type symmetric second order dual models which are not studied in the literature. In addition to validation of various duality theorems under suitable second order convexity/generalized second order convexity, an attempt is also made to identify self duality for this pair of programs under additional restrictions on the kernel functions involved.

2. **Pre-requisites and Definitions**

Let R^n denoted the n-dimensional Euclidean space. The following ordering relations in R^n are recalled for our use. If $x, y \in R^n$, then

$$x < y \Leftrightarrow x_i < y_i, \quad (i = 1, 2, \ldots, n)$$
$$x \leqq y \Leftrightarrow x_i \leq y_i, \quad (i = 1, 2, \ldots, n)$$
$$x \leq y \Leftrightarrow x_i \leq y_i, \quad (i = 1, 2, \ldots, n), \text{ but } \ x \neq y$$
$$x \nleq y \text{ is the negation of } \ x \leq y$$

For $x, y \in R$, $x \leq y$ and $x < y$ have the usual meaning.

Let $\phi(x, y)$ be twice differentiable real-valued function defined on $R^n \times R^n$. Let $\nabla_x \phi(\bar{x}, \bar{y})$ and $\nabla_y \phi(\bar{x}, \bar{y})$ denote the gradient vectors with respect to x and y, respectively evaluated at $(\bar{x}, \bar{y})$. Also let $\nabla_x^2 \phi(\bar{x}, \bar{y})$ and $\nabla_y^2 \phi(\bar{x}, \bar{y})$ debits the Hessian matrix of second order partial derivatives of ϕ with respect to x and y, respectively evaluated at $(\bar{x}, \bar{y})$. The symbols $\nabla_{xx} \phi(\bar{x}, \bar{y})$ and $\nabla_{yy} \phi(\bar{x}, \bar{y})$ are similarly defined. The symbols $\nabla_y(\nabla_x^2 \phi(\bar{x}, \bar{y})q)$ and $\nabla_x(\nabla_y^2 \phi(\bar{x}, \bar{y})p)$ denote the matrices whose (i, j)th elements are respectively given as $\frac{\partial}{\partial y_i}(\nabla_x^2 \phi(\bar{x}, \bar{y})q)_j$, with $q \in R^n$ and $\frac{\partial}{\partial x_i}(\nabla_y^2 \phi(\bar{x}, \bar{y})p)_j$ with $p \in R^m$.

Definition 2.1. The function ϕ us said to be bonvex in first variable x at $u \in R^m$, if for all $v \in R^n$, $q \in R^n$, $x \in R^n$ and for fixed y,

$$\begin{aligned} &\phi(x, v) - \phi(u, v) \\ &\quad \geqq \ (x - u)^T [\nabla_x \phi(u, v) + \nabla_x^2 (u, v) q] - \frac{1}{2} q^T \nabla_x^2 \phi(u, v) q\,. \end{aligned}$$

and $\phi(x, y)$ is used to be boncave in the second variable y at v, if for all $u \in R^m$, $p_i \in R^m$, $y \in R^m$ and for fixed $x \in R^n$,

$$\begin{aligned} &\phi(x, v) - f(x, y) \\ &\quad \leqq \ (v - y)^T [\nabla_y \phi(x, y) + \nabla_y^2 \phi(x, y) p] - \frac{1}{2} p^T \nabla_y^2 \phi(x, y) p\,. \end{aligned}$$

Definition 2.2. The function ϕ is said to be pseudobonvex in the first variable x at $u \in R^n$, if for all $v \in R^n, q_i \in R^n$ and $x \in R^n$ and for fixed y,

$$\begin{aligned} & (x - u)^T \left[\nabla_x \phi(u, v) + \nabla_x^2 \phi(u, v) q \right] \geqq 0 \\ \Rightarrow \qquad & \phi(x, v) \geqq \phi(u, v) - \frac{1}{2} q^T \nabla_x^2 \phi(u, v) q \end{aligned}$$

and ϕ us said to be pseudoboncave in the second variable y at $v \in R^n$, if for all $u \in R^m$, $p \in R^m$ and $y \in R^m$ and for fixed $x \in R^n$

$$\begin{aligned} & (v - y)^T [\nabla_y \phi(x, y) + \nabla_y^2 \phi(x, y) p] \leqq 0 \\ \Rightarrow \qquad & \phi(x, v) \leqq \phi(x, y) - \frac{1}{2} p^T \nabla_y^2 \phi(x, y) p\,. \end{aligned}$$

Definition 2.3. The function ϕ is said to be skew symmetric, when both x and $y \in R^n$, and

$$\phi(x,y) = -\phi(y,x),$$

for all in the domain of ϕ.

Consider the following usual unconstrained multiobjective optimization programs;

(VP) Minimize $\psi(x) = (\psi^1(x), \psi^2(x), \ldots, \psi^k(x))$

subject to $x \in X = \{x \in R^n \mid h(x) \leqq 0\}$

where $\psi : R^n \to R^k$ and $h : R^n \to R^m$.

Definition 2.4. (a) A point $\bar{x} \in X$ is said to be an efficient solution of (VP) if there exists no other $x \in X$ such that $\psi(x) \leq \psi(\bar{x})$.

(b) A point $\bar{x} \in X$ is said to be properly efficient solution of (VP) if it is efficient and if there exists a scalar $M > 0$ such that for each $i \in \{1,2,\ldots,k\}$ and $x \in X$ satisfying $\psi^1(x) < \psi^1(\bar{x})$

$$\frac{\psi^1(\bar{x}) - \psi^1(x)}{\psi^1(x) - \psi^1(\bar{x})} \leq M,$$

for some j such that $\psi^1(x) > \psi^1(\bar{x})$.

(c) An efficient point $\bar{x} \in X$ that is not properly efficient is said to be improperly efficient. Then $\bar{x}$ is improperly efficient means that every scale $M > 0$ (no matter how large), then point $x \in X$ and an i such that $\psi^1(x) < \psi^1(\bar{x})$ and

$$\frac{\psi^1(\bar{x}) - \psi^1(x)}{\psi^1(x) - \psi^1(\bar{x})} > M,$$

for all j satisfying $\psi^1(x) > \psi^1(\bar{x})$.

(d) A point $\bar{x} \in X$ is said to be weak minimum of (VP) if there does not exists any feasible point x such that $\phi(\bar{x}) > \phi(x)$.

Any efficient solution of (VP) is obviously a weak minimum of (VP).

3. **Mixed Type Second Order Multi-objective Duality**

For $N = \{1,2,\ldots,n\}$ and $M = \{1,2,\ldots,m\}$, let $J_1 \subseteq N$ and $K_1 \subseteq M$ and $J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in the subset J_1. The other symbols $|J_2|$, $|K_1|$ and $|K_2|$ are defined similarly. Let $x^1 \in R^{|J_1|}$ and $x^2 \in R^{|J_2|}$, then any $x \in R$ can be written as $x = (x^1, x^2)$. Similarly for $y^1 \in R^{|K_1|}$ and $y^2 \in R^{|K_2|}$, can be written as $y = (y^1, y^2)$. Let $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ be twice differentiable functions. It is to be noticed here that if J_1 is an empty set, the $J_2 = N$, $|J_1| = 0$ and $|J_2| = n$. Then $R^{|J_1|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be the zero-dimensional and $|K_1|$-dimensional vectors respectively. Similarly we can describe the cases K_1 an empty set, K_2 an empty set and J_2, as an empty set.

We now introduce the following pair of nonlinear programs and study its second order symmetric duality by the following theorems:

Primal Problem:

$$
\begin{aligned}
\text{(SMP):}\ \ \text{Minimize}\ \ & F(x^1,x^2,y^1,y^2,p,r) \\
& \quad = (F_1(x^1,x^2,y^1,y^2,p,r),\dots,F_k(x^1,x^2,y^1,y^2,p,r)) \\
\text{subject to}\ \ & \nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p \leqq 0, && (3.1)\\
& \nabla_{y^2}(\lambda^T g)(x^2,y^2)+\nabla^2_{y^2}(\lambda^T g)(x^2,y^2)r \leqq 0, && (3.2)\\
& (y^2)^T[\nabla_{y^2}(\lambda^T g)(x^2,y^2)+\nabla^2_{y^2}(\lambda^T g)(x^2,y^2)r] \geqq 0, && (3.3)\\
& x^1,x^2 \geqq 0, && (3.4)\\
& \lambda\in\Lambda^+. && (3.5)
\end{aligned}
$$

Dual Problem:

$$
\begin{aligned}
\text{(SMD):}\ \ \text{Maximize}\ \ & G(u^1,u^2,v^1,v^2,q,s) \\
& \quad = (G_1(u^1,u^2,v^1,v^2,q,s),\dots,G_k(u^1,u^2,v^1,v^2,q,s)) \\
\text{subject to}\ \ & \nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q \geqq 0, && (3.6)\\
& \nabla_{x^2}(\lambda^T g)(u^2,v^2)+\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)s \geqq 0, && (3.7)\\
& (u^2)^T[\nabla_{x^2}(\lambda^T g)(u^2,v^2)+\nabla^2_{x^2}(\lambda^T g)(u^2,v^2)s] \leqq 0, && (3.8)\\
& x^1,x^2 \geqq 0, && (3.9)\\
& \lambda\in\Lambda^+. && (3.10)
\end{aligned}
$$

where

(i) $$
\begin{aligned}
F_i(x^1,x^2,y^1,y^2,p,r) = {} & f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1}f_i(x^1,y^1)p \\
& -(y^1)^T\{\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p\} \\
& +g_i(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}g_i(x^2,y^2)r
\end{aligned}
$$

(ii) $$
\begin{aligned}
G_i(u^1,u^2,v^1,v^2,q,s) = {} & f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1}f_i(u^1,v^1)q \\
& -(u^1)^T\{\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\} \\
& +g_i(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}g_1(u^2,v^2)s
\end{aligned}
$$

(iii) $p\in R^{|K_1|}$, $r\in R^{|K_2|}$, $q\in R^{|J_1|}$, $s\in R^{|J_2|}$, and $\lambda=(\lambda_1,\dots,\lambda_k)^T$ with $\lambda_1\in R$, $i=1,2,\dots,k$.

(iv) $\Lambda^+=\left\{\lambda\in R^k|\lambda>0,\sum_{i=1}^{k}\lambda_1\right\}$

Theorem 3.1 (**Weak duality**). *For* $(x^1, x^2, y^1, y^2, \lambda, p, r)$ *be feasible for* (SMP) *and* $(u^1, u^2, v^1, v^2, \lambda, q, s)$ *feasible for* (SMD), *let*

(i) *for each* $i \in \{1, 2, \ldots, k\}$; $f_1(\cdot, y^1)$ *be bonvex at* u^1 *for fixed* y^1 *and* $f_i(x^1, \cdot)$ *be boncave at* y^1 *for fixed* x^1, *and*
(ii) $\lambda^T g(\cdot, y^2)$ *be pseudoconvex at* u^2 *for fixed* y^2, *and* $\lambda^T g(x^2, \cdot)$ *be pseudoboncave at* y^2 *for fixed* x^2.

Then $F(x^1, x^2, y^1, y^2, p, r) \leq G(u^1, u^2, v^1, v^2, q, s)$.

Proof. By the convexity-boncavity of f_i, $i \in \{1, 2, \ldots, k\}$,

$$f_i(x^1, v^1) - f_i(u^1, v^1) \geqq (x^1 - u^1)^T [\nabla_{x^1} f_i(u^1, v^1) + \nabla^2_{x^1} f_i(u^1, v^1) q] - \frac{1}{2} q^T \nabla^2_{x^1} f_i(u^1, v^1) q \tag{3.11}$$

and

$$f_i(x^1, v^1) - f_i(x^1, y^1)^T \geqq (v^1 - y^1)[\nabla_{y^1} f_i(x^1, y^1) + \nabla^2_{y^1} f_i(x^1, y^1) p] - \frac{1}{2} p^T \nabla^2_{y^1} f_i(x^1, y^1) p\,. \tag{3.12}$$

Multiplying (3.12) by (-1) and adding resulting inequality to (3.11), we have

$$\begin{aligned}
&f_i(x^1, y^1) - \frac{1}{2} p^T \nabla^2_{y^1} f_i(x^1, y^1) p - (y^1)^T \{\nabla_{y^1}(\lambda^T f)(x^1, y^1) + \nabla^2_{y^1}(\lambda^T f)(x^1, y^1) p\} \\
&\left(\frac{\pi}{2} - \theta\right) - \left[f_i(u^1,v^1) - \frac{1}{2} q^T \nabla^2_{x^1} f_i(u^1,v^1) q - (u^1)^T \{\nabla_{x^1} f_i(u^1,v^1) + \nabla^2_{x^1} f_i(u^1,v^1) q\}\right] \\
&\quad \geqq (x^1)^T \{\nabla^1_{x^1} f_i(u^1,v^1) + \nabla^2_{x^1} f_i(u^1,v^1) q\} - (v^1)^T \{\nabla_{y^1} f_i(x^1,y^1) + \nabla^2_{y^1} f_i(x^1,y^1) p\}
\end{aligned}$$

Using (3.5) and (3.10), this inequality becomes,

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i \bigg[f_i(x^1, y^1) - \frac{1}{2} p^T \nabla^2_{y^1} f_i(x^1, y^1) p \\
&\qquad - (y^1)^T \{\nabla_{y^1}(\lambda^T f)(x^1, y^1) + \nabla^2_{y^1}(\lambda^T f)(x^1, y^1) p\} \bigg] \\
&- \sum_{i=1}^{k} \lambda_i \bigg[f_i(u^1, v^1) - \frac{1}{2} q^T \nabla^2_{x^1} f_i(u^1, v^1) q \\
&\qquad - (u^1)^T \{\nabla_{x^1}(\lambda^T f)(u^1, v^1) + \nabla^2_{x^1}(\lambda^T f)(u^1, v^1) q\} \bigg] \\
&\qquad\quad \geqq (x^1)^T \{\nabla_{x^1}(\lambda^T f)(u^1, v^1) + \nabla^2_{x^1}(\lambda^T f)(u^1, v^1) q\} \\
&\qquad\qquad - (v^1)^T \{\nabla_{y^1}(\lambda^T f)(x^1, y^1) + \nabla^2_{y^1}(\lambda^T f)(x^1, y^1) p\}.
\end{aligned}$$

This, in view of (3.6) with (3.4), and (3.1) with (3.9), yields,

$$\sum_{i=1}^{k}\lambda_i\Bigg[f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1}f_i(x^1,y^1)p$$
$$-(y^1)^T\{\nabla_{y^1}(\lambda^Tf)(x^1,y^1)+\nabla^2_{y^1}(\lambda^Tf)_i(x^1,y^1)p\}\Bigg]$$
$$\geqq\sum_{i=1}^{k}\lambda_i\Bigg[f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1}f_i(u^1,v^1)q$$
$$-(u^1)^T\{\nabla_{x^1}(\lambda^Tf)(u^1,v^1)+\nabla^2_{x^1}(\lambda^Tf)(u^1,v^1)q\}\Bigg]. \quad (3.13)$$

From (3.4), (3.7) and (3.8), we have

$$(x^2-u^2)^T[\nabla_{x^2}(\lambda^Tg)(u^2,v^2)+\nabla^2_{x^2}(\lambda^Tg)(u^2,v^2)s]\geqq 0.$$

Also from (3.9), (3.2) and (3.3), we have

$$(v^2-y^2)^T[\nabla_{y^2}(\lambda^Tg)(x^2,y^2)+\nabla^2_{y^2}(\lambda^Tg)(x^2,y^2)r]\leqq 0.$$

By pseudobonvexity of $\lambda^Tg(\cdot,y^2)$ at u^2, we have

$$\lambda^Tg(x^2,v^2)\geqq\left[(\lambda^Tg)(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}(\lambda^Tg)(u^2,v^2)s\right] \quad (3.14)$$

and by pseudoboncavity $\lambda T_g(\cdot,y^2)$ at y^2, we have,

$$\lambda^Tg(x^2,v^2)\leqq\lambda^Tg(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}(\lambda^Tg)(x^2,y^2)r\,. \quad (3.15)$$

From (3.14) and (3.15), we have

$$\lambda^Tg(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}(\lambda^Tg)(x^2,y^2)r$$
$$\geqq\lambda^Tg(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{x^2}(\lambda^Tg)(x^2,v^2)s \quad (3.16)$$

Combing (3.13) and (3.16), we have

$$\sum_{i=1}^{k}\lambda_i\Bigg[f_i(x^1,y^1)-\frac{1}{2}p^T\nabla^2_{y^1}f_i(x^1,y^1)p$$
$$-(y^1)^T\{\nabla_{y^1}(\lambda^Tf)(x^1,y^1)+\nabla^2_{y^1}(\lambda^Tf)(x^1,y^1)p\}$$
$$+g_i(x^2,y^2)-\frac{1}{2}r^T\nabla^2_{y^2}\lambda^Tg_i(x^2,y^2)r\Bigg]$$
$$\geqq\sum_{i=1}^{k}\lambda_i\Bigg[f_i(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1}f_i(u^1,v^1)q$$

$$- (u^1)^T \{\nabla_{x^1}(\lambda^T f)(u^1,v^1) + \nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q\} \\ + \lambda^T g_i(u^2,v^2) - \frac{1}{2} s^T \nabla^2_{x^2} g_i(u^2,v^2)s \Big]$$

or

$$\sum_{i=1}^{k} \lambda_i F_i(x^1,x^2,y^1,y^2,p,r) \geqq \sum_{i=1}^{k} \lambda_i G_i(u^1,u^2,v^1,v^2,q,s)$$

or

$$\lambda^T F(x^1,x^2,y^1,y^2,p,r) \geqq \lambda^T G(u^1,u^2,v^1,v^2,q,s).$$

This implies

$$F(x^1,x^2,y^1,y^2,p,r) \nleq G(u^1,u^2,v^1,v^2,q,s). \qquad \square$$

Theorem 3.2 (**Strong duality**). *Let for each $i \in \{1,2,\dots,k\}$, f^i be thrice differentiable on $R^n \times R^m$. Let $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})$ be a properly efficient solution of (SMP); fix $\lambda = \bar{\lambda}$. Assume that*

(A_1): *the set $(\nabla^2_{y^1} f_1, \nabla^2_{y^1} f_2, \dots, \nabla^2_{y^1} f_k)$ is linearly independent,*

(A_2): *the set $(\nabla^2_{y^2} g_1, \nabla^2_{y^2} g_2, \dots, \nabla^2_{y^2} g_k)$ is linearly independent,*

(A_3): *both the Hessian matrices $\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda}^T f)p)$ and $\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda}^T g)r)$, are either positive or negative definite,*

(A_4): *the set $(\nabla_{y^2} g_1 + \nabla^2_{y^2} g_1 \bar{r}, \nabla_{y^2} g_2 + \nabla^2_{y^2} g_2 \bar{r}, \dots, \nabla_{y^2} g_k + \nabla^2_{y^2} g_k \bar{r})$ is linearly independent and*

(A_5): *the set $\{\nabla_{y^1} f_1 + \nabla^2_{y^1} f_1 \bar{p}, \nabla_{y^1} f_2 + \nabla^2_{y^1} f_2 \bar{p}, \dots, \nabla_{y^1} f_k + \nabla^2_{y^1} f_k \bar{p}\}$ is linearly independent.*

Where $f_1 = f_1(\bar{x}^1,\bar{y}^1)$, $g_1 = g_1(\bar{x}^1,\bar{y}^1)$, $i = 1,2,\dots,k$.
Then $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q}=0,\bar{s}=0)$ is feasible for (SMD) *and $F(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r}) = G(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{q},\bar{s})$.*

Proof. Since $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})$ is a properly efficient solution of (SMP), it is also a weak minimum. Hence there exist $\alpha \in R^k$, with $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_k)$ and $\mu \in R^k$, $\beta \in R^{|K_1|}$, $\theta \in R^{|K_2|}$, $\delta^1 \in R^{|J_1|}$, $\delta^2 \in R^{|J_2|}$ and $\eta \in R$ such that the following Fritz John optimality condition [12] are satisfied at $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda},\bar{p},\bar{r})$

$$\nabla_{x^1}(\alpha^T f) + (\beta - (\alpha^T e)\bar{y}^1)^T \nabla_{y^1 x^1}(\lambda^T f) \\ + \sum_{i=1}^{k} \left\{ (\beta - (\alpha^T e)\bar{y}^1)\bar{\lambda}_1 - \frac{\alpha_1 \bar{p}}{2} \right\} \nabla_{x^1}(\nabla^2_{y^1} f_1 \bar{p}) = \delta^1, \tag{3.17}$$

$$\nabla_{x^2}(\alpha^T g) + (\theta - \eta \bar{y}^2) \nabla_{y^2 x^2}(\lambda^T g) \\ + \sum_{i=1}^{k} \left\{ (\theta - \eta \bar{y})\bar{\lambda}_1 - \frac{\alpha_1 \bar{r}}{2} \right\} \nabla_{x^2}(\nabla^2_{y^2} g_1 \bar{r}) = \delta^2, \tag{3.18}$$

$$\nabla_{y^1}(\alpha-(\alpha^T e)\bar\lambda)^T f+\left(\beta-(\alpha^T e)\bar y^1-(\alpha^T e)\bar p\right)\nabla^2_{y^1}(\bar\lambda^T f)$$
$$+\sum_{i=1}^k\left\{(\beta-(\alpha^T e)\bar y^1)\bar\lambda_1-\frac{\alpha_1\bar p}{2}\right\}\nabla_{y^1}(\nabla^2_{y^1}f_1\bar p)=0\,, \tag{3.19}$$
$$\nabla_{y^2}(\alpha-\eta\bar\lambda)^T g+(\theta-\eta\bar y^2-\eta\bar r)^T\nabla^2_{y^2}(\bar\lambda^T g)$$
$$+\left\{(\theta-\eta\bar y^2)\bar\lambda_1-\frac{\alpha^T\bar r}{2}\right\}\nabla_{y^2}(\nabla^2_{y^2}g\bar r)=0\,, \tag{3.20}$$
$$\sum_{i=1}^k((\beta-(\alpha^T e)\bar y^1)\bar\lambda_1-\alpha_1\bar p)\nabla^2_{y^1}f_1=0\,, \tag{3.21}$$
$$\sum_{i=1}^k((\theta-\eta\bar y^2)\bar\lambda_{_1}-\alpha_1\bar r)\nabla^2_{y^2}g_1=0\,, \tag{3.22}$$
$$(\beta-(\alpha^T e)\bar y^1)^T[\nabla_{y^1}f_1+\nabla^2_{y^1}f\bar p]+(\theta-\eta\bar y^2)^T[\nabla_{y^2}g+\nabla^2_{y^2}g\bar r]+\mu=0\,, \tag{3.23}$$
$$\beta[\nabla_{y^1}(\bar\lambda^T f_1)+\nabla^2_{y^1}(\bar\lambda^T f)\bar p]=0\,, \tag{3.24}$$
$$\theta^T[\nabla_{y^2}(\bar\lambda^T g)+\nabla^2_{y^2}(\bar\lambda^T g)\bar r]=0\,, \tag{3.25}$$
$$\eta(\bar y^2)^T[\nabla_{y^2}(\bar\lambda^T g)+\nabla^2_{y^2}(\bar\lambda^T g)\bar r]=0\,, \tag{3.26}$$
$$\delta^1\bar x^1=0\,, \tag{3.27}$$
$$\delta^2\bar x^2=0\,, \tag{3.28}$$
$$\mu^T\bar\lambda=0\,, \tag{3.29}$$
$$(\alpha,\beta,\theta,\eta,\delta^1,\delta^2,\mu)\geq 0\,, \tag{3.30}$$
$$(\alpha,\beta,\theta,\eta,\delta^1,\delta^2,\bar\lambda,\mu)\neq 0\,. \tag{3.31}$$

Since $\bar\lambda>0$, from (3.29), we have

$$\mu=0\,. \tag{3.32}$$

From (3.21) along with the assumption (A_1) and (3.22) along with the assumption (A_2), we obtain

$$(\beta-(\alpha^T e)\bar y^1)\bar\lambda_1=\alpha_1\bar p,\quad i=1,2,\dots,k \tag{3.33}$$

and

$$(\theta-\eta\bar y^2)\bar\lambda_1=\alpha_1\bar r,\quad i=1,2,\dots,k. \tag{3.34}$$

Multiplying (3.23) by $\bar\lambda$ and using (3.29), we get

$$(\beta-(\alpha^T e)\bar y^1)^T(\nabla_{y^1}(\bar\lambda^T f)+\nabla^2_{y^1}(\bar\lambda^T f)\bar p)$$
$$+(\theta^T-\eta\bar y^2)[\nabla_{y^2}(\bar\lambda^T g)+\nabla^2_{y^2}(\bar\lambda^T g)\bar r]=0\,. \tag{3.35}$$

From (3.25) and (3.26), we have

$$(\theta^T - \eta\bar{y}^2)[\nabla_{y^2}(\bar{\lambda}^T g) + \nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r}] = 0 \tag{3.36}$$

i.e.,

$$(\theta^T - \eta\bar{y}^2)[\nabla_{y^2}(\eta\bar{\lambda})^T g + \nabla^2_{y^2}(\eta\bar{\lambda})^T g\bar{r}] = 0. \tag{3.37}$$

Using (3.36) i.e. in (3.35), we have

$$(\beta - (\alpha^T e)\bar{y}^1)^T(\nabla_{y^1}(\bar{\lambda}^T f) + \nabla_{y^1}(\bar{\lambda}^T f)\bar{p}) = 0 \tag{3.38}$$

i.e.,

$$(\beta - (\alpha^T e)\bar{y}^1)^T(\nabla_{y^1}((\alpha^T e)\bar{\lambda}^T f) + \nabla^2_{y^1}((\alpha^T e)\bar{\lambda}^T f) = 0. \tag{3.39}$$

Using (3.33) in (3.19) and (3.34) in (3.20), we obtain

$$(\alpha - (\alpha^T e)\bar{\lambda})^T(\nabla_{y^1} f + \nabla^2_{y^1} f\bar{p}) + \frac{1}{2}(\beta - (\alpha^T e)\bar{y}^1)^T\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda} f)\bar{p}) = 0, \tag{3.40}$$

$$(\alpha - \eta^T\bar{\lambda})^T(\nabla_{y^2} g + \nabla^2_{y^2} g\bar{r}) + \frac{1}{2}(\theta - \eta\bar{y}^2)\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda} g)\bar{r}) = 0. \tag{3.41}$$

On multiplying (3.40) by $(\beta - (\alpha^T e)^{-1} y)$ and (3.41) by $(\theta - \eta\bar{\lambda})^T$ and then adding, we obtain

$$\begin{aligned}
&(\beta - (\alpha^T e)\bar{y}^1)\{\nabla_{y^1}(\alpha - (\alpha^T e)\bar{\lambda})^T f + \nabla^2_{y^1}((\alpha - (\alpha^T e)\bar{\lambda})^T f\bar{p}\}\\
&(\theta - \eta^T\bar{y}^2)^T\{\nabla_{y^2}(\alpha - \eta^T\bar{\lambda})g + (\nabla^2_{y^2}(\alpha - \eta^T\bar{\lambda})g\bar{r}\}\\
&\quad + \frac{1}{2}(\beta - (\alpha^T e)\bar{y}^1)^T\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda}^T f)\bar{p}) + (\beta - (\alpha^T e)\bar{y}^1)\\
&\quad + \frac{1}{2}(\theta - \eta\bar{y}^2)^T\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r}) + (\theta - \eta\bar{y}^2) = 0.
\end{aligned} \tag{3.42}$$

Using (3.32) and then multiply (3.23) by α, we have

$$\begin{aligned}
&(\beta - (\alpha^T e)\bar{y}^1)\{\nabla_{y^1}(\alpha - (\alpha^T e)\bar{\lambda})^T f + \nabla^2_{y^1}((\alpha - (\alpha^T e)\bar{\lambda})^T f\bar{p}\}\\
&\quad + (\theta - \eta\bar{y}^2)^T\{\nabla_{y^2}(\alpha^T g) + \nabla^2_{y^2}(\alpha^T g)\bar{r}\} = 0.
\end{aligned}$$

Summing (3.37) and (3.39) from this inequality we have

$$\begin{aligned}
&(\beta - (\alpha^T e)\bar{y}^1)\{\nabla_{y^1}(\alpha - (\alpha^T e)\bar{\lambda})^T f + \nabla^2_{y^1}((\alpha - (\alpha^T e)\bar{\lambda})^T f\bar{p}\}\\
&\quad + (\theta - \eta^T\bar{y}^2)^T\{\nabla_{y^2}(\alpha - \eta^T\bar{\lambda})g + (\nabla^2_{y^2}(\alpha - \eta^T\bar{\lambda})g\bar{r}\} = 0.
\end{aligned} \tag{3.43}$$

Using (3.43) in (3.42), we have

$$\begin{aligned}
&(\beta - (\alpha^T e)\bar{y}^1)\nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda}^T f)\bar{p})^T(\beta - (\alpha^T e)\bar{y}^1)\\
&\quad + (\theta - \eta\bar{y}^2)^T\nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r})^T(\theta - \eta\bar{y}^2) = 0.
\end{aligned} \tag{3.44}$$

But by the assumption (A_3), we have

$$(\beta - (\alpha^T e)\bar{y}^1)^T \nabla_{y^1}(\nabla^2_{y^1}(\bar{\lambda} f)\bar{p})(\beta - (\alpha^T e)\bar{y}^1) = 0$$

and

$$(\theta - \eta \bar{y}^2)^T \nabla_{y^2}(\nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r})^T(\theta - \eta \bar{y}^2) = 0$$

which respectively gives

$$\beta - (\alpha^T e)\bar{y}^1 = 0 \tag{3.45}$$

and

$$\theta - \eta \bar{y}^2 = 0\,. \tag{3.46}$$

From (3.40) together with (3.45), we have

$$(\alpha - (\alpha^T e)\bar{\lambda}^1)^T(\nabla_{y^1} f + \nabla^2_{y^1} f \bar{p}) = 0$$

which because (A_5) gives

$$\alpha - (\alpha^T e)\bar{\lambda}^1 = 0\,. \tag{3.47}$$

The relation (3.41) together with (3.46) gives

$$(\alpha - \eta \bar{\lambda}^1)^T \{\nabla_{y^2} g + \nabla^2_{y^2} g \bar{r}\} = 0$$

which because of (A_4) implies

$$\alpha - \eta \bar{\lambda} = 0\,. \tag{3.48}$$

If possible, let $\eta = 0$. Then from (3.48), we have $\alpha = 0$ and from (3.45) and (3.46) we have $\theta = 0 = \beta$. From (3.17) and (3.18), we get $\delta^1 = 0$ and $\delta^2 = 0$. Contradicting (3.31).

Hence $\eta > 0$. From (3.48) we have $\alpha > 0$. From (3.45) and (3.46) we obtain

$$\bar{y}^1 \geqq 0, \quad \bar{y}^2 \geqq 0. \tag{3.49}$$

From (3.17) along with (3.33) and (3.47), we get

$$\nabla_{x^1}(\bar{\lambda}^T f) = \delta^1.$$

This along with (3.30) and (3.27), we obtain

$$\nabla_{x^1}(\bar{\lambda}^T f) \geq 0 \tag{3.50}$$

and

$$(\bar{x}^1)^T \nabla_{x^1}(\bar{\lambda}^T f) = 0. \tag{3.51}$$

From (3.18) along with (3.34) and (3.47), yields

$$\nabla_{x^2}(\bar{\lambda}^T g) = \delta^2.$$

This along with (3.30) and (3.28) yields

$$\nabla_{x^2}(\bar{\lambda}^T g) \geq 0 \tag{3.52}$$

and

$$(\bar{x}^2)^T \nabla_{x^2}(\bar{\lambda}^T g) = 0. \tag{3.53}$$

From (3.33) along with (3.45) and $\alpha_1 > 0$, and from (3.34) along with (3.46) $\alpha_1 > 0$, respectively, we have

$$\bar{p} = 0 = \bar{r}.$$

From (3.49), (3.50), (3.52) and (3.53), it implies that $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, q = 0, s = 0)$ is feasible for (SMD).

From (3.24) along with (3.45) and $\alpha > 0$ and (3.26) with $\eta > 0$, we have respectively,

$$(\bar{y}^1)^T(\nabla_{y^1}(\bar{\lambda}^T f) + \nabla^2_{y^1}(\bar{\lambda}^T f)\bar{p}) = 0 \tag{3.54}$$

and

$$(\bar{y}^2)^T(\nabla_{y^2}(\bar{\lambda}^T g) + \nabla^2_{y^2}(\bar{\lambda}^T g)\bar{r}) = 0\,. \tag{3.55}$$

Consider

$$\begin{aligned} F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) \;=\; & f_i(\bar{x}^1, \bar{y}^1) - \frac{1}{2}\bar{p}\nabla^2_{y^1} f_i(\bar{x}^1, \bar{y}^1)\bar{p} \\ & - (\bar{y}^1)^T\{\nabla_{y^1}(\bar{\lambda} f)^T(\bar{x}^1, \bar{y}^1) + \nabla^2_{y^1}(\bar{\lambda} f)^T(\bar{x}^1, \bar{y}^1)\bar{p}\} \\ & + g_i(\bar{x}^2, \bar{y}^2) - \frac{1}{2}\bar{r}_i^T \nabla^2_{y^2} g_i(\bar{x}^2, \bar{y}^2)\bar{r}\,. \end{aligned}$$

This, along with (3.54) and $\bar{p} = 0 = \bar{r}$, becomes

$$F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2), \quad i = 1, 2, \ldots, k. \tag{3.56}$$

Again consider,

$$\begin{aligned} G_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s}) \;=\; & f_i(\bar{x}^1, \bar{y}^1) - \frac{1}{2}\bar{q}\nabla^2_{x^1} f_i(\bar{x}^1, \bar{y}^1)\bar{q} \\ & - (\bar{x}^1)^T\{\nabla_{x^1}(\bar{\lambda} f)^T(\bar{x}^1, \bar{y}^1) + \nabla^2_{x^1}(\bar{\lambda} f)^T(\bar{x}^1, \bar{y}^1)\bar{q}\} \\ & + g_i(\bar{x}^2, \bar{y}^2) - \frac{1}{2}\bar{s}^T \nabla^2_{x^2} g_i(\bar{x}^2, \bar{y}^2)\bar{s}\,. \end{aligned}$$

This along with (3.51) and $\bar{q} = \bar{s} = 0$, becomes

$$G_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s}) = f_i(\bar{x}^1, \bar{y}^1) + g_i(\bar{x}^2, \bar{y}^2), \quad i = 1, 2, \ldots, k \tag{3.57}$$

From (3.56) and (3.57), we have

$$F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = G_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s}), \text{ for all } i \in \{1, 2, \ldots, k\}.$$

This implies

$$F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = G_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s}). \tag{3.58}$$

That is, the objective values of (SMP) and (SMD) are equal.

If $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ is not efficient, then there exists $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s})$ such that

$$G(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) \geq G(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$$

which because of (3.58) gives

$$G(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) \geq F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}).$$

This contradicts the weak duality (Theorem 3.1).

If $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ were improperly efficient, then for some feasible $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s})$ and some i

$$G_i(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) - F_i(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) > M$$

and so is

$$\bar{\lambda}^T G_i(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{\lambda}, \bar{q}, \bar{s}) > \bar{\lambda}^T F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}).$$

This again contradicts Theorem 3.1. Hence $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ is, indeed, a properly efficient solution of (SMD).

Theorem 3.3 (**Converse duality**). *Let for each $i \in \{1, 2, \ldots, k\}$, f_i be thrice differentiable on $R^n \times R^m$. Let $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$ be a properly efficient solution of* (SMD)*; fix $\lambda = \bar{\lambda}$ Assume that*

(C_1): *the set $\nabla^2_{x^1} f_1, \nabla^2_{x^1} f_2, \ldots, \nabla^2_{x^1} f_k$ is linearly independent,*
(C_2): *the set $\nabla^2_{x^2} g_1, \nabla^2_{x^2} g_2, \ldots, \nabla^2_{x^2} g_k$ is linearly independent,*
(C_3): *both the Hessian matrices $\nabla_{x^1}(\nabla^2_{x^1}(\lambda^T f)\bar{q})$ and $\nabla_{x^2}(\nabla^2_{x^2}(\lambda^T g)\bar{r})$ are either positive or negative definite,*
(C_4): *the set $\{\nabla_{x^1} f_1 + \nabla^2_{x^1} f_1 \bar{q}, \nabla_{x^1} f_2 + \nabla^2_{x^1} f_2 \bar{q}, \ldots, \nabla_{x^1} f_k + \nabla^2_{x^1} f_k \bar{q}\}$ is linearly independent; and*
(C_5): *the set $\{\nabla_{x^2} g_1 + \nabla^2_{x^2} g_1 \bar{s}, \nabla_{x^2} g_2 + \nabla^2_{x^2} g_2 \bar{s}, \ldots, \nabla_{x^2} g_k + \nabla^2_{x^2} g_k \bar{s}\}$ is linearly independent.*

where $f_i = f_i(\bar{x}^1, \bar{y}^1)$, $g_i = g_i(\bar{x}^1, \bar{y}^1)$, $i = 1, 2, \ldots, k$. *Then* $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p} = 0, \bar{r} = 0)$ *is feasible for* (SMP) *and* $F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = G(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{q}, \bar{s})$.

Moreover, if the hypothesis of Theorem 3.1 are satisfied for all feasible solutions of (SMP) and (SMD), then $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r})$ is a properly efficient solution of (SMP).

4. **Mixed Type Second Order Multiobjective Self Duality**

In this section, we now prove the following self-duality theorem. A mathematical program is said to be self-dual, if it is formally identical with its dual, that is, if the dual is recast in the form of the primal, the new program so obtained is the same as the primal. In general the program (SMP) and (SMD) are not self dual without an added restriction on $f_i(x, y)$ and $f_i(y, x)$, $i \in \{1, 2, \ldots, k\}$. The functions $f_i : R^{|J_1|} \times R^{|J_2|} \to R$ and $g_i : R^{|J_1|} \times R^{|J_2|} \to R$, $i \in \{1, 2, \ldots, k\}$ is the skew symmetric if for all $x, y \in R^n$,

$$f_i(x^1, y^1) = -f_i(y^1, x^1), i \in \{1, 2, \ldots, k\} \text{ and } g_i(x^2, y^2) = g_i(y^2, x^2).$$

We describe the programs (SMP) and (SMD) as dual program if the conclusion of Theorem 3.2 hold.

Theorem 4.1 (**Self duality**). *If the kernel function* $f_i(\bar{x}^1, \bar{y}^1)$ *and* $g_i(\bar{x}^1, \bar{y}^2)$ *for* $i \in \{1, 2, \ldots, k\}$ *are skew symmetric, then* (SMP) *is self-dual. If also* (SMP) *and* (SMD) *are dual program, and* $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r})$ *is a joint optimal solution, then so is* $(\bar{y}^1, \bar{y}^2, \bar{x}^1, \bar{x}^2, \bar{\lambda}, \bar{p}, \bar{r})$ *and* $F(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda}, \bar{p}, \bar{r}) = 0$.

Proof. Consider (SMP) and note that (SMD) can be written:

$$\begin{aligned}
\text{Minimize } & -G(x^1, x^2, y^1, y^2, q, s) \\
& \quad = (-G_i(x^1, x^2, y^1, y^2, q, s), \ldots, -G_k(x^1, x^2, y^1, y^2, q, s)) \\
\text{subject to } & -(\nabla_{x^1}(\lambda^T f)(x^1, y^1) + \nabla_{x^1}(\lambda^T f)(x^1, y^1)q) \leqq 0, \\
& -(\nabla_{x^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{x^2}(\lambda^T g)(x^2, y^2)s) \leqq 0, \\
& -(x^2)^T(\nabla_{x^2}(\lambda^T g)(x^2, y^2) + \nabla^2_{x^2}(\lambda^T g)(x^2, y^2)s) \geqq 0, \\
& y^1, y^2 \geqq 0, \\
& {}^*\lambda \in \Lambda^+,
\end{aligned}$$

where

$$\begin{aligned}
G_i(x^1, x^2, y^1, y^2, \lambda, q, s) = {} & f_i(x^1, y^1) - \frac{1}{2} q^T \nabla^2_{x^1} f_i(x^1, y^1) q \\
& - (x^1)^T(\nabla_{x^1}(\lambda^T f)(x^1, y^1) + \nabla^2_{x^1}(\lambda^T f)(x^1, y^1)q) \\
& + g_i(x^2, y^2) - \frac{1}{2} s^T \nabla^2_{x^2} g_i(x^2, y^2) s.
\end{aligned}$$

Since for each $i \in \{1,2,\dots,k\}$, f_i and g_i are skew symmetric,

$$\nabla_{x^1} f_i(x^1,y^1) = -\nabla_{y^1} f_i(y^1,x^1), \nabla_{x^2} g_i(x^2,y^2) = -\nabla_{y^2} f_i(y^2,x^2),$$
$$\nabla_{x^2}(\lambda^T g)(x^2,y^2) = -\nabla_{y^2}(\lambda^T g)(y^2,x^2), \nabla^2_{x^2}(\lambda^T g)(x^2,y^2) = -\nabla^2_{y^2}(\lambda^T g)(y^2,x^2),$$
$$\nabla^2_{x^1}(\lambda^T f)(x^1,y^1) = -\nabla^2_{y^1}(\lambda^T f)(y^1,x^1), \nabla_{x^1}(\lambda^T f)(x^1,y^1) = -\nabla_{y^1}(\lambda^T f)(y^1,x^1),$$

and program (SMD) becomes.

$$\begin{aligned} \text{Minimize } & G(y^1,y^2,x^1,x^2,q,s) \\ & \quad = (G_1(y^1,y^2,x^1,x^2,q,s),\dots,G_k(y^1,y^2,x^1,x^2,q,s)) \\ \text{subject to } & (\nabla_{y^1}(\lambda^T f)(y^1,x^1) + \nabla^2_{y^1}(\lambda^T f)(y^1,x^1)q) \leqq 0, \\ & -(\nabla_{y^2}(\lambda^T g)(y^2,x^2) + \nabla^2_{y^2}(\lambda^T g)(y^2,x^2)s) \leqq 0, \\ & (x^2)^T(\nabla_{y^2}(\lambda^T g)(y^2,x^2) + \nabla^2_{y^2}(\lambda^T g)(y^2,x^2)s) \geqq 0, \\ & y^1, y^2 \geqq 0, \\ & \lambda \in \Lambda^+, \end{aligned}$$

where

$$\begin{aligned} G_i(y^1,y^2,x^1,x^2,\lambda,q,s) \;=\; & f_i(y^1,x^1) + \frac{1}{2} q^T \nabla^2_{y^1} f_i(y^1,x^1) q \\ & -(x^1)^T\{\nabla_{y^1}(\lambda^T f)^T(y^1,x^1) + \nabla^2_{y^1}(\lambda^T f)(y^1,x^1)q\} \\ & + g_i(y^2,x^2) - \frac{1}{2} s^T \nabla^2_{x^2} g_i(y^2,x^2)s, \quad i = 1,2,\dots,k. \end{aligned}$$

This is just (SMP).

The remainder of the proof follows on the line of [3] and [4]. □

5. Special Cases

If $k = 1$, $\lambda = 1$, $f_i = f$ and $g_i = g$, the second order symmetric multiobjective dual programs (SMP) and (SMD) to the following program, studied by Husain and Abha [9]:

Primal Program:

$$\begin{aligned} \text{(SP)} : \text{Minimize } F(x^1,y^1,y^1,y^2,p,r) = & f(x^1,y^1) - \frac{1}{2} p^T \nabla^2_{y^1} f(\bar{x}^1,\bar{y}^1)p \\ & + (y^1)^T\{\nabla_{y^1} f(x^1,y^1) + \nabla^2_{y^1} f(x^1,y^1)p\} \\ & + g(x^2,y^2) - \frac{1}{2} r^T \nabla^2_{y^2} g(x^2,y^2) r \end{aligned}$$

$$\begin{aligned} \text{subject to } & \nabla_{y^1} f(x^1,y^1) + \nabla^2_{y^1} f(x^1,y^1)p \leqq 0, \\ & \nabla_{y^2} g(x^2,y^2) + \nabla^2_{y^2} g(x^2,y^2) r \leqq 0 \end{aligned}$$

$$(y^2)^T\{\nabla_{y^2} g(x^2,y^2)+\nabla^2_{y^2} g(x^2,y^2)r \geqq 0$$
$$x^1,x^2 \geqq 0.$$

Dual Program:

$$\begin{aligned}
\text{(SD)} : \text{Maximize } G(u^1,u^2,v^1,v^2,q,s) = {} & f(u^1,v^1)-\frac{1}{2}q^T\nabla^2_{x^1} f(u^1,v^1)q \\
& +(u^1)^T\{\nabla_{x^1} f(u^1,v^1)+\nabla^2_{x^1} f(u^1,v^1)q\} \\
& +g(u^2,v^2)-\frac{1}{2}s^T\nabla^2_{y^2} g(u^2,v^2)s
\end{aligned}$$
$$\begin{aligned}
\text{subject to } \; & \nabla_{x^1} f(u^1,v^1)+\nabla^2_{x^1} f(u^1,v^1)q \geqq 0, \\
& \nabla_{x^2} g(u^2,v^2)+\nabla^2_{x^2} g(u^2,v^2)s \geqq 0, \\
& (u^2)^T\{\nabla_{x^2} g(u^2,v^2)+\nabla^2_{x^2} g(u^2,v^2)s\} \leqq 0, \\
& v^1,v^2 \geqq 0.
\end{aligned}$$

If $J_2=\phi$ and $K_2=\phi$, the programs (SMP) and (SMD) reduce to the following pair of Wolfe type second order multiobjective dual programs which are not explicitly studied in the literature

Primal Program:

$$\begin{aligned}
\text{(SWP)} : \text{Minimize } & F^1(x^1,y^1,p) = (F^1_1(x^1,y^1,p),\dots,F^1_k(x^1,y^1,p)) \\
\text{subject to } \; & \nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p \leqq 0, \\
& x^1 \geq 0, \\
& \lambda \in \Lambda^+ .
\end{aligned}$$

Dual Program:

$$\begin{aligned}
\text{(SWD)} : \text{Minimize } & G^1(u^1,v^1,q) = (G^1_1(u^1,v^1,q),\dots,G^1_k(u^1,v^1,q)) \\
\text{subject to } \; & \nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q \geqq 0, \\
& y^1 \geq 0, \\
& \lambda \in \Lambda^+,
\end{aligned}$$

where, for each $i \in \{1,2,\dots,k\}$,

(i) $F^1_i(x^1,y^1,p) = f_i(x^1,y^2)-(y^T)[\nabla_{y^1}(\lambda^T f)(x^1,y^1)+\nabla^2_{y^1}(\lambda^T f)(x^1,y^1)p]$
$$-\frac{1}{2}p^T\nabla^2_{y^1} f_i(x^1,y^1)p,$$

(ii) $G^1_i(u^1,v^1,q) = f_i(u^1,v^1)-(u^T)[\nabla_{x^1}(\lambda^T f)(u^1,v^1)+\nabla^2_{x^1}(\lambda^T f)(u^1,v^1)q]$
$$-\frac{1}{2}q^T\nabla^2_{y^1} f_i(u^1,v^1)q.$$

If $J_1 \neq \phi$ and $K_1 = \phi$, the programs (SMP) and (SMD) become the Mond-Weir second order multiobjective dual program which are reported in mathematical programming.

Primal Program:

$$\text{(SMWP)}: \text{Minimize } F^2(x^2,y^2,r) = (F_1^2(x^2,y^2,r),\ldots,F_k^2(x^2,y^2,r))$$

$$\text{subject to } \nabla_{y^2}(\lambda^T g)(x^2,y^2) + \nabla^2_{y^2}(\lambda^T g)(x^2,y^2)r \leqq 0\,,$$

$$(y^2)^T[\nabla_{y^2}(\lambda^T g)(x^2,y^2) + \nabla^2_{y^2}(\lambda^T g)(x^2,y^2)r] \geqq 0\,,$$

$$x^2 \geqq 0\,,$$

$$\lambda > 0\,.$$

Dual Program:

$$\text{(SMWD)}: \text{Maximize } G^2(u^2,v^2,s) = (G_1^2(u^2,v^2,s),\ldots,G_k^2(u^2,v^2,s))\,,$$

$$\text{subject to } \nabla_{x^2}(\lambda^T g)(u^2,v^2) + \nabla^2_{x^2}(\lambda^T g)(u^2,v^2)s \geqq 0\,,$$

$$(u^2)^T[\nabla_{x^2}(\lambda^T g)(u^2,v^2) + \nabla^2_{x^2}(\lambda^T g)(u^2,v^2)s] \leqq 0\,,$$

$$v^2 \geqq 0\,,$$

$$\lambda > 0\,.$$

where, for each $i \in \{1,2,\ldots,k\}$,

(i) $F_i(x^2,y^2,r) = g_i(x^2,y^2) - \dfrac{1}{2} r^T \nabla^2_{y^2} g_i(x^2,y^2)r$,

(ii) $G_i(u^2,v^2,s) = g_i(u^2,v^2) - \dfrac{1}{2} s^T \nabla^2_{x^2} g_i(u^2,v^2)s$.

If $p = q = s = r = 0$, then the programs (SMP) and (SMD) reduce to the mixed type first-order symmetric multiobjective programs studied by Bector, Chandra and Abha [4].

References

[1] M.S. Bazarra and J.J. Goode, On symmetric dual nonlinear programming, *Operations Research* **21**(1)(1973), 1–9.

[2] C.R. Bector and S. Chandra Generalized bonvex functions and second order duality in mathematical programming, Department of Act. and Management Services, *Research Report* 2–85, (1985), University of Manitoba, Winnipeg.

[3] C.R. Bector and S. Chandra, Second order symmetric and self dual programs, *Opsearch* **23**(1986), 89–95.

[4] C.R. Bector, S. Chandra and Abha, On mixed symmetric duality in multiobjective programming, *Opsearch* **36**(4)(1999), 399–407.

[5] S. Chandra, I. Husain and Abha, On mixed symmetric duality in mathematical programming, *Opsearch* **36**(2)(1999), 165–171.

[6] G.B. Dantzig, Eisenberg and R.W. Cottle, Symmetric dual nonlinear programs, *Pacific Jounal of Mathematics* **15**(1965), 809–812.

[7] W.S. Dorn, A symmetric dual theorem for quadratic programs, *Journal of Operational Society of Japan* **2**(1960), 93–97.
[8] D. Gale, H.W. Kuhn and A.W. Tucker, Linear programming and theory of games, Activity analysis of production and allocation, *Cowles Commission Monographs No.13*, John Wiley and Sons, Inc., New York, Chapman and Hall. Ltd., London, (1951), 317-319.
[9] I. Husain and Abha, Second order mixed symmetric and self duality in mathematical programming, *Recent Developments in Operational Research*, Narosa Publication House, New Delhi, India (2001), 137–147.
[10] O.L. Mangasarian and S. Fromovitz, The Fritz John necessary optimality conditions in the presence of equality and inequality constraints, *J. Math. Anal. Appl.* **17**(1967), 37–47.
[11] O.L. Mangasarian, Second and higher order duality in nonlinear programming, *Journal of Mathematical Analysis and Applications* **51**(1975), 607–620.
[12] B. Mond, Second order duality for nonlinear programs, *Opsearch* **11**(1974), 90–99.
[13] B. Mond and R.W. Cottle. Self duality in mathematical programming, *Siam J. App. Math.* **14**(1966), 420–423.
[14] B. Mond and T. Weir, Generalized concavity and duality, in *Generalized Concavity in Optimization and Economics*, S. Schaible and W.T. Ziemba (editors), Academic Press, New York, 1981
[15] B. Mond, A symmetric dual theorem for non-linear programs, *Quarterly Journal of Applied Mathematics* **23**(1965), 265–269.
[16] V. Neumann, On the theory of games of strategy 1959, Contribution to theory of Games, Vol. IV, *Annals of Mathematics Studies*, #40, Princeton University Press, Princeton (1959).
[17] S.K. Suneja, C.S. Lalitha and Seema Khurana, Second order symmetric duality in multiobjective programming, *European Journal of Operational Research* **144**(2000), 492–500.
[18] P. Wolfe, A duality theorem for nonlinear programs, *Quart. Appl. Math.* **19**(1961), 239–244.

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