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Research Article

Stochastic Integrals in Formulating Discounting Models for Implementations of Strategic Operations

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Abstract. In many cases of formulating stochastic models the principal components of such models are stochastic integrals. In consequence, stochastic discounting integrals are used as principal components of stochastic discounting models. The present paper makes use of a positive random variable and a stochastic discounting integral for the formulation of a stochastic discounting model. Moreover, the paper provides interpretation of the formulated stochastic discounting model in developing and investigating strategic operations.

Keywords. Strategic operation, Stochastic integral, Characteristic function, Model discounting

Mathematics Subject Classification (2020). 60E10, 60H05, 90B50, 91B70

1. Introduction

Stochastic integrals constitute fundamental concepts of probability theory with very significant applications in a wide variety of theoretical and practical disciplines (Chung and Williams [4], and McKean [9]). In particular, stochastic integrals are generally recognized as very useful structural elements in the discipline of stochastic modeling (Pinsky and Karlin [11]). The present paper is devoted to the formulation, investigation and interpretation of a stochastic model incorporating a stochastic integral as principal component. More precisely, the incorporated stochastic integral is a stochastic discounting integral and hence the formulated stochastic model is a stochastic discounting model. Below, we present the incorporated stochastic integral.

We consider the stochastic process $\{X(t), t \geq 0\}$, the positive random variable $L = X(t+1) - X(t)$, the characteristic function $\varphi_L(u)$ and the stochastic integral

$$\int_0^\infty e^{-t/a} dX(t), \quad a > 0. \tag{1.1}$$

The sufficient conditions for establishing the function

$$\exp\left\{a \int_0^u \frac{\log \varphi_L(w)}{w} dw\right\} \tag{1.2}$$

as the characteristic function of the stochastic integral (1.1) are known (Harrison [6], and Jurek and Vervaat [7]).

Moreover, we consider another stochastic integral strongly facilitating the investigation and interpretation of the stochastic integral (1.1). We consider the stochastic process $\{Y(t), t \geq 0\}$, the positive random variable $S = Y(t+1) - Y(t)$, the characteristic function $\varphi_s(u)$, and the stochastic integral

$$\int_0^1 t^{1/a} dY(t). \tag{1.3}$$

The sufficient conditions for establishing the function

$$\exp\left\{\frac{a}{u^a} \int_0^u \log \varphi_s(w) w^{a-1} dw\right\} \tag{1.4}$$

as the characteristic function of the stochastic integral (1.3) are known (Lukacs [8]). The main theoretical and practical contribution of the present paper is based on the significant property of equality in distribution (Riedel [12]). More precisely, the equality in distribution between the stochastic integral (1.1) and the stochastic integral (1.3) is incorporated. Such an equality in distribution facilitates the formulation of a stochastic discounting model with (1.2) as its characteristic function.

2. Incorporating Stochastic Integrals in Models

It is generally adopted the theoretical and practical importance of making use in stochastic modeling formulations of stochastic integrals (Artikis and Artikis [2], and Harrison [6]). The present section concentrates on the formulation of stochastic discounting models having as principal components the very well known stochastic integrals appearing in the first section of the present paper. It is of some theoretical interest to mention that the extremely important concept of equality in distribution constitutes the main structural factor for supporting the implementation of the contribution of the present section.

The first part of this section is devoted to the contribution of the stochastic integral

$$\int_0^\infty e^{\frac{-t}{a}} dX(t)$$

in the formulation of a stochastic discounting model.

Theorem 2.1. *We suppose that the random variable L is independent of the stochastic integral*

$$\int_0^\infty e^{\frac{-t}{a}} dX(t)$$

then

$$\int_0^\infty e^{\frac{-t}{a}} dX(t) \stackrel{d}{=} \int_0^1 t^{\frac{1}{a}} dY(t)$$

if, and only if,

$$L+\int_0^\infty e^{\frac{-t}{a}}\,dX(t)\overset{d}{=}S\,, \tag{2.1}$$

where $\overset{d}{=}$ *denotes equality in distribution.*

Proof. Only the sufficiency condition will be proved since the necessity condition can be proved by reversing the argument. From the assumption of independence of the random variable L and the stochastic integral

$$\int_0^\infty e^{\frac{-t}{a}}\,dX(t)$$

and the introduction of characteristic functions in (2.1) we get the integral equation

$$\varphi_L(u)\exp\left\{a\int_0^u \frac{\log\varphi_L(w)}{w}dw\right\}=\varphi_s(u)$$

or equivalently the integral equation

$$\log\varphi_L(u)+a\int_0^u \frac{\log\varphi_L(w)}{w}dw=\log\varphi_s(u). \tag{2.2}$$

From the integral equation (2.2) we get the integral equation

$$u^{a-1}\log\varphi_L(u)+au^{a-1}\int_0^u \frac{log\varphi_L(w)}{w}dw=u^{a-1}\log\varphi_s(u). \tag{2.3}$$

It is readily understood that the integral equation (2.3) can be written in the form

$$\frac{d}{du}\left(u^a\int_0^u \frac{\log\varphi_L(w)}{w}dw\right)=\frac{d}{du}\int_0^u \log\varphi_s(w)w^{a-1}dw\,. \tag{2.4}$$

If we integrate (2.4) we get the integral equation

$$u^a\int_0^u \frac{\log\varphi_{L(W)}}{w}dw=\int_0^u \log\varphi_s\,(w)w^{a-1}dw$$

which can be written in the form

$$a\int_0^u \frac{\log\varphi_L(w)}{w}dw=\frac{a}{u^a}\int_0^u \log\varphi_s(w)w^{a-1}dw \tag{2.5}$$

for $u\neq 0$. From (2.5) we get that

$$\exp\left\{a\int_0^u \frac{\log\varphi_L(w)}{w}dw\right\}=\exp\left\{\frac{a}{u^a}\int_0^u \log\varphi_s(w)w^{a-1}dw\right\}. \tag{2.6}$$

If we use stochastic integrals in (2.6) we get the equality in distribution

$$\int_0^\infty e^{\frac{-t}{a}}\,dX(t)\overset{d}{=}\int_0^1 t^{\frac{1}{a}}\,dY(t).$$

It is particularly useful to consider the practical significance of the above theoretical result. More precisely, we shall make clear that the stochastic model (1.4) can be interpreted as a stochastic discounting model. It is known that the stochastic integral

$$\int_0^\infty e^{\frac{-t}{a}}\,dX(t)$$

denotes the present value of income, as viewed from the time point 0, produced by an organization during its infinite lifetime, where $\frac{1}{a}$ is the force of interest rate (Harrison [6]). Moreover, if the random variable L denotes a cash flow arising at the time point 0 then

$$S=L+\int_0^\infty e^{\frac{-t}{a}}\,dX(t)$$

is a stochastic discounting model. This stochastic discounting model can become more realistic if we replace the random variable L by a random variable D distributed as the random variable L and D denotes a cash flow arising at the time point 0. The second part of the present section is devoted to the contribution of the stochastic integral

$$\int_0^1 t^{\frac{1}{a}} dY(t)$$

in the formulation of a stochastic model. □

Theorem 2.2. *We suppose that the random variable L is independent of the stochastic integral*

$$\int_0^1 t^{\frac{1}{a}} dY(t)$$

then

$$\int_0^\infty e^{\frac{-t}{a}} dX(t) \overset{d}{=} \int_0^1 t^{\frac{1}{a}} dY(t)$$

if, and only if,

$$L + \int_0^1 t^{\frac{1}{a}} dY(t) \overset{d}{=} S. \tag{2.7}$$

Proof. Only the sufficient condition will be proved since the necessity condition can be proved by reversing the argument. From the independence of the random variable L and the stochastic integral

$$\int_0^1 t^{\frac{1}{a}} dY(t)$$

and the introduction of characteristic functions $\varphi_L(u)$, $\varphi_s(u)$ and the characteristic function (1.4) in (2.7) we get the integral equation

$$\varphi_L(u) \exp\left\{\frac{a}{u^a}\int_0^u \log\varphi_s(w) w^{a-1} dw\right\} = \varphi_s(u). \tag{2.8}$$

From (2.8) we get the integral equation

$$\log\varphi_L(u) + \frac{a}{u^a}\int_0^u \log\varphi_s(w) w^{a-1} dw = \log\varphi_s(u). \tag{2.9}$$

Moreover (2.9) can be written in the form

$$a\log\varphi_L(u) = -\frac{a^2}{u^a}\int_0^u \log\varphi_s(w) w^{a-1} dw + a\log\varphi_s(u)$$

or equivalently in the form

$$a\frac{\log\varphi_L(u)}{u} = -\frac{a^2}{u^{a+1}}\int_0^u \log\varphi_s(w) w^{a-1} dw + a\frac{\log\varphi_s(u)}{u} \tag{2.10}$$

for $u \neq 0$. It is readily seen that (2.10) implies that

$$a\int_0^u \frac{\log\varphi_L(w)}{w} dw = \frac{a}{u^a}\int_0^u \log\varphi_s(w) w^{a-1} dw. \tag{2.11}$$

The integral equation (2.11) can be written in the form

$$\exp\left\{a\int_0^u \frac{\log\varphi_L(w)}{w} dw\right\} = \exp\left\{\frac{a}{u^a}\int_o^u \log\varphi_s(w) w^{a-1} dw\right\}. \tag{2.12}$$

If we use stochastic integrals in (2.12) we get the equality in distribution

$$\int_0^{\infty} e^{\frac{-t}{a}} dX(t) \stackrel{d}{=} \int_0^1 t^{\frac{1}{a}} dY(t).$$

It can be said that the main practical contribution of the above theoretical result is the indirect interpretation of the stochastic integral

$$\int_0^1 t^{\frac{1}{a}} dY(t)$$

in the area of stochastic discounting modeling. Moreover, it is obvious that the equality in distribution

$$\int_0^{\infty} e^{\frac{-t}{a}} dX(t) \stackrel{d}{=} \int_0^1 t^{\frac{1}{a}} dY(t)$$

is the fundamental reason for considering such an indirect interpretation. In addition, the above equality in distribution is quite easily adopted as the structural factor for the formulation and investigation of the stochastic model

$$S = L + \int_0^1 t^{\frac{1}{a}} dY(t).$$

It is obvious that the above stochastic model establishes a connection between the random variable L and the random variable S or equivalently the stochastic process

$$\{X(t), t \geq 0\}$$

and the stochastic process

$$\{Y(t), t \geq 0\}.$$

Such a connection makes necessary the undertaking of further research activities for the direct interpretation of the stochastic integral

$$\int_0^1 t^{\frac{1}{a}} dY(t)$$

in the area of stochastic discounting modeling (Riedel [12], and Righter[13]). □

3. Interpretations in Proactive Activities

Decision making under conditions of uncertainty is strongly supported by making use of fundamental probabilistic concepts. In particular, proactive decision making under conditions of uncertainty can be substantially improved by the formulation, investigation, and implementation of stochastic discounting models (Bühlmann [3]). Moreover, strategic decision making under conditions of uncertainty can be extremely useful when researchers and practitioners in strategic management make extensive use of stochastic discounting models quite suitable for describing and implementing significant proactive applications (Stepanov [14]).

The contribution of the paper consists of the establishment of two results by formulating, investigating, and interpreting in practice of two stochastic models. The formulation of the first stochastic model means a sum incorporating a positive random variable denoting a present value and a stochastic discounting integral. The interpretation in stochastic discounting of the first stochastic model is direct. In consequence, the formulation of the first stochastic model, the investigation of the stochastic discounting integral, and the direct interpretation in discounting of the first stochastic model strongly supports the incorporation of that stochastic model in strategic thinking, strategic decision making, and strategic management. The formulation of the second stochastic model means a sum incorporating a positive random variable and a stochastic integral which has not a direct interpretation in stochastic discounting operations. It is of some importance to mention that the investigation of the property of equality in distribution between the two stochastic models. More precisely, the investigation of the second stochastic model can contribute to the investigation of the first model under the assumption of equality in distribution. In conclusion, the theoretical contribution of the paper makes very clear the significant role of stochastic discounting integrals in the implementation of thinking, decision making, and management in a strategic manner (Fred [5], and Pidd [10]).

4. Conclusion

It is readily adopted that the contribution of the present paper consists of the establishment of four results. The first result is the formulation of a stochastic model having as principal component the first stochastic integral in the first section of the paper. The second result is the interpretation of the formulated stochastic model as a stochastic discounting model. The third result is the formulation of a stochastic model having as principal component the second stochastic integral in the first section of the paper. The fourth result is the incorporation of the property of equality in distribution between the stochastic integrals for investigating the probabilistic behavior of the formulated stochastic models. It is of particular practical importance to make a comment on the interpretation of the formulated stochastic model incorporating the stochastic discounting integral in the area of strategic operations. This model is the sum of a positive random variable and the stochastic discounting integral of the first section. The positive random variable denotes a cash flow arising at the time point zero. Moreover, the stochastic discounting integral denotes the present value, as viewed from the time point zero, of the income generated by an organization during its indefinite lifetime. In consequence, the interpretation of the positive random variable and the interpretation of the stochastic discounting integral imply that the sum of these probabilistic concepts is a stochastic discounting model. It is readily recognized that the principal component of the formulated stochastic discounting model is the stochastic discounting integral. Such a recognition, make quite obvious the suitability of the formulated stochastic discounting model for the description, investigation, and interpretation of strategic thinking, strategic decision making, strategic management and other fundamental strategic operations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. T. Artikis and P. T. Artikis, *Probability Distributions in Risk Management Operations*, Intelligent Systems Reference Library series (ISRL, Volume 83), Springer, xiii + 317 pages (2015), DOI: 10.1007/978-3-319-14256-2.

[2] C. T. Artikis and P. T. Artikis, Stochastic integrals in discounting proactive operations, *Journal of Informatics and Mathematical Sciences* **13**(1) (2021), 25 – 30, DOI: 10.26713/jims.v13i1.1511.

[3] H. Bühlmann, Stochastic discounting, *Insurance: Mathematics and Economics* **11**(2) (1992), 113 – 127, DOI: 10.1016/0167-6687(92)90048-G.

[4] K. L. Chung and R. J. Williams, *Introduction to Stochastic Integration*, 2nd edition, Birkhaeuser, New York, xvii + 277 pages (2014), DOI: 10.1007/978-1-4614-9587-1.

[5] D. Fred, *Strategic Management: Concepts and Cases*, Prentice Hall, New Jersey, xxxi + 290 pages (2011).

[6] J. H. Harrison, Ruin problems with compounding assets, *Stochastic Processes and their Applications* **5**(1) (1977), 67 – 79, DOI: 10.1016/0304-4149(77)90051-5.

[7] Z. J. Jurek and W. Vervaat, An integral representation for selfdecomposable Banach space valued random variables, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **62** (1983), 247 – 262.

[8] E. Lukacs, Characterization of stable processes, *Journal of Applied Probability* **6**(2) (1969), 409 – 418, DOI: 10.2307/3212010.

[9] H. P. McKean (1969), *Stochastic Integrals*, Academic Press, New York — London, xiii + 140 pages, DOI: 10.1016/C2013-0-12301-X.

[10] M. Pidd, *Tools for Thinking: Modelling in Management Science*, Wiley, New York, xi + 304 pages (2009).

[11] M. Pinsky and S. Karlin,, *An Introduction to Stochastic Modeling*, Academic Press, Oxford, 563 pages (2011), DOI: 10.1016/C2009-1-61171-0.

[12] M. Riedel, Characterization of stable processes by identically distributed stochastic integrals, *Advances in Applied Probability* **12**(3) (1980), 689 – 709, DOI: 10.2307/1426427.

[13] R. Righter, Stochastic comparison of discounted rewards, *Journal of Applied Probability* **48**(1) (2011), 293 – 294, DOI: 10.1239/jap/1300198151.

[14] S. S. Stepanov, *Stochastic World*, Springer, Cham., 337 pages (2013), DOI: 10.1007/978-3-319-00071-8.

[15] F. W. Steutel and K. van Harn, *Infinite Divisibility of Probability Distributions on the Real Line*, 1st edition, CRC Press, Boca Raton, 550 pages (2004), DOI: 10.1201/9780203014127.

[16] B. Wahlstrom, Models, modelling and modellers: an application to risk analysis, *European Journal of Operational Research* **75**(3) (1994), 477 – 487, DOI: 10.1016/0377-2217(94)90290-9.