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On Minimal and Vertex Minimal Dominating Graph

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Abstract. In this paper, we obtain the bounds on the number of edges, vertices, domatic number, and domination number of the minimal dominating graph and vertex minimal dominating graph of a graph G.

1. Introduction

The graph considered here are finite, undirected without loops or multiple edges. Any undefined term in this paper may be found in [2, 3, 4].

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D is a minimal dominating set if no proper subset $D' \subset D$ is a dominating set. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. The upper domination number $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set in G.

Domatic number $d(G)$ of a graph G to be the largest order of a partition of $V(G)$ into dominating set of G.

The minimal dominating graph $MD(G)$ of a graph G is the intersection graph defined on the family of all minimal dominating sets of vertices of G (see [5]).

The vertex minimal dominating graph $M_vD(G)$ of a graph G is a graph with $V(M_vD(G)) = V' = V \cup S$, where S is the collection of all minimal dominating sets of G with two vertices $u, v \in V'$ are adjacent if either they are adjacent in G or $v = D$ is a minimal dominating set of G containing u (see [6]).

In Figure 1, a graph G, its minimal dominating graph $MD(G)$ and vertex minimal dominating graph $M_vD(G)$ are shown.

The following results are useful to prove our next results.

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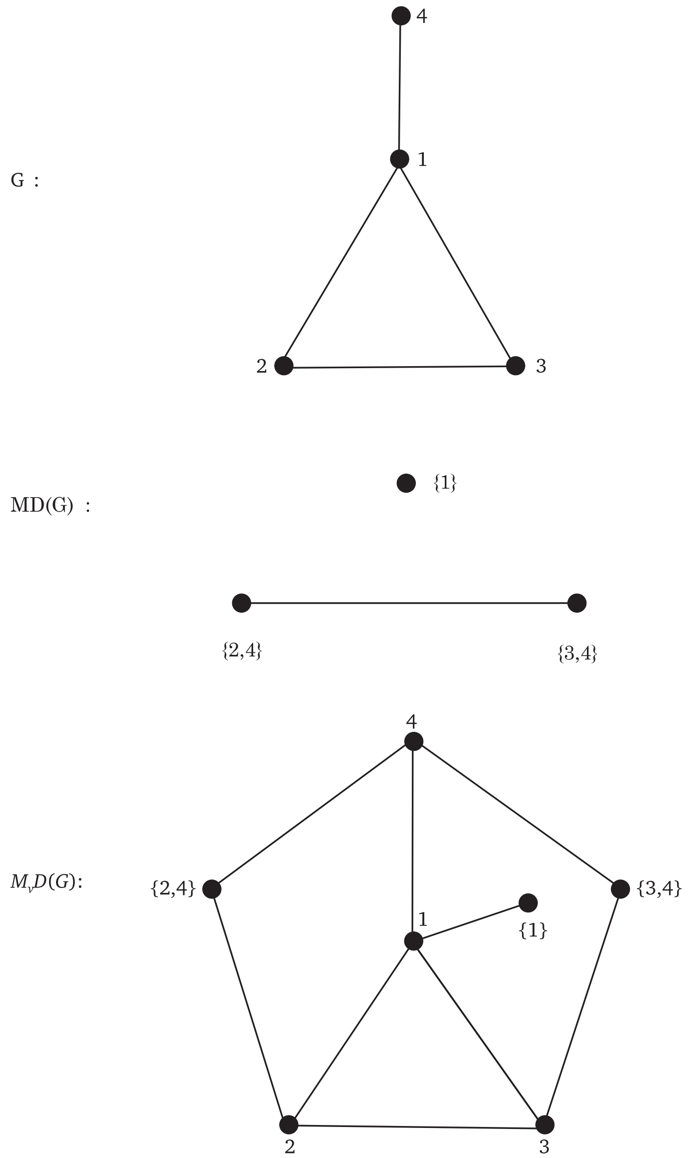

Figure 1

Remark 1. The degree of the vertices of vertex minimal dominating graph $M_vD(G)$ is given by,

(i) $\deg_{M_vD(G)}(D_i) = |D_i|$,
(ii) $\deg_{M_vD(G)}(v_j) = \deg_G(v_j) + t_j$

where D_i, $1 \leq i \leq n$ denotes the minimal dominating sets of G and t_j, $1 \leq j \leq p$ denotes the number of minimal dominating sets containing v_j in G.

Remark 2. For any graph G, the set $S = \{S_1, S_2, \ldots, S_n\}$ is independent set of $M_vD(G)$. Where S_i, $1 \le i \le n$ denotes the all minimal dominating sets of G.

Theorem A ([5]). *For any graph G,*

$$\gamma(MD(G)) = P$$

if and only if every independent set of G is a dominating set.

Theorem B ([5]). *For any graph G, $MD(G)$ is complete if an only if G contains an isolated vertex.*

Theorem C ([6]). *For any graph G, $M_vD(G)$ is tree if and only if $G = \overline{K}_p$ or K_2*

$$S(\overline{G}) \subset D(G).$$

Theorem D ([7]). *If $\Gamma(G) \le 2$, then*

$$S(\overline{G}) \subset D(G),$$

where $S(G)$ is the subdivision graph of G.

Theorem E ([6]). *For any graph G,*

$$D(G) \subseteq M_vD(G).$$

Further, the equality holds if and only if $G = \overline{K}_P$.

Theorem F ([1]). (i) $d(K_p) = p$; $d(\overline{K}_p) = 1$,
(ii) *for any tree T, with $p \ge 2$ vertices, $d(T) = 2$.*

2. **Minimal Dominating Graph**

Theorem 1. *For any graph G,*

$$d(G) \le n \le p(p-1)/2,$$

where n denotes the number of vertices of $MD(G)$. Further the lower bound attained if an only if $G = K_p$ or $\overline{K}_p$ or $K_{1,p-1}$ and the upper bound is attained if and only if G is $(p-2)$- regular.

Proof. The lower bound follows from the fact that every graph has at least $d(G)$ number of minimal dominating sets of G and the upper bound follows from the fact that every vertex is in at most $(p-1)$ minimal dominating sets of G.

Suppose the lower bound is attained. Then every vertex is in exactly one minimal dominating set of G, and hence every minimal dominating set is independent. Thus, there exist two minimal dominating sets D and D' in a graph G such that every vertex in D is adjacent to every vertex in D'. This implies the necessity.

Conversely, suppose $G = K_p$ or $\overline{K}_p$ or $K_{1,p-1}$. Then, by Theorem F, $d(K_p) = p$ or $d(\overline{K}_p) = 1$ or $d(K_{1,p-1}) = 2$ which implies that order of $MD(G)$ is p or one or two respectively.

Suppose the upper bound is attained. Then each vertex is in exactly $(p-1)$ minimal dominating sets and hence G is $(p-2)$-regular.

Converse is obvious. □

Theorem 2. *For any graph G,*

$$0 \leq m \leq p(p-1),$$

where m is the number of edges in $MD(G)$, further the lower bound attained if and only if $G = K_P$ or $\overline{K}_P$ or $K_{1,P-1}$ and the upper bound is attained if and only if G is $(p-2)$-regular.

Proof. Suppose the lower bound attains. Then $MD(G)$ is totally disconnected or K_1. Consequently $G = K_P$ or $\overline{K}_P$ or $K_{1,P-1}$.

Conversely, suppose $G = K_P$, then each vertex of G is a minimal dominating set of G. Hence $MD(G)$ is totally disconnected.

Suppose if $G = K_{1,p-1}$, then clearly, G has only two minimal dominating sets with no element in common. Hence $MD(G)$ is disconnected.

Suppose $G = \overline{K}_P$. Then $V(G)$ is the minimal dominating set of G. Hence $MD(G) = K_1$.

Suppose the upper bound is attained. Then each vertex of G is in exactly $(p-1)$ minimal dominating sets and hence G is $(p-2)$-regular.

Conversely, suppose G is $(p-2)$-regular. Then clearly each vertex of G is in exactly $(p-1)$ minimal dominating sets of G and in G we have p number vertices, which implies $MD(G)$ has $p(p-1)$ edges. □

Theorem 3. *For any graph G,*

$$\gamma(G) + \gamma(MD(G)) = p + 1$$

if and only if every independent set of G is a dominating set or $G = \overline{K}_p$.

Proof. Suppose every independent set of G is dominating set. Then each $\{v\} \subseteq V$ is a minimal dominating set of G, this prove that $MD(G) = \overline{K}_p$. Hence the result.

Suppose $G = \overline{K}_p$. Then $V(G)$ is a minimal dominating set of G. This implies $MD(G) = K_1$. Hence the result.

Conversely, suppose $\gamma(G)+\gamma(MD(G)) = p+1$ holds. If $G \neq K_p$. Then there exist at least two nonadjacent vertices u and v in G. Clearly each vertex $w \in V(G)$ other than u and v form a minimal dominating set of G. Also the set $\{u, v\}$ form minimal dominating set of G. Consequently this gives $\gamma(G) = 1$ and $\gamma(MD(G)) = (p-1)$, which is a contradiction.

Also if $G \neq \overline{K}_p$, then there exist at least one non-trivial component G_1 in G. In G we have two minimal dominating sets of order $(p-1)$, consequently this gives $\gamma(G) = p-1$ and $\gamma(MD(G)) = 1$, which is a contradiction. Therefore $G = \overline{K}_p$. □

Theorem 4. *For any graph G,*

$$d(MD(G)) = |V(MD(G))|$$

if and only if G contains an isolated vertex.

Proof. Suppose $d(MD(G)) = |V(MD(G))|$ holds. Then by Theorem F, $MD(G)$ is complete. And also by the Theorem B, $MD(G)$ is complete if and only if G contains an isolated vertex. Thus G contains isolated vertices.

Conversely, suppose G contains an isolated vertex. Then by Theorem B, $MD(G)$ is complete and also by Theorem F, we have $d(MD(G)) = |V(MD(G))|$. □

Theorem 5. *For any graph G,*

$$\gamma(MD(G)) = 1$$

if and only if G contains an isolated vertex.

Proof. Suppose $\gamma(MD(G)) = 1$. Then, $MD(G)$ is complete. And also by Theorem B, $MD(G)$ complete if and only if G contains an isolated vertex. Hence G contains isolated vertex.

Conversely, suppose G contains an isolated vertex, then by Theorem B, $MD(G)$ is complete which implies $\gamma(MD(G)) = 1$. This completes the proof. □

3. Vertex minimal dominating graph

Theorem 6. *For any graph $G, M_vD(G)$ is bipartite if and only if $G = \overline{K}_p$ or $K_{1,P-1}$.*

Proof. Suppose $M_vD(G)$ is bipartite, then we have to prove that $G = \overline{K}_p$ or $K_{1,P-1}$. On the contrary, if $G \neq \overline{K}_p$, then there exists a component G_1 of G which is not trivial. Then, clearly $M_vD(G)$ contains a cycle of length five, which is a contradiction. Hence $G = \overline{K}_p$.

Suppose if $G \neq K_{1,P-1}$, then there exist a cycle in G. Since, G is subgraph of $M_vD(G)$, this implies that $M_vD(G)$ contains a cycle of odd length (length three), which is again a contradiction. Hence $G = K_{1,P-1}$.

Conversely, suppose $G = \overline{K}_p$, then clearly by Theorem C, $M_vD(G)$ is tree this implies $M_vD(G)$ is bipartite.

Suppose $G = K_{1,p-1}$, then there exist exactly two minimal dominating sets D and D'. D contains a vertex u of degree $(p-1)$ and D' contains the $V(G)-u$ vertices of degree one. Clearly, by definition of $M_vD(G)$, we get the bipartite graph. □

Theorem 7. *For any graph G,*

$$\kappa(M_vD(G)) = \min\left\{\min_{1\leq i\leq n}\left\{\deg_{M_vD(G)}(D_i)\right\},\ \min_{1\leq j\leq p}\left\{\deg_{M_vD(G)}(v_j)\right\}\right\}.$$

Proof. We consider the following cases:

Case 1. Let u be the vertex of $M_vD(G)$ which corresponds to the minimal dominating set of G and is of minimum degree among all the vertices of $M_vD(G)$ then, by deleting the vertices adjacent to u, a disconnected graph is obtained. Thus, $\kappa(M_vD(G)) = \min_{1\leq i\leq n}\left\{\deg_{M_vD(G)}(D_i)\right\}$

Case 2. Let w be the vertex of $M_vD(G)$ which corresponds to the vertex of G and is of minimum degree among all the other vertices of $M_vD(G)$. Then by deleting vertices adjacent to w results into a disconnected graph. Thus, $\kappa(M_vD(G)) = \min_{1\leq j\leq p}\left\{\deg_{M_vD(G)}(v_j)\right\}$. □

Theorem 8. *For any graph G,*

$$\lambda(M_vD(G)) = \min\left\{\min_{1\leq i\leq n}\left\{\deg_{M_vD(G)}(D_i)\right\},\ \min_{1\leq j\leq p}\left\{\deg_{M_vD(G)}(v_j)\right\}\right\}.$$

Proof. The proof is similar to the proof of Theorem 7. □

Theorem 9. *For any graph G,*

$$\gamma(M_vD(G)) = p \text{ if and only if } G = K_p.$$

Proof. Suppose $\gamma(M_vD(G)) = p$. On the contrary, if $G \neq K_p$, then there exist at least two non-adjacent vertices u and v in G. Clearly, each vertex $w \in V(G)$ other than u and v form a minimal dominating set of G. Also the set $\{u, v\}$ form a minimal dominating set of G. Consequently, $\gamma(M_vD(G)) = (p-1)$, which is a contradiction. Hence $G = K_P$.

Conversely, suppose $G = K_P$, then each $\{v\} \subseteq V(G)$ is a minimal dominating set of G. By the definition, each vertex is adjacent to exactly one minimal dominating set, hence it follows that $\gamma(M_vD(G)) = p$. □

Theorem 10. *For any graph G,*

$$d(M_vD(G)) = 2$$

if and only if $G = \overline{K}_P$ *or* K_2.

Proof. Suppose $d(M_vD(G)) = 2$. Then, by Theorem F, $M_vD(G)$ is a tree and also, by Theorem C, we have $G = \overline{K}_P$ or K_2.

Conversely, suppose $G = \overline{K}_P$ or K_2. Then, by Theorem C, $M_vD(G)$ is a tree. Also, by Theorem F, $d(M_vD(G)) = 2$. □

Theorem 11. *If* $\Gamma(G) = 2$*, then*

$$S(\overline{G}) \subseteq M_vD(G)\,.$$

Further, the equality holds if and only if $G = \overline{K}_2$.

Proof. By Theorem D, $S(\overline{G}) \subset D(G)$. Also, by Theorem E, $D(G) \subseteq M_vD(G)$. This implies $S(\overline{G}) \subseteq M_vD(G)$.

Now, we have to prove second part.

Suppose, the equality holds. On the contrary, if $G = \overline{K}_P$, for $p \geq 3$ then there exist a minimal dominating set in G with at least three vertices, a contradiction. Hence $G = \overline{K}_2$.

Conversely, suppose $G = \overline{K}_2$, then there exist a minimal dominating set D containing two vertices, say u and v of G. By definition of $M_vD(G), u$ and v are adjacent to D in $M_vD(G)$. Clearly which gives the path P_3. Also we know that $\overline{G} = K_2$ and $S(\overline{G}) = P_3$. Therefore we have $S(\overline{G}) = M_vD(G)$. □

Theorem 12. *For any graph G,*

$$\chi(M_vD(G)) = \begin{cases} \chi(G)+1 & \text{if vertices of any minimal dominating set} \\ & \text{are colored with } \chi(G) \text{ colors,} \\ \chi(G) & \text{otherwise.} \end{cases}$$

Proof. Let G be a graph with $\chi(G) = k$, and D be the set of all minimal dominating sets of G. By Remark 2, D is independent. In the coloring of $M_vD(G)$, either we can make use of the colors which are used to color G, that is $\chi(M_vD(G)) = k = \chi(G)$.

Or, we should have to use one more new color. In particular, if the vertices of any minimal dominating set x of G have colored with k colors. Then we require one more new color to color x in $M_vD(G)$. Hence in this case we required $k + 1$ colors to color $M_vD(G)$. Therefore,

$$\begin{aligned} & \chi(M_vD(G)) = k+1 \\ \Rightarrow \quad & \chi(M_vD(G)) = \chi(G)+1\,. \end{aligned}$$ □

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