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Research Article

Some Coincidence and Common Fixed Point Theorems for Two Pairs of Self-Mappings Satisfying a Rational Inequality in Complex Valued Metric Spaces

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Abstract. In this paper, we prove some coincidence and common fixed point theorems for two pairs of weakly compatible mappings satisfying a rational inequality in the framework of complex valued metric spaces. The proved results generalizes and extends some well known results in the literature.

Keywords. Complex valued metric spaces; Common fixed point; Weakly compatible

Mathematics Subject Classification (2020). 47H10; 54H25

1. Introduction and Preliminaries

Recently, Azam *et al.* [1] introduced the complex valued metric space, which is more general than the well-known metric spaces. After then, many authors have studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions in the framework of complex valued metric spaces (e.g. [2–4, 7–11]).

The purpose of this paper is to study common fixed points for two pairs of self-mappings satisfying a rational inequality in complex valued metric spaces. Consistent with Azam *et al.* [1], the following definitions and results will be needed in the sequel.

A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first coordinate is called $\Re(z)$ and second coordinate is $\Im(z)$.

Let $\mathbb{C}$ be the set of complex number and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \le \mathrm{Im}(z_2)$. It follows that $z_1 \precsim z_2$, if one of the following holds:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;

(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$;

(iii) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$;

(iv) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii) , (iii) and (iv) is satisfied and we will write $z_1 \prec z_2$ if only (iv) is satisfied.

Remark 1.1. We obtained that the following statements hold:

(1) If $a, b \in \mathbb{R}$ with $a \le b$, then $az \precsim bz$ for all $z \in \mathbb{C}$.

(2) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.

(3) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

Definition 1.2 ([1]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies

(1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.3. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = 2i|z_1 - z_2|$ for all $z_1, z_2 \in X$. Then (X, d) is a complex valued metric space.

Definition 1.4. Let(X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X.

(1) If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \prec c$, for all $n \ge N$ then $\{x_n\}$ is said to be convergent to $x \in X$ and we denote this by $x_n \to x$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = x$.

(2) If for every $c \in \mathbb{C}$ with $0 \prec c$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_{n+m}) \prec c, \quad \text{for all } n \ge N,$$

where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.

(3) If for every Cauchy sequence in X is convergent, then (X, d) is said to be Complete complex valued metric space.

Definition 1.5 ([5]). Let S and I be self-mappings of a set X. If $w = Sx = Ix$ for some $x \in X$, then x is called a coincidence point of S and I, and w is called a point of coincidence of S and I.

Definition 1.6 ([6]). S and T be two self-mappings defined on a set X. S and T are said to be weakly compatible if they commute at their coincidence points.

In [1], Azam *et al*. established the following two lemmas.

Lemma 1.7 ([1]). *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 1.8 ([1]). *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $m,n \to \infty$.*

2. Main Results

In this section, we prove some common fixed point results with rational type contraction conditions.

Theorem 2.1. *Let S,T,f and g be self-mappings defined on a complex-valued metric space (X,d) satisfying $TX \subseteq fX$, $SX \subseteq gX$ and*

$$\lambda d(Sx,Ty) \precsim a\frac{[d(fx,Sx)d(fx,Ty)+d(gy,Ty)d(gy,Sx)]}{d(fx,Ty)+d(gy,Sx)} + b\frac{d(fx,Ty)d(gy,Sx)}{d(fx,Sx)+d(gy,Ty)}, \tag{2.1}$$

for all $x,y \in X$, where $\lambda,a,b \in \mathbb{C}_+$ and $0 \prec a+b \prec \lambda$.

If one of SX,TX,fX or gX is complete subspace of X, then

(a) *both pairs $\{S,f\}$ and $\{T,g\}$ have a unique point of coincidence in X;*

(b) *if both pairs $\{S,f\}$ and $\{T,g\}$ are weakly compatible, then S,T,f and g have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. Since $SX \subseteq gX$ we find a point x_1 in X such that $Sx_0 = gx_1$. Also, since $TX \subseteq fX$, we choose a point x_2 with $Tx_1 = fx_2$. Thus, in general, for the point x_{2n-2} one can find a point x_{2n-1} such that $Sx_{2n-2} = gx_{2n-1}$ and then a point x_{2n} with $Tx_{2n-1} = fx_{2n}$ for $n = 1,2,\ldots$.

Repeating such arguments one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$y_{2n-1} = Sx_{2n-2} = gx_{2n-1},$$
$$y_{2n} = Tx_{2n-1} = fx_{2n},$$

for $n = 1,2,\ldots$.

From inequality (2.1), we have

$$\begin{aligned}\lambda d(Sx_{2n},Tx_{2n+1}) &\precsim a\frac{[d(fx_{2n},Sx_{2n})d(fx_{2n},Tx_{2n+1})+d(gx_{2n+1},Tx_{2n+1})d(gx_{2n+1},Sx_{2n})]}{d(fx_{2n},Tx_{2n+1})+d(gx_{2n+1},Sx_{2n})}\\ &\quad + b\frac{d(fx_{2n},Tx_{2n+1})d(gx_{2n+1},Sx_{2n})}{d(fx_{2n},Sx_{2n})+d(gx_{2n+1},Tx_{2n+1})}\\ \lambda d(y_{2n+1},y_{2n+2}) &\precsim a\frac{[d(y_{2n},y_{2n+1})d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+2})d(y_{2n+1},y_{2n+1})]}{d(y_{2n},y_{2n+2})+d(y_{2n+1},y_{2n+1})}\\ &\quad + b\frac{d(y_{2n},y_{2n+2})d(y_{2n+1},y_{2n+1})}{d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})}\end{aligned}$$

so that

$$|\lambda|\,|d(y_{2n+1},y_{2n+2})| \le |a|\,|d(y_{2n},y_{2n+1})|,$$
$$|d(y_{2n+1},y_{2n+2})| \le \left|\frac{a}{\lambda}\right| |d(y_{2n},y_{2n+1})|$$

or

$$|d(y_{2n+1},y_{2n+2})| \le k_1|d(y_{2n},y_{2n+1})|, \tag{2.2}$$

where $k_1 = \left|\frac{a}{\lambda}\right|$.
Since $\lambda, a \in \mathbb{C}_+$ and $0 \prec a \prec \lambda$ then $k_1 = \left|\frac{a}{\lambda}\right| < 1$.
Again, using inequality (2.1),

$$\lambda d(Sx_{2n},Tx_{2n-1}) \precsim \frac{a[d(fx_{2n},Sx_{2n})d(fx_{2n},Tx_{2n-1}) + d(gx_{2n-1},Tx_{2n-1})d(gx_{2n-1},Sx_{2n})]}{d(fx_{2n},Tx_{2n-1}) + d(gx_{2n-1},Sx_{2n})} + b\frac{d(fx_{2n},Tx_{2n-1})d(gx_{2n-1},Sx_{2n})}{d(fx_{2n},Sx_{2n}) + d(gx_{2n-1},Tx_{2n-1})}$$

or

$$\lambda d(y_{2n+1},y_{2n}) \precsim \frac{a[d(y_{2n},y_{2n+1})d(y_{2n},y_{2n}) + d(y_{2n-1},y_{2n})d(y_{2n-1},y_{2n+1})]}{d(y_{2n},y_{2n}) + d(y_{2n-1},y_{2n+1})} + b\frac{d(y_{2n},y_{2n})d(y_{2n-1},y_{2n+1})}{d(y_{2n},y_{2n+1}) + d(y_{2n-1},y_{2n})}.$$

Therefore,

$$|\lambda|\,|d(y_{2n+1},y_{2n})| \le |a|\,|d(y_{2n-1},y_{2n})|,$$
$$|d(y_{2n+1},y_{2n})| \le k_1|d(y_{2n},y_{2n-1})|, \tag{2.3}$$

where $k_1 = \left|\frac{a}{\lambda}\right|$.
Combining (2.2) and (2.3), we have

$$|d(y_{2n+1},y_{2n+2})| \le k|d(y_{2n},y_{2n-1})|,$$

where $k = k_1^2$.
Continuing this process, we get

$$|d(y_{2n+1},y_{2n+2})| \le k_1^{2n}|d(y_1,y_2)|. \tag{2.4}$$

By using inequality (2.1), we have

$$|d(y_{2n+3},y_{2n+2})| \le \left|\frac{a}{\lambda}\right| |d(y_{2n+2},y_{2n+1})| = k_1|d(y_{2n+2},y_{2n+1})|. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$|d(y_{2n+3},y_{2n+2})| \le k_1^{2n+1}|d(y_1,y_2)|. \tag{2.6}$$

From (2.4) and (2.6), we get

$$|d(y_n,y_{n+1})| \le \frac{\max\{1,k_1\}}{k_1^2}k_1^n|d(y_1,y_2)|, \quad \text{for } n = 2,3,\ldots.$$

Since $0 < k_1 < 1$, for m,n $(m > n)$, we have

$$|d(y_n, y_m| \leq \left[\frac{k_1^n}{k_1^2(1-k_1)}\right] \max\{1, k_1\}|d(y_1, y_2)t| \to 0 \ \text{ as } \ m,n \ \to \infty.$$

By Lemma 1.8, the sequence $\{y_n\}$ is a Cauchy sequence in (X,d). Now assume fX is a complete subspace of X, then the subsequence $y_{2n} = Tx_{2n-1} = fx_{2n}$ converges to some u in fX. That is

$$y_{2n} = fx_{2n} = Tx_{2n-1} \to u \ \text{ as } \ n \to \infty. \tag{2.7}$$

As $\{y_n\}$ is a Cauchy sequence which contains a convergent Subsequence $\{y_{2n}\}$, we can find $v \in X$ such that

$$fv = u. \tag{2.8}$$

We claim that $Sv = u$. Using inequalities (2.1) and (2.8), we have

$$\begin{aligned}
\lambda d(Sv, y_{2n}) = \lambda d(Sv, Tx_{2n-1}) \\
&\precsim a \frac{[d(fv,Sv)d(fv,Tx_{2n-1}) + d(gx_{2n-1},Tx_{2n-1})d(gx_{2n-1},Sv)]}{d(fv,Tx_{2n-1}) + d(gx_{2n-1},Sv)} \\
&\quad + b\frac{d(Sv,Tx_{2n-1})d(gx_{2n-1},Sv)}{d(fv,Sv) + d(gx_{2n-1},Tx_{2n-1})} \\
&= a\frac{[d(u,Sv)d(u,y_{2n}) + d(y_{2n-1},y_{2n})d(y_{2n-1},Sv)]}{d(u,y_{2n}) + d(y_{2n-1},Sv)} \\
&\quad + b\frac{d(Sv,y_{2n})d(y_{2n-1},Sv)}{d(u,Sv) + d(y_{2n-1},Sv)}.
\end{aligned}$$

Letting $n \to \infty$ in above inequality, using (2.7), we get

$$\lambda d(Sv, u) \precsim 0.$$

Since $0 \prec \lambda$, which implies that $d(Sv, u) = 0$, that is

$$Sv = u. \tag{2.9}$$

Now, combining (2.8) and (2.9), we have

$$fv = Sv = u,$$

that is u is a point of coincidence of f and S.

By similar manner, we can show that u is a point of coincidence of g and T.

Since $u = Sv \in SX \subseteq gX$, there exists $w \in X$ such that

$$u = gw. \tag{2.10}$$

We claim that $Tw = u$. Using inequality (2.1), we have

$$\begin{aligned}
&\lambda d(u, Tw) = \lambda d(Sv, Tw), \\
&\quad \precsim a\frac{[d(fv,Sv)d(fv,Tw) + d(gw,Tw)d(gw,Sv)]}{d(fv,Tw) + d(gw,Sv)} + b\frac{d(fv,Tw)d(gw,Sv)}{d(fv,Sv) + d(gw,Tw)}, \\
&\lambda d(u, Tw) \precsim 0.
\end{aligned}$$

Since $0 \prec \lambda$, which implies that $d(u, Tw) = 0$, that is

$$u = Tw. \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$u = gw = Tw,$$

that is, u is a point of coincidence of g and T.
Now suppose that u' is another point of coincidence of f and S, that is

$$u' = fv' = Sv'.$$

For some $v' \in X$. Using inequality (2.1), we have

$$\lambda d(u',u) = \lambda d(Sv',Tw) \precsim a\frac{[d(fv',Sv')d(fv',Tw)+d(gw,Tw)d(gw,Sv')]}{d(fv',Tw)+d(gw,Sv')} + b\frac{d(fv',Tw)d(gw,Sv')}{d(fv',Sv')+d(gw,Tw)},$$

by using $0 \prec \lambda$, this implies that

$$d(u',u) = 0, \text{ that is } u' = u.$$

It is clear that u is unique point of coincidence of $\{S,f\}$ and $\{T,g\}$.
Now, we prove that S,T,f and g have a unique common fixed point in X.
Since $\{S,f\}$ and $\{T,g\}$ are weakly compatible and $u = fv = Sv = gw = Tw$, we can write

$$Su = S(fv) = f(Sv) = fu = w_1 \text{ (say)} \quad \text{and} \quad Tu = T(gw) = g(Tw) = gu = w_2 \text{ (say)}.$$

By using inequality (2.1), we get

$$\lambda d(w_1,w_2) = \lambda d(Su,Tu) \precsim a\frac{[d(fu,Su)d(fu,Tu)+d(gu,Tu)d(gu,Su)]}{d(fu,Tu)+d(gu,Su)} + b\frac{d(fu,Tu)d(gu,Su)}{d(fu,Su)+d(gu,Tu)},$$

by using $0 \prec \lambda$, this implies that

$$w_1 = w_2$$

that is $Su = fu = Tu = gu$, by using inequality (2.1) implies that

$$\lambda d(Sv,Tu) \precsim a\frac{[d(fv,Sv)d(fv,Tu)+d(gu,Tu)d(gu,Sv)]}{d(fv,Tu)+d(gu,Sv)} + b\frac{d(fv,Tu)d(gu,Sv)}{d(fv,Sv)+d(gu,Tu)}.$$

Hence, we deduce (by using $0 \prec \lambda$) that $Sv = Tu$, that is, $u = Tu$. This implies that

$$u = Su = fu = Tu = gu.$$

So u is unique common fixed point of S,T,f and g. □

Putting $f = g = I_X$, where I_X is the identity mapping from X into X in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let S,T be self-mappings defined on a complex-valued metric space (X,d) satisfying*

$$\lambda d(Sx,Ty) \precsim a\frac{[d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)]}{d(x,Ty)+d(y,Sx)} + b\frac{d(x,Ty)d(y,Sx)}{d(x,Sx)+d(y,Ty)}, \tag{2.12}$$

for all $x,y \in X$, where $\lambda,a,b \in \mathbb{C}_+$ and $0 \prec a+b \prec \lambda$.
If one of SX or TX is a complete subspace of X, then S and T have a unique common fixed in X.

Theorem 2.3. *Let S,T,f and g be self-mappings defined on a complex-valued metric space (X,d) satisfying $TX \subseteq fX$, $SX \subseteq gX$ and*

$$\lambda d(Sx,Ty) \precsim \alpha d(fx,gy) + \beta \frac{d(fx,Sx)d(Ty,gy)}{1+d(fx,gy)} + \gamma \frac{d(fx,Ty)d(Sx,gy)}{1+d(fx,gy)} + \eta \frac{d(fx,Sx)d(fx,gy)}{1+d(fx,gy)} + \xi \frac{d(Tx,fy)d(Ty,gy)}{1+d(Tx,fy)}, \quad \text{for all } x,y \in X \tag{2.13}$$

where $\lambda,\alpha,\beta,\gamma,\eta,\xi \in \mathbb{C}_+$ and $0 \prec \alpha+\beta+\gamma+\eta+\xi \prec \lambda$.

If one of SX,TX,fX or gX is a complete subspace of X, then

(a) *both pairs $\{S,f\}$ and $\{T,g\}$ have a unique point of coincidence in X.*

(b) *if both pairs $\{S,f\}$ and $\{T,g\}$ are weakly compatible then S,T,f and g have a unique common fixed point in X.*

Proof. The proof of this theorem is similar to that of Theorem 2.1. □

3. Conclusion

In this attempt, we have proved some coincidence and common fixed point theorems for two pairs of self-mappings satisfying a rational inequality in Complex valued metric spaces. These results generalizes and improves the recent results of Rouzkard [10], Rouzkard *et al*. [11] and Kumar *et al*. [8], in the sense that in our results, we are using different type rational inequality for two pairs of self maps in complex valued metric spaces, which extends the further scope of our results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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