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Research Article

Coupled Fixed Point Problem in Abstract Convex Spaces

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Abstract. We establish some existence results for a generalized coupled coincidence point problem (for short, (GCCP)) in abstract convex spaces. The solvability of the GCCP is presented by using our KKM theory. Also, we derive the results on coupled coincidence points and coupled fixed points, which were studied by Lakshmikantham and Ćirić, Amini-Harandi, and Mitrović.

Keywords. Abstract convex space; KKM theorem; Partial KKM space; Coupled fixed point problem

MSC. 46A03; 47H10; 49J53; 54C60; 54H25; 91A11; 91B02

1. Introduction

Since 2006, we have extended hundreds of the KKM theoretic results to the corresponding ones in abstract convex spaces. Such previous results were related to particular spaces such as simplices, topological vector spaces, Lassonde's convex spaces, Horvath spaces, generalized (G-) convex spaces, and a large number of variants of them.

Independently to the above progress, one of the newest branches of fixed point theory is concerned with the study of coupled fixed points, brought by Guo and Lakshmikantham [5] in 1987. In [4], Bhaskar and Lakshmikantham established some fixed and coupled fixed point theorems for contractions in two variables defined on partially ordered metric spaces with applications to ordinary differential equations. Thereafter, these results were extended by several authors in 2006-2015 (see Karapinar *et al*. [7]). Especially, it was extended by

Lakshmikantham and Ćirić [9] in 2009 by introducing the coupled fixed point problem. Moreover, Mitrović [10] in 2010 showed the existence of a coupled coincidence point and coupled fixed point in normed vector spaces by using the KKM technique. In 2011, Amini-Harandi [2] gave a best approximation theorem in certain convex metric spaces and applied it to some best and coupled best approximations and coupled coincidence point results in normed spaces and hyperconvex metric spaces.

In 2013, Mitrović [11] introduced the following generalized coupled coincidence point problem (GCCP): finding $(x,y)\in K\times K$ such that

$$F_1(x,y)\cap G_1(x)\neq\emptyset,\quad F_2(y,x)\cap G_2(y)\neq\emptyset,$$

where $F_i : K\times K\multimap Y$, $G_i : K\multimap Y$, $i=1,2$, are multimaps, X and Y are topological vector spaces and K a nonempty subset of X.

The aim of this paper is to obtain a result of existence of (GCCP) in abstract convex spaces using the KKM technique. In fact, we extend the results of [11] on topological vector spaces to partial KKM spaces, a subclass of abstract convex spaces.

This paper is organized as follows: Section 2 is a short preliminary on abstract convex spaces. In Section 3, we derive some preliminary results for only abstract convex spaces of the form $(X;\Gamma)$ for simplicity. Section 4 deals with the main existence result for a generalized coupled coincidence point problem (for short, (GCCP)) in abstract convex spaces using a KKM type theorem. Finally, in Section 5, we list a number of old or new examples of partial KKM spaces which can be applied our main result. Consequently, many applications given in [11] on coupled coincidence points and coupled fixed points can be extended to a large number of partial KKM spaces.

2. Preliminaries

A multimap (or simply, a map) $F : X\multimap Y$ is a set-valued map with non-empty values from a set X into the power set of a set Y. For $A\subset X$, let $F(A)=\bigcup\{F(x) : x\in A\}$. For any $B\subset Y$, the lower inverse and upper inverse of B under F are defined by

$$F^-(B)=\{x\in X : F(x)\cap B\neq\emptyset\}\ \text{ and }\ F^+(B)=\{x\in X : F(x)\subset B\},\ \text{ resp.}$$

A map F is upper [resp. lower] semicontinuous on X if and only if for every open $V\subset Y$, the set $F^+(V)$ [resp. $F^-(V)$] is open. Note that, a map F is lower [resp. upper] semicontinuous on X if and only if for every closed $V\subset Y$, the set $F^+(V)$ [resp. $F^-(V)$] is closed. A map F is continuous if and only if it is upper and lower semicontinuous,

For the concepts on abstract convex spaces and partial KKM spaces, we follow [15–17] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle\multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for each $A\in\langle D\rangle$. The triple $(E,D;\Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any

$D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \cup\{\Gamma_A : A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E,D;\Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$.

In case $E = D$, let $(E;\Gamma) := (E,E;\Gamma)$.

Definition 2.2. Let $(E,D;\Gamma)$ be an abstract convex space. A *KKM map* $G : D \multimap E$ is the one satisfying

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \quad \text{for all } A \in \langle D \rangle.$$

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E,D;\Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

There are plenty of examples of KKM spaces; see [15, 16, 21] and the references therein.

Now, we have the following diagram for subclasses of abstract convex spaces $(E,D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde's convex space

$\Longrightarrow$ Horvath space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space

$\Longrightarrow$ KKM space $\Longrightarrow$ Partial KKM space

$\Longrightarrow$ Abstract convex space.

Note that Horvath spaces are new class including c-spaces due to Horvath (see [20]).

We need the following in [14]:

Proposition 2.1. *For an abstract convex space $(E,D;\Gamma)$ and a nonempty subset D' of D, let X be a Γ-convex subset of E relative to D' and $\Gamma' : \langle D' \rangle \multimap X$ a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \quad \textit{for } A \in \langle D' \rangle.$$

Then $(X,D';\Gamma')$ itself is an abstract convex space called a subspace relative to D'.

Under an additional requirement, we have the whole intersection property for the map-values of a KKM map:

Proposition 2.2. *Let $(E,D;\Gamma)$ be a partial KKM space [resp. a KKM space], and $G : D \to 2^E$ a multimap satisfying*

(a) *G has closed [resp. open] values;*

(b) $\Gamma_N \subset G(N)$ *for any* $N \in \langle D \rangle$ *(that is,* G *is a KKM map); and*

(c) $\bigcap_{z \in M} \overline{G(z)}$ *is compact for some* $M \in \langle D \rangle$.

Then, we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Therefore, for the partial KKM principle, if $G(y)$ is compact for at least one y, then the family $\{G(y)\}_{y \in D}$ has the whole intersection property.

3. Abstract Convex Space $(X;\Gamma)$

From now on, we consider only abstract convex spaces of the form $(X;\Gamma)$, for simplicity.

Example 3.1. A convexity space $(E,\mathscr{C})$ in the classical sense consists of a nonempty set E and a family $\mathscr{C}$ of subsets of E such that E itself is an element of $\mathscr{C}$ and $\mathscr{C}$ is closed under arbitrary intersection. For any subset $X \subset E$, its $\mathscr{C}$-convex hull is defined and denoted by $\mathrm{Co}_{\mathscr{C}} X := \bigcap\{Y \in \mathscr{C} : X \subset Y\}$. We say that X is *$\mathscr{C}$-convex* if $X = \mathrm{Co}_{\mathscr{C}} X$. Now, we can consider the map $\Gamma : \langle E \rangle \multimap E$ given by $\Gamma_A := \mathrm{Co}_{\mathscr{C}} A$ for each $A \in \langle E \rangle$. Then $(E,\mathscr{C})$ becomes our abstract convex space $(E;\Gamma)$.

Example 3.2 (Ben-El-Mechaiekh *et al.* [3], Altwaijry *et al.* [1])**.** An *L-structure* on E is given by a nonempty set-valued map $\Gamma : \langle E \rangle \multimap E$ verifying:

($*$) For every $A = \{x_0, x_1, \ldots, x_n\} \in \langle E \rangle$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0,1,\ldots,n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j : j \in J\})$.

The pair (E,Γ) is then called an *L-space* and $X \subset E$ is said to be *L-convex* if $\forall\ A \in \langle X \rangle$, $\Gamma(A) \in X$.

These authors stated repeatedly that our original G-convex spaces $(E,D;\Gamma)$ are their L-spaces (E,Γ). How can a triple reduces to a pair? Moreover, they gave fifteen examples of their L-space in [1].

Note that L-spaces are G-convex spaces in the later definition and hence KKM spaces.

Example 3.3 (Kulpa and Szymanski [8])**.** An *L^*-operator* on a topological space X is a function $\Lambda : \langle X \rangle \multimap X$ that satisfies the following condition:

($*$) If $A \in \langle X \rangle$ and $\{U_x : x \in A\}$ is an open cover of X, then there exists $B \in \langle X \rangle$ such that $B \subseteq A$ and $\Lambda(B) \cap \bigcap\{U_x : x \in B\} \neq \emptyset$.

A topological space X with an L^*-operator Λ on it is referred to as an *L^*-space* and it is denoted by (X,Λ). For an L^*-space (X,Λ), a set $Y \subseteq X$ is said to be *L^*-convex* if $\Lambda(A) \subseteq Y$ for each $A \in \langle Y \rangle$. The family $\mathrm{CON}(X,\Lambda)$ of all L^*-convex subsets of X constitutes a *convexity structure* on X.

From now on we consider abstract convex spaces of the form $(X;\Gamma)$ satisfying $\Gamma_{\{x\}} = \Gamma(\{x\}) = \{x\}$ for $x \in X$, for simplicity.

Lemma 3.1. *Let $(X;\Gamma)$ be a partial KKM space, K a nonempty Γ-convex subset of X. Then*

(i) *K can be made into a partial KKM space; and*

(ii) *$K\times K$ can be made into a partial KKM space.*

Proof. (i) Since K is a Γ-convex subset of X, by definition, we have $\mathrm{co}_\Gamma(K)=\mathrm{co}_\Gamma(K\cap X)\subset K$, hence it is Γ-convex relative to K. Therefore, we have a space $(K;\Gamma')$ by letting $\Gamma'=\Gamma|_{\langle K\rangle}$.

Let $G:K\multimap K$ be a closed-valued KKM map, and let $G':X\multimap X$ be the extension of G such that $G'(x)=K$ for $x\in X\setminus K$ and $G'(x)=G(x)$ for $x\in K$. Then $G':X\multimap X$ is a closed-valued KKM map on the partial KKM space $(X;\Gamma)$, and hence $\{G'(x)\}_{x\in X}$ has the finite intersection property. Therefore, $\{G(x)\}_{x\in K}$ has the same property.

(ii) We show that $(K\times K;(\Gamma\times\Gamma)')=(K;\Gamma')\times(K;\Gamma')$ is a partial KKM space, that is, for a closed-valued KKM map $G:K\times K\multimap K\times K$, its values $\{G(x,y):(x,y)\in K\times K\}$ has the finite intersection property. Therefore, for any $M=\{x_i:1\le i\le n\}\subset K$, $N=\{y_i:1\le i\le n\}\subset K$, and $x_0,y_0\in K$, we have

$$\Gamma'_M\times\Gamma'_{\{y_0\}}=\Gamma'_M\times\{y_0\}\subset\bigcup_{i=1}^n G(x_i,y_0)\quad\text{and}\quad\Gamma'_{\{x_0\}}\times\Gamma'_N=\{x_0\}\times\Gamma'_N\subset\bigcup_{i=1}^n G(x_0,y_i).$$

Hence $G_{y_0}=G(\cdot,y_0):K\multimap K$ is a closed-valued KKM map. Since K is a partial KKM space by (i), we have

$$x_*\in\bigcap_{i=1}^n G_{y_0}(x_i).$$

Similarly, from $G_{x_0}=G(x_0,\cdot):K\multimap K$, we have

$$y_*\in\bigcap_{i=1}^n G_{x_0}(y_i).$$

Therefore, $(x_*,y_*)\in\bigcap_{i=1}^n G(x_i,y_i)$. □

We need the following extended forms of definitions and results in [11]:

Let $(X;\Gamma)$ and $(Y;\Lambda)$ be abstract convex spaces.

Definition 3.1. A multimap $G:X\multimap Y$ is said to be *quasiconvex* if, for all Λ-convex set $S\subset Y$ and $N\in\langle X\rangle$, we have

$$G(x)\cap S\neq\emptyset\ \ \forall\, x\in N\quad\Longrightarrow\quad G(x')\cap S\neq\emptyset\ \ \forall\, x'\in\Gamma_N\,.$$

This definition is originated from Nikodem [13].

Lemma 3.2. *A map $G:X\multimap Y$ is quasiconvex if and only if the set $G^-(S)$ is Γ-convex for all Λ-convex sets $S\subset Y$.*

Proof. Necessity. For any Λ-convex set $S\subset Y$, let $N\in\langle G^-(S)\rangle$ and $x\in N$. Then

$$G(x)\cap S\neq\emptyset\ \ \forall\, x\in N\quad\Longrightarrow\quad G(x')\cap S\neq\emptyset\ \ \forall\, x'\in\Gamma_N$$

since G is quasiconvex. Therefore, $\Gamma_N\subset G^-(S)$.

Sufficiency. Suppose $G(x)\cap S \neq \emptyset$ for each $x \in N \in \langle X\rangle$. Since $G^-(S)$ is Γ-convex, we have

$$x \in G^-(S) \Longrightarrow N \subset G^-(S) \Longrightarrow \Gamma_N \subset G^-(S) \Longrightarrow\ G(x')\cap S \neq \emptyset \quad \forall\ x' \in \Gamma_N.$$

Therefore, $G: X \multimap Y$ is quasiconvex. □

Lemma 3.3. *For $F: X\times X \multimap Y$ and $G: X \multimap Y$,*

$$(z,t) \notin F^+(Y\setminus G(x)) \text{ if and only if } x \in G^-(F(z,t))$$

for all $x,z,t \in X$.

Proof. Note that

$$\begin{aligned} & (z,t)\notin F^+(Y\setminus G(x)) = \{(z,t)\in X\times X : F(z,t)\not\subset Y\setminus G(x)\} \\ \Longleftrightarrow\quad & \exists\ y \in F(z,t)\ \text{ such that } y \in G(x) \\ \Longleftrightarrow\quad & \exists\ y \in F(z,t)\ \text{ such that } x \in G^-(y) \\ \Longleftrightarrow\quad & x \in G^-(F(z,t)). \end{aligned}$$

This completes our proof. □

4. Main Result

In this section, we extend the main results of [11] for topological vector spaces to partial KKM spaces:

Theorem 4.1. *Let $(X;\Gamma)$ be a partial KKM space satisfying $\Gamma_{\{x\}} = \{x\}$ for $x \in X$, K a nonempty compact Γ-convex subset of X, $F_i : K\times K \multimap X$, $i = 1,2$, lower semicontinuous maps with Γ-convex values and $G_i : K \multimap X$, $i = 1,2$, quasiconvex maps with open values. If*

$$F_1(x,y)\cap G_1(K)\neq\emptyset \ \text{ and } \ F_2(y,x)\cap G_2(K)\neq\emptyset$$

for each $(x,y)\in K\times K$, then there exists $(x_0,y_0)\in K\times K$ such that

$$F_1(x_0,y_0)\cap G_1(x_0)\neq\emptyset, \quad F_2(y_0,x_0)\cap G_2(y_0)\neq\emptyset.$$

Proof. Suppose the conclusion is not true. Then for each $(x,y)\in K\times K$,

$$F_1(x,y)\cap G_1(x) = \emptyset \ \text{ or } \ F_2(y,x)\cap G_2(y)=\emptyset. \tag{4.1}$$

Define a map $H: K\times K \multimap K\times K$ by

$$H(z,t) = \{(x,y)\in K\times K : (x,y)\in F_1^+(X\setminus G_1(z)) \text{ or } (y,x)\in F_2^+(X\setminus G_2(t))\},$$

for each $(z,t)\in K\times K$. We have that $(z,t)\in H(z,t)$, hence $H(z,t)$ is nonempty for all $(z,t)\in K\times K$. The maps F_1 and F_2 are lower semicontinuous and we have that $H(z,t)$ is closed for each $(z,t)\in K\times K$. Since $K\times K$ is a compact set we have that $H(z,t)$ is a compact set for each $(z,t)\in K\times K$.

Note that $K\times K$ can be made into an abstract convex space. In fact, $(K\times K;(\Gamma\times\Gamma)') = (K;\Gamma')\times(K;\Gamma')$ is a partial KKM space by Lemma 3.1(ii). We denote $(\Gamma\times\Gamma)'$ by simply Λ.

We claim that the map H is a KKM map. In fact, suppose for any $(z_i,t_i)\in K\times K$, $i\in\{1,\dots,n\}$, there exists

$$(z_0,t_0)\in \mathrm{co}_\Lambda\{(z_i,t_i): i\in\{1,\dots,n\}\},$$

such that

$$(z_0,t_0)\notin \bigcup_{i=1}^n H(z_i,t_i).$$

From this, we obtain

$$(z_0,t_0)\notin F_1^+(X\setminus G_1(z_i)) \text{ and } (t_0,z_0)\notin F_2^+(X\setminus G_2(t_i)),$$

for each $i\in\{1,\dots,n\}$. From Lemma 3.3 we obtain

$$z_i\in G_1^-(F_1(z_0,t_0)) \text{ and } t_i\in G_2^-(F_2(t_0,z_0)),$$

for each $i\in\{1,\dots,n\}$. Since the sets $F_1(z_0,t_0)$ and $F_2(t_0,z_0)$ are Γ-convex and the maps G_1 and G_2 are quasiconvex, from Lemma 3.2 we have that

$$z_0\in G_1^-(F_1(z_0,t_0)) \text{ and } t_0\in G_2^-(F_2(t_0,z_0)),$$

so,

$$G(z_0)\cap F_1(z_0,t_0)\neq\emptyset \text{ and } G(t_0)\cap F_2(t_0,z_0)\neq\emptyset,$$

which contradicts (4.1). Hence H is a KKM map having closed compact values. Hence, by Proposition 2.2, there exists $(x_0,y_0)\in K\times K$ such that

$$(x_0,y_0)\in H(z,t) \text{ for all } (z,t)\in K\times K.$$

From this, we obtain

$$F_1(x_0,y_0)\cap G_1(z)=\emptyset \text{ or } F_2(y_0,x_0)\cap G_2(t)=\emptyset,$$

for each $(z,t)\in K\times K$, which contradicts the condition (4.1). Hence the conclusion of Theorem 4.1 holds. □

Remark 4.1. We followed the proof of [11, Theorem 3.1] with some minor corrections. We give an example extending [11, Theorem 3.1] as follows:

Definition 4.1. A convex subset X of a vector space is called a *convex space* (in the sense of Lassonde) if X has a topology that induces the Euclidean topology on the convex hulls of any $N\in\langle X\rangle$. Note that X can be represented by $(X;\Gamma)$ where $\Gamma:\langle X\rangle\to X$ is the convex hull operator.

Corollary 4.1. *Let X be a convex space, K a nonempty compact convex subset of X, $F_i: K\times K\multimap X$, $i=1,2$, lower semicontinuous maps with convex values and $G_i: K\multimap X$, $i=1,2$, quasiconvex maps with open values. If*

$$F_1(x,y)\cap G_1(K)\neq\emptyset \text{ and } F_2(y,x)\cap G_2(K)\neq\emptyset$$

for each $(x,y)\in K\times K$, then there exists $(x_0,y_0)\in K\times K$ such that

$$F_1(x_0,y_0)\cap G_1(x_0)\neq\emptyset,\quad F_2(y_0,x_0)\cap G_2(y_0)\neq\emptyset.$$

5. Applications

Mitrović [11, Theorem 2.1] obtained Theorem 4.1 for the case where X is a topological vector space and K is a nonempty compact convex subset of X. Moreover, he applied his theorem to Corollaries 3.1-3.6 for various types of topological vector spaces, to Corollary 3.7 for hyperconvex spaces, and to Corollary 3.8 for normed spaces. These corollaries are solutions of coupled coincidence problems and coupled fixed point theorems, and can be also extended to partial KKM spaces by following our Theorem 4.1.

Moreover, in our previous works [16–20] and others, we gave various concrete examples of KKM spaces other than the subclasses in the diagram in Section 2 as follows:

(1) Normed vector spaces

(2) Hyperconvex metric spaces of Aronszajn and Panitchpakdi, 1956

(3) Hyperbolic metric spaces — Kirk, 1982; Reich and Shafrir, 1990

(4) Topological semilattices — Horvath and Llinares-Ciscar, 1996

(5) E-convex spaces — Youness, 1999

(6) Bayoumi's KKM spaces — Bayoumi, 2003

(7) Γ-convex spaces — Zafarani, 2004

(8) $\mathbb{R}$-tree — Kirk and Panyanak, 2007

(9) Connected linearly ordered spaces — Park, 2007

(10) Horvath's convex space — Horvath, 2008

(11) $\mathbb{B}$-spaces — Briec and Horvath, 2008

(12) Extended long line L^* — Park, 2008

(13) Complete continuous midpoint metric spaces — Horvath, 2009

(14) Metric spaces with global nonpositive curvature (NPC) — Niculescu-Rovenţa, 2009

(15) $\mathbb{R}$-KKM spaces — Sankar Raj and Somasundaram, 2012

(16) KKM spaces of Chaipunya-Kumam — Chaipunya and Kumam, 2015

(17) Complete finite dimensional Riemannian manifolds — Park, 2019

Further, in [21], we added a number of new examples of partial KKM spaces.

Therefore, our Theorem 4.1 can be applied all of these spaces, and we can have a large number of Corollaries as like as Corollaries 3.1–3.8 of [11].

Finally, we note that Karapinar *et al*. [7] in 2020 investigated the existence of a unique coupled fixed point for α-admissible mapping which is of α-admissible $F(\psi_1,\psi_2)$-contractions in M-metric space and its application.

6. Conclusion

Since we began to initiate the KKM theory in 1992, there have appeared hundreds of works on the theory. Moreover, from 2006, we have established a large number of the KKM theoretic results on abstract convex spaces.

Independently to such progress, a number of authors are working on coupled fixed point problems on very particular types of spaces than abstract convex spaces. As we have shown in Section 5, there exist a large number of spaces which can be applied our new results in this article.

Therefore, the readers are cordially advised to obtain useful results on such several new spaces.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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