



# Fixed Point Theorems of $\alpha_*$ - $\psi$ -Common Rational Type Contractive Order Closed Set-Valued Mappings on Generalized Metric Spaces

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**Abstract.** Alsulami *et al.* [2] introduced the notion of  $(\alpha-\psi)$ -rational type contractive mappings. They have been established some fixed point theorems for the mappings in complete generalized metric spaces. In this paper, we introduce the notion of some fixed point theorems of  $\alpha_*$ - $\psi$ -common rational type contractive order closed set-valued mappings on generalized metric spaces with application to fractional integral equations and give a common fixed point result about fixed points of the set-valued mappings. Also, we give a result about common fixed points of self-maps on a partially ordered set and on complete metric satisfy a contractive condition.

**Keywords.** Common fixed points;  $\alpha_*$ -common admissible;  $\alpha_*$ - $\psi$ -common rational type contractive; Partially ordered set; Weakly increasing

**MSC.** Primary 47H04, 47H10; Secondary 54H25

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## 1. Introduction

As we already know that the fixed point theory has many applications and was extended by several authors from different views (see, e.g. [1–20]). In 2008, Harandi *et al.* [5] introduced the best proximity pairs for upper semi continuous set-valued maps in hyper convex metric spaces. In 2012, Samet *et al.* [18] introduced the notion of  $\alpha$ - $\psi$ -contractive type mappings. Denote with  $\Psi$  the family of upper semi-continuous, strictly increasing functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  such that

$\{\psi^n(t)\}_{n \in \mathbb{N}}$  converges to 0 as  $n \rightarrow \infty$  and  $\psi(t) < t$  for all  $t > 0$  where  $\psi^n$  is the  $n$ -th iterate of  $\psi$  and  $\psi \in \Psi$  [2]. In the years 2012 and 2013, Asl *et al.* ([14], [13]) introduced the notion of common fixed point theorems for  $\alpha_*$ - $\psi$ -contractive multifunction. In 2015, Alsulami *et al.* [2] introduced the notion of  $(\alpha-\psi)$ -rational type contractive mappings. In 2015, Farajzadeh *et al.* [9] introduced the on fixed point theorems for  $(\xi, \alpha, \eta)$ -expansive mappings in complete metric spaces and [10] introduced the some fixed Point Theorems for Generalized  $\alpha$ - $\eta$ - $\psi$ -Geraghty contractive type mappings in partial  $b$ -metric spaces. The aim of this paper is to introduce the notion of some Fixed points of  $\alpha_*$ - $\psi$ -common rational type contractive order closed multi-valued mappings on generalized metric spaces with application to fractional integral equations.

## 2. Preliminaries

We list some fundamental definitions. Let  $2^X$  denote the family of all nonempty subsets of  $X$ .

**Definition 2.1** ([6]). Let  $X$  be a nonempty set and  $d : X \times X \rightarrow [0, \infty)$  satisfy the following conditions, for all  $x, y \in X$  and all distinct  $u, v \in X$  each of which is different from  $x$  and  $y$ :

(GMS1)  $d(x, y) = 0$  if and if  $x = y$ ,

(GMS2)  $d(x, y) = d(y, x)$ ,

(GMS3)  $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ .

Then the map  $d$  is called a generalized metric and abbreviated as  $GM$ . Here, the pair  $(X, d)$  is called a generalized metric space and abbreviated as  $GMS$ .

In the above definition, if  $d$  satisfies only (GMS1) and (GMS2), then it is called a semi-metric (see, e.g. [19]).

A sequence  $\{x_n\}$  in a  $GMS (X, d)$  is  $GMS$  convergent to a limit  $x$  if and only if  $d(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ .

A sequence  $\{x_n\}$  in a  $GMS (X, d)$  is  $GMS$  Cauchy if and if for every  $\epsilon > 0$  there exists a positive integer  $N(\epsilon)$  such that  $d(x_n, x_m) < \epsilon$ , for all  $n > m > N(\epsilon)$ .

A  $GMS (X, d)$  is called complete if every  $GMS$  Cauchy sequence in  $X$  is  $GMS$  convergent.

A mapping  $T : (X, d) \rightarrow (X, d)$  is continuous if for any sequence  $\{x_n\}$  in  $X$  such that  $d(x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ , we have  $d(Tx_n, Tx) \rightarrow 0$  as  $n \rightarrow \infty$ .

The following assumption was suggested by Wilson [19] to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points  $u, v$  there is number  $r_{u,v} > 0$  such that for every  $z \in X$ ,  $r_{u,v} < d(u, z) + d(z, v)$ .

**Proposition 2.1** ([17]). *In a semi-metric space, the assumption (W) is equivalent to the assertion that the limits are unique.*

**Proposition 2.2** ([17]). *Suppose that  $\{x_n\}$  is a Cauchy sequence in a  $GMS (X, d)$  with*

$$\lim_{n \rightarrow \infty} d(x_n, u) = 0, \quad \exists u \in X.$$

Then  $\lim_{n \rightarrow \infty} d(x_n, z) = d(u, z)$ ,  $\forall z \in X$ . In particular, the sequence  $\{x_n\}$  does not converge to  $z$  if  $z \neq u$ .

**Definition 2.2** ([13]). Let  $(X, d)$  be a GMS and  $T, S : X \rightarrow 2^X$  with given set-valued,  $\alpha : X \times X \rightarrow [0, +\infty)$ ,  $\alpha_* : 2^X \times 2^X \rightarrow [0, +\infty)$ ,  $\alpha_*(A, B) = \inf\{\alpha(a, b) : a \in A, b \in B\}$ ,  $\psi \in \Psi$ ,  $D(s, Ts) = \inf\{d(s, z) / z \in Ts\}$ ,  $H$  is the Hausdorff metric, and let

$$M(Ax, By) = \max \left\{ d(x, y), D(x, Ax), D(y, By), \frac{D(x, Ax)D(y, By)}{1 + d(x, y)}, \frac{D(x, Ax)D(y, By)}{1 + H(Ax, By)} \right\},$$

$$H(Ax, By) = \max \left\{ \sup_{a \in Ax} D(a, By), \sup_{b \in By} D(Ax, b) \right\}. \quad (1)$$

One says that  $T, S$  are  $\alpha_*$ - $\psi$ -common rational type contractive set-valued mappings whenever

$$\alpha_*(Ax, By)H(Ax, By) \leq \psi(M(Ax, By)), \quad A, B = T \text{ or } S \quad \forall x, y \in X. \quad (2)$$

**Definition 2.3** ([13]). Let  $T, S : X \rightarrow 2^X$  and  $\alpha : X \times X \rightarrow [0, +\infty)$ . One says that  $T, S$  are an  $\alpha_*$ -common admissible if  $\alpha(x, y) \geq 1 \Rightarrow \alpha_*(Ax, By) \geq 1$ ,  $A, B = T$  or  $S$  and for all  $x, y \in X$ .

**Definition 2.4.** A subset  $B \subseteq X$  is said to be an approximation if for each given  $y \in X$ , there exists  $z \in B$  such that  $D(B, y) = d(z, y)$ .

**Definition 2.5.** A set-valued mapping  $T : X \rightarrow 2^X$  is said to have approximate values in  $X$  if  $Tx$  is an approximation for each  $x \in X$ .

**Definition 2.6** ([11]). A set-valued operator  $T : X \rightarrow 2^X$  is called order closed if for monotone sequences  $x_n \in X$  and  $y_n \in Tx_n$ , with  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$  and  $\lim_{n \rightarrow \infty} d(y_n, y) = 0$  implies  $y \in Tx$ .

**Definition 2.7.** Let  $(X, d)$  be a metric space. If  $T : X \rightarrow 2^X$  is a set-valued mapping, then  $x \in X$  is called fixed point for  $T$  if and only if  $x \in F(x)$ , the set

$$Fix(T) := \{x \in X \mid x \in Tx\}$$

is called the fixed point set of  $T$ .

Throughout this paper, we always assume that all set-valued operators have approximate values. We have the following result.

Finally, we should emphasize that throughout this paper we suppose that all set-valued mappings on a metric space  $(X, d)$  have closed values.

### 3. Main Results

Now, we are ready to state and prove our main results. Fixed point theorems for order closed set-valued mappings.

**Lemma 3.1.** Let  $(X, d)$  be a GMS. Suppose that  $T, S : X \rightarrow 2^X$  are  $\alpha_*$ - $\psi$ -common rational type contractive set-valued mappings satisfies the following conditions:

- (i)  $T, S$  are  $\alpha_*$ -common admissible;

(ii) there exists  $x_0 \in X$  such that

$$\alpha_*({x_0}, Tx_0) \geq 1 \quad \text{or} \quad \alpha_*(Tx_0, {x_0}) \geq 1.$$

Then  $\text{Fix}(T) = \text{Fix}(S)$ .

*Proof.* We first show that any fixed point of  $T$  is also a fixed point of  $S$  and conversely. Since  $\text{Fix}(T) \neq \text{Fix}(S)$ , we may assume there exists  $x^* \in X$  such that  $x^* \in \text{Fix}(T)$ , but  $x^* \notin \text{Fix}(S)$ , since  $D(x^*, Sx^*) > 0$ . Let  $x_0 \in X$  such that  $\alpha_*({x_0}, Tx_0) \geq 1$ . Define the sequence  $\{x_n\}$  in  $X$  by  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$  for all  $n \in \mathbf{N}_0$ . If  $x_{n_0} = x_{n_0+1}$  for some  $n_0 > 1$ , then  $x^* = x_{n_0}$  are a common fixed point for  $T, S$ . So, we can assume that  $x_{2n} \notin Tx_{2n}$  and  $x_{2n+1} \notin Sx_{2n+1}$  for all  $n \in \mathbf{N}_0$ . Since  $T, S$  are  $\alpha_*$ -common admissible, we have

$$\alpha(x_0, x_1) \geq \alpha_*({x_0}, Tx_0) \geq 1 \Rightarrow \alpha_*(Tx_0, Sx_1) \geq 1;$$

$$\alpha(x_1, x_2) \geq \alpha_*(Tx_0, Sx_1) \geq 1 \Rightarrow \alpha_*(Sx_1, Tx_2) \geq 1;$$

$$\alpha(x_2, x_3) \geq \alpha_*(Sx_1, Tx_2) \geq 1 \Rightarrow \alpha_*(Tx_2, Sx_3) \geq 1.$$

Inductively, we have

$$\alpha(x_{2n}, x_{2n+1}) \geq 1 \Rightarrow \alpha_*(Tx_{2n}, Sx_{2n+1}) \geq 1$$

and

$$\alpha(x_{2n+1}, x_{2n+2}) \geq 1 \Rightarrow \alpha_*(Sx_{2n+1}, Tx_{2n+2}) \geq 1$$

for all  $n \in \mathbf{N}_0$ . We obtain

$$\begin{aligned} M(Tx^*, Sx^*) &= \max \left\{ d(x^*, x^*), D(x^*, Tx^*), D(x^*, Sx^*), \frac{D(x^*, Tx^*)D(x^*, Sx^*)}{1 + d(x^*, x^*)}, \right. \\ &\quad \left. \frac{D(x^*, Tx^*)D(x^*, Sx^*)}{1 + H(Tx^*, Sx^*)} \right\} \\ &= D(x^*, Sx^*) \end{aligned} \quad (3)$$

and

$$\begin{aligned} D(x^*, Sx^*) &\leq H(Tx^*, Sx^*) \leq \alpha_*(Tx^*, Sx^*)H(Tx^*, Sx^*) \leq \psi(M(Tx^*, Sx^*)) \\ &\leq \psi(D(x^*, Sx^*)) < D(x^*, Sx^*). \end{aligned} \quad (4)$$

This contradiction establishes that  $\text{Fix}(T) \subseteq \text{Fix}(S)$ . A similar argument establishes the reverse containment, and therefore  $\text{Fix}(T) = \text{Fix}(S)$ .  $\square$

**Theorem 3.2.** Let  $(X, d)$  be a complete GMS,  $T, S : X \rightarrow 2^X$  be a  $\alpha_*$ - $\psi$ -common rational type contractive set-valued mappings and satisfies the following conditions:

- (i)  $T, S$  are  $\alpha_*$ -common admissible;
- (ii) there exists  $x_0 \in X$  such that  $\alpha_*({x_0}, Tx_0) \geq 1, \alpha_*({x_0}, STx_0) \geq 1$ ;
- (iii)  $T$  or  $S$  is order closed.

Then  $T, S$  have common fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$  converges to the common fixed point of  $T, S$ .

*Proof.* By Lemma 3.1, we have  $Fix(T) = Fix(S)$ . Since  $T, S$  are  $\alpha_*$ -common admissible, we have

$$\begin{aligned} \alpha(x_0, x_1) \geq \alpha_*({x_0}, Tx_0) \geq 1 &\Rightarrow \alpha_*(Tx_0, Sx_1) \geq 1; \text{ then} \\ \alpha(x_1, x_2) \geq \alpha_*(Tx_0, Sx_1) \geq 1 &\Rightarrow \alpha_*(Sx_1, Tx_2) \geq 1. \end{aligned} \tag{5}$$

Inductively, we have  $\alpha(x_n, x_{n+1}) \geq 1, \forall n \in \mathbf{N}_0$ . By similar arguments, since  $\alpha_*({x_0}, STx_0) \geq 1$ , we have

$$\begin{aligned} \alpha(x_0, x_2) \geq \alpha_*({x_0}, STx_0) \geq 1 &\Rightarrow \alpha_*(Tx_0, Tx_2) \geq 1; \text{ then} \\ \alpha(x_1, x_3) \geq \alpha_*(Tx_0, Tx_2) \geq 1 &\Rightarrow \alpha_*(Sx_1, Sx_3) \geq 1. \end{aligned} \tag{6}$$

Inductively, we have  $\alpha(x_n, x_{n+2}) \geq 1, \forall n \in \mathbf{N}_0$ . Consider equations (1), (2) with  $x = x_{2n+1}$  and  $y = x_{2n+2}$ . Clearly, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) \leq H(Tx_{2n}, Sx_{2n+1}) &\leq \alpha_*(Tx_{2n}, Sx_{2n+1})H(Tx_{2n}, Sx_{2n+1}) \\ &\leq \psi(M(Tx_{2n}, Sx_{2n+1})), \end{aligned} \tag{7}$$

where

$$\begin{aligned} M(Tx_{2n}, Sx_{2n+1}) &= \max \left\{ d(x_{2n}, x_{2n+1}), D(x_{2n}, Tx_{2n}), D(x_{2n+1}, Sx_{2n+1}), \right. \\ &\quad \left. \frac{D(x_{2n}, Tx_{2n})D(x_{2n+1}, Sx_{2n+1})}{1 + d(x_{2n}, x_{2n+1})}, \frac{D(x_{2n}, Tx_{2n})D(x_{2n+1}, Sx_{2n+1})}{1 + D(Tx_{2n}, Sx_{2n+1})} \right\} \\ &= \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \right. \\ &\quad \left. \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})}, \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \right\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}, \end{aligned} \tag{8}$$

since

$$\begin{aligned} \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+1})} &= \frac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1})} \times d(x_{2n+1}, x_{2n+2}) \\ &\leq d(x_{2n+1}, x_{2n+2}) \end{aligned} \tag{9}$$

and

$$\begin{aligned} \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} &= \frac{d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \times d(x_{2n}, x_{2n+1}) \\ &\leq d(x_{2n}, x_{2n+1}). \end{aligned} \tag{10}$$

If

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2}).$$

So, in general,

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),$$

which is contradiction since  $d(x_{2n+1}, x_{2n+2}) > 0$  thus

$$d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n}, x_{2n+1})).$$

Similarly,

$$d(x_{2n}, x_{2n+1}) \leq \psi(d(x_{2n-1}, x_{2n})).$$

We have

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \leq \dots \leq \psi^n(d(x_0, x_1)), \tag{11}$$

for all  $n \in \mathbb{N}$ . From the property of  $\psi$ , we conclude that  $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ , for all  $n \in \mathbb{N}$ , it is clear that  $\lim_{n \rightarrow \infty} d(x_{n+1}, x_{n+2}) = 0$ . Consider equation (1), (2) with  $x = x_{2n-1}$  and  $y = x_{2n+1}$ . Clearly, we have

$$\begin{aligned} d(x_{2n}, x_{2n+2}) &\leq H(Sx_{2n-1}, Sx_{2n+1}) \leq \alpha_*(Sx_{2n-1}, Sx_{2n+1})H(Sx_{2n-1}, Sx_{2n+1}) \\ &\leq \psi(M(Sx_{2n-1}, Sx_{2n+1})), \end{aligned} \tag{12}$$

where

$$\begin{aligned} M(Sx_{2n-1}, Sx_{2n+1}) &= \max \left\{ d(x_{2n-1}, x_{2n+1}), D(x_{2n-1}, Sx_{2n-1}), D(x_{2n+1}, Sx_{2n+1}), \right. \\ &\quad \frac{D(x_{2n-1}, Sx_{2n-1})D(x_{2n+1}, Sx_{2n+1})}{1 + d(x_{2n-1}, x_{2n+1})}, \\ &\quad \left. \frac{D(x_{2n-1}, Sx_{2n-1})D(x_{2n+1}, Sx_{2n+1})}{1 + H(Sx_{2n-1}, Sx_{2n+1})} \right\} \\ &= \max \left\{ d(x_{2n-1}, x_{2n+1}), d(x_{2n-1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), \right. \\ &\quad \left. \frac{d(x_{2n-1}, x_{2n})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n-1}, x_{2n+1})}, \frac{d(x_{2n-1}, x_{2n})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+2})} \right\}. \end{aligned} \tag{13}$$

From (12), (13) we have  $d(x_{2n+1}, x_{2n+2}) < d(x_{2n-1}, x_{2n})$ . Define  $a_{2n} = d(x_{2n}, x_{2n+2})$  and  $b_{2n} = d(x_{2n}, x_{2n+1})$ . Then

$$M(Sx_{2n-1}, Sx_{2n+1}) = \max \left\{ a_{2n-1}, b_{2n-1}, \frac{b_{2n-1}b_{2n+1}}{1 + a_{2n-1}}, \frac{b_{2n-1}b_{2n+1}}{1 + a_{2n}} \right\}. \tag{14}$$

If  $M(Sx_{2n-1}, Sx_{2n+1}) = b_{2n-1}$ , or  $\frac{b_{2n-1}b_{2n+1}}{1 + a_{2n-1}}$  or  $\frac{b_{2n-1}b_{2n+1}}{1 + a_{2n}}$  then taking  $\limsup$  as  $n \rightarrow \infty$  in (13) and using (14) and upper semi-continuity of  $\psi$ , we get

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} a_{2n} \leq \limsup_{n \rightarrow \infty} \psi(M(Sx_{2n-1}, Sx_{2n+1})) \\ &= \psi \left( \limsup_{n \rightarrow \infty} M(Sx_{2n-1}, Sx_{2n+1}) \right) = \psi(0) = 0 \end{aligned} \tag{15}$$

and hence,  $\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} d(x_{2n}, x_{2n+2}) = 0$ . If  $M(Sx_{2n-1}, Sx_{2n+1}) = a_{2n-1}$ , then (15) implies  $a_{2n} \leq \psi(a_{2n-1}) < a_{2n-1}$  and similarly  $a_{2n+1} \leq \psi(a_{2n}) < a_{2n}$ . By induction, we get  $a_n \leq \psi(a_{n-1}) < a_{n-1}$ , due to the property of  $\psi$ . In other words, the sequence  $a_n$  is positive monotone decreasing, and hence, it converges to some  $t \geq 0$ . Assume that  $t > 0$ . Now, by (15), we get

$$t = \limsup_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} \psi(a_n) = \psi \left( \limsup_{n \rightarrow \infty} a_{n-1} \right) = \psi(t) < t \tag{16}$$

which is a contradiction. Therefore,  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} d(x_n, x_{n+2}) = 0$ . Now, we shall prove that  $x_n \neq x_m$  for all  $n \neq m$ . Assume on the contrary that  $x_n = x_m$  for some  $m, n \in \mathbb{N}$  with  $n \neq m$ . Since  $d(x_p, x_{p+1}) > 0$  for each  $p \in \mathbb{N}$ , without loss of generality, we may assume that  $m > n + 1$ ,  $m = 2k$

and  $n = 2l$  for  $k, l \in \mathbf{N}$ . Substitute again  $x = x_{2l} = x_{2k}$  and  $y = x_{2l+1} = x_{2k+1}$  in (1), (2) which yields

$$\begin{aligned} d(x_{2l}, x_{2l+1}) &= d(x_{2k}, x_{2k+1}) \\ &\leq H(Sx_{2k-1}, Tx_{2k}) \\ &\leq \alpha_*(Sx_{2k-1}, Tx_{2k})H(Sx_{2k-1}, Tx_{2k}) \\ &\leq \psi(M(Sx_{2k-1}, Tx_{2k})), \end{aligned} \tag{17}$$

where

$$\begin{aligned} M(Sx_{2k-1}, Tx_{2k}) &= \max \left\{ d(x_{2k-1}, x_{2k}), D(x_{2k-1}, Sx_{2k-1}), D(x_{2k}, Tx_{2k}), \right. \\ &\quad \left. \frac{D(x_{2k-1}, Sx_{2k-1})D(x_{2k}, Tx_{2k})}{1 + d(x_{2k-1}, x_{2k})}, \frac{D(x_{2k-1}, Sx_{2k-1})D(x_{2k}, Tx_{2k})}{1 + H(Sx_{2k-1}, Tx_{2k})} \right\} \\ &= \max \left\{ d(x_{2k-1}, x_{2k}), d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+1}), \right. \\ &\quad \left. \frac{d(x_{2k-1}, x_{2k})d(x_{2k}, x_{2k+1})}{1 + d(x_{2k-1}, x_{2k})}, \frac{d(x_{2k-1}, x_{2k})d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1})} \right\} \\ &= \max\{d(x_{2k-1}, x_{2k}), d(x_{2k}, x_{2k+1})\}. \end{aligned} \tag{18}$$

If  $M(Sx_{2k-1}, Tx_{2k}) = d(x_{2k-1}, x_{2k})$ , then from (22), implies

$$d(x_{2l}, x_{2l+1}) \leq \psi(d(x_{2k-1}, x_{2k})) \leq \psi^{2k-2l}(d(x_{2l}, x_{2l+1})). \tag{19}$$

If on the other hand  $M(Sx_{2k-1}, Tx_{2k}) = d(x_{2k}, x_{2k+1})$ , then from (17) we have

$$d(x_{2l}, x_{2l+1}) \leq \psi(d(x_{2k}, x_{2k+1})) \leq \psi^{2k-2l+1}(d(x_{2l}, x_{2l+1})). \tag{20}$$

Using the property of  $\psi$ , the two inequalities (19) and (20) imply  $d(x_{2l}, x_{2l+1}) < d(x_{2l}, x_{2l+1})$ , which is impossible. Now, we shall prove that  $\{x_n\}$  is a Cauchy sequence, that is,

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+k}) = 0, \quad \text{for all } k \in \mathbf{N}.$$

We have already proved the cases for  $k = 1$  and  $k = 2$  in (17) and (18), respectively. Take arbitrary  $k \geq 3$ . We discuss two cases.

*Case 1.* Suppose that  $k = 2m + 1$ , where  $m \geq 1$ . Using the quadrilateral inequality (GMS3), we have

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \dots \leq \psi^n(d(x_0, x_1)), \quad \text{for all } n \in \mathbf{N}_0 \tag{21}$$

and

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + \dots + d(x_{n+2m}, x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \psi^p(d(x_0, x_1)) \\ &\leq \sum_{p=n}^{+\infty} \psi^p(d(x_0, x_1)) \rightarrow 0 \end{aligned} \tag{22}$$

as  $n \rightarrow \infty$ .



Case 2. Suppose that  $k = 2m$ , where  $m \geq 2$ . Using the quadrilateral inequality (GMS3), we have

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \dots + d(x_{n+2m-1}, x_{n+2m}) \\ &\leq d(x_n, x_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p(f(d(x_0, x_1))) \\ &\leq d(x_n, x_{n+2}) + \sum_{p=n}^{+\infty} \psi^p(f(d(x_0, x_1))) \rightarrow 0 \end{aligned} \quad (23)$$

as  $n \rightarrow \infty$ . In both of the above cases, we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+k}) = 0,$$

for all  $k \geq 3$ . Fix  $\epsilon > 0$  and let  $n(\epsilon) \in \mathbf{N}_0$  such that

$$\sum_{n=n(\epsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \epsilon. \quad (24)$$

Let  $n, m \in \mathbf{N}_0$  with  $m > n > n(\epsilon)$ . Using the quadrilateral inequality (GMS3), we obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_m) \\ &\leq d(x_n, x_{n+1}) + \dots + d(x_{m-1}, x_m) \\ &= \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) \\ &\leq \sum_{n=n(\epsilon)}^{\infty} \psi^n(d(x_0, x_1)) < \epsilon. \end{aligned} \quad (25)$$

Thus we proved that  $\{x_n\}$  is a Cauchy sequence in the metric space  $(X, d)$ . Since  $(X, d)$  is complete metric space, there exists  $x^* \in X$  such that  $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$ . From the order closed of  $T$ , it follows that  $x_{2n+1} \in Tx_{2n} \rightarrow Tx^*$  as  $\lim_{n \rightarrow \infty} D(x_{2n+1}, Tx^*) = 0$ , due to Proposition 2.1, we conclude that  $x^* \in Tx^*$ . Similarly if  $S$  is order closed, we have  $x^* \in Sx^*$ .  $\square$

**Corollary 3.3.** Let  $(X, d)$  be a complete GMS,  $T : X \rightarrow 2^X$  be a  $\alpha_*$ - $\psi$ -rational type contractive set-valued mappings and satisfies the following conditions:

- (i)  $T$  is  $\alpha_*$ -admissible;
- (ii) there exists  $x_0 \in X$  such that  $\alpha_*({x_0}, Tx_0) \geq 1$ ,  $\alpha_*({x_0}, T^2x_0) \geq 1$ ;
- (iii)  $T$  is order closed.

Then  $T$  has fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{n+1} \in Tx_n$  converges to the fixed point of  $T$ .

**Example 3.1.** Let  $X$  be a finite set defined as  $X = \{1, 2, 3, 4\}$ . Define  $d : X \times X \rightarrow [0, \infty)$  as:

$$\begin{aligned} d(1, 1) &= d(2, 2) = d(3, 3) = d(4, 4) = 0, \\ d(1, 2) &= d(2, 1) = 3, \end{aligned}$$



$$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = 1, \text{ and}$$

$$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = \frac{1}{2}.$$

The function  $d$  is not a metric on  $X$ . Indeed, note

$$3 = d(1, 2) \geq d(1, 3) = d(3, 2) = 1 + 1 = 2,$$

that is, the triangle inequality is not satisfied. However,  $d$  is a generalized metric on  $X$  and moreover  $(X, d)$  is a complete generalized metric space. Define  $T, S : X \rightarrow 2^X$  as:  $T1 = T2 = T3 = \{2, 4\}$ ,  $T4 = \{1, 3\}$  and  $S1 = S2 = S4 = \{2, 3\}$ ,  $S3 = \{1, 2\}$ ,  $\alpha : X \times X \rightarrow [0, +\infty)$ ,  $\alpha_* = \inf \alpha$  as  $\alpha(x, y) = 1$ ,  $\psi(t) = \frac{2}{3}t$ . Clearly,  $T, S$  satisfies the conditions of Theorem 3.2 and has a common fixed point  $x = 2$ .

Now, we prove the following result for self-maps.

**Corollary 3.4.** *Let  $(X, d)$  be a complete GMS,  $T, S : X \rightarrow X$  be a  $\alpha$ - $\psi$ -common rational type contractive mappings and satisfies the following conditions:*

- (i)  $T, S$  are  $\alpha$ -common admissible;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1, \alpha(x_0, STx_0) \geq 1$ ;
- (iii)  $T$  or  $S$  is continuous.

Then  $T, S$  have common fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{2n+1} = Tx_{2n}$  and  $x_{2n+2} = Sx_{2n+1}$  converges to the common fixed point of  $T, S$ .

**Corollary 3.5** ([2]). *Let  $(X, d)$  be a complete GMS,  $T : X \rightarrow X$  be a  $\alpha$ - $\psi$ -rational type contractive mappings and satisfies the following conditions:*

- (i)  $T$  are  $\alpha$ -admissible;
- (ii) there exists  $x_0 \in X$  such that  $\alpha(x_0, Tx_0) \geq 1, \alpha(x_0, T^2x_0) \geq 1$ ;
- (iii)  $T$  is continuous.

Then  $T$  has fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{n+1} = Tx_n$  converges to the fixed point of  $T$ .

#### 4. Fixed Point Theorems for Weakly Increasing and Order Closed Set-Valued Mappings

In the following we provide set-valued versions of the preceding theorem. The results are related to those in ([11]).

Let  $X$  be a topological space and  $\leq$  be a partial order on  $X$ .

**Definition 4.1** ([11]). Let  $A, B$  be two nonempty subsets of  $X$ , the relations between  $A$  and  $B$  are defines follows:

- ( $r_1$ ) If for every  $a \in A$ , there exists  $b \in B$  such that  $a \leq b$ , then  $A <_1 B$ .
- ( $r_2$ ) If for every  $b \in B$  there exists  $a \in A$ , such that  $a \leq b$ , then  $A <_2 B$ .
- ( $r_3$ ) If  $A <_1 B$  and  $A <_2 B$ , then  $A < B$ .

**Definition 4.2** ([7], [8]). Let  $(X, \leq)$  be a partially ordered set. Two mappings  $f, g : X \rightarrow X$  are said to be weakly increasing if  $fx \leq gfx$  and  $gx \leq fgx$  hold for all  $x \in X$ .

Note that, two weakly increasing mappings need not be nondecreasing.

**Example 4.3.** Let  $X = R^+$  endowed with usual ordering. Let  $f, g : X \rightarrow X$  defined by

$$fx = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x < \infty \end{cases} \quad \text{and} \quad gx = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } 1 < x < \infty \end{cases}$$

then it is obvious that  $fx \leq gfx$  and  $gx \leq fgx$  for all  $x \in X$ . Thus  $f$  and  $g$  are weakly increasing mappings. Note that both  $f$  and  $g$  are not nondecreasing.

**Definition 4.4.** ([3]) Let  $(X, \leq)$  be a partially ordered set. Two mapping  $F, G : X \rightarrow 2^X$  are said to be weakly increasing with respect to  $<_1$  if for any  $x \in X$  we have  $Fx <_1 Gy$  for all  $y \in Fx$  and  $Gx <_1 Fy$  for all  $y \in Gx$ . Similarly, two maps  $F, G : X \rightarrow 2^X$  are said to be weakly increasing with respect to  $<_2$  if for any  $x \in X$  we have  $Gy <_2 Fx$  for all  $y \in Fx$  and  $Fy <_2 Gx$  for all  $y \in Gx$ .

Now, we give some examples.

**Example 4.5.** ([3]) Let  $X = [1, \infty)$  and  $\leq$  be usual order on  $X$ . Consider two mappings  $F, G : X \rightarrow 2^X$  defined by  $Fx = [1, x^2]$  and  $Gx = [1, 2x]$  for all  $x \in X$ . Then the pair of mappings  $F$  and  $G$  are weakly increasing with respect to  $<_2$  but not  $<_1$ . Indeed, since

$$Gy = [1, 2y] <_2 [1, x^2] = Fx \quad \text{for all } y \in Fx$$

and

$$Fy = [1, y^2] <_2 [1, 2x] = Gx \quad \text{for all } y \in Gx$$

so  $F$  and  $G$  are weakly increasing with respect to  $<_2$  but  $F2 = [1, 4] >_1 [1, 2] = G1$  for  $1 \in F2$ , so  $F$  and  $G$  are not weakly increasing with respect to  $<_1$ .

**Example 4.6.** ([3]) Let  $X = [1, \infty)$  and  $\leq$  be usual order on  $X$ . Consider two mappings  $F, G : X \rightarrow 2^X$  defined by  $Fx = [0, 1]$  and  $Gx = [x, 1]$  for all  $x \in X$ . Then the pair of mappings  $F$  and  $G$  are weakly increasing with respect to  $<_1$  but not  $<_2$ . Indeed, since

$$Fx = [0, 1] <_1 [y, 1] = Gy \quad \text{for all } y \in Fx$$

and

$$Gx = [x, 1] <_1 [0, 1] = Fy \quad \text{for all } y \in Gx$$

so  $F$  and  $G$  are weakly increasing with respect to  $<_1$  but  $G1 = 1 >_2 0, 1 = F1$  for  $1 \in F1$ , so  $F$  and  $G$  are not weakly increasing with respect to  $<_2$ .

**Theorem 4.1.** Let  $(X, \leq, d)$  be a partially ordered complete GMS. Suppose that  $T, S : X \rightarrow 2^X$  are set-valued mappings and satisfies the following conditions:

- (i)  $H(Ax, By) \leq \psi(M(Ax, By))$  for all  $A, B = T$  or  $S$ ;
- (ii)  $T$  and  $S$  be a weakly increasing pair on  $X$  with respect to  $<_1$ ;

- (iii) there exists  $x_0 \in X$  such that  $\{x_0\} <_1 Tx_0$  and  $\{x_0\} <_1 STx_0$ ;
- (iv)  $T$  or  $S$  is order closed.

Then  $T, S$  have common fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$  converges to the common fixed point of  $T, S$ .

*Proof.* Define the sequence  $x_n$  in  $X$  by  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$  for all  $n \in N_0$ . If  $x_n = x_{n+1}$  for some  $n \in N_0$ , then  $x^* = x_n$  is a common fixed point for  $T, S$ . Using that the pair of set-valued mappings  $T$  and  $S$  is weakly increasing and by define  $\alpha : X \times X \rightarrow [0, +\infty)$

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases}$$

It can be easily shown that the sequence  $x_n$  is nondecreasing with respect to  $\leq$  i.e; and

$$\alpha_*(\{x_0\}, Tx_0) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_0, x_1) \geq 1 \Rightarrow x_0 \leq x_1.$$

Now since  $T$  and  $S$  are weakly increasing with respect to  $<_1$ , we have  $x_1 \in Tx_0 <_1 Sx_1$ . Thus there exist some  $x_2 \in Sx_1$  such that  $x_1 \leq x_2$ . Again since  $T$  and  $S$  are weakly increasing with respect to  $<_1$ , we have  $x_2 \in Sx_1 <_1 Tx_2$ . Thus there exist some  $x_3 \in Tx_2$  such that  $x_2 \leq x_3$ . Continue this process, we will get a nondecreasing sequence  $\{x_n\}_{n=1}^\infty$  which satisfies  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$ ,  $n = 0, 1, 2, 3, \dots$  We get

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_{2n} \leq x_{2n+1} \leq x_{2n+2} \leq \dots$$

In particular  $x_n, x_{n+k}$  are comparable for all  $k \in N$ .  $\alpha(x_n, x_{n+k}) \geq 1$  for all  $n \in N_0$  and by (4) we have  $\lim_{n \rightarrow \infty} d(x_n, x_{n+k}) = 0$ . Following the proof of Theorem 3.2, we know that  $\{x_n\}$  is a Cauchy sequence in the partially ordered complete GMS  $(X, \leq, d)$ . There exists  $x^* \in X$  such that  $\lim_{n \rightarrow +\infty} d(x_n, x^*) = 0$ . In the case, suppose that, for example,  $T$  is a order closed set-valued mappings then we have that  $\lim_{n \rightarrow +\infty} d(Tx_n, Tx^*) = 0$ , which (taking  $n$  even) implies that  $x^* \in Tx^*$ . The proof is similar when  $S$  is a order closed set-valued mappings. Then  $x^*$  is a common fixed point of  $T, S$ . □

**Theorem 4.2.** Let  $(X, \leq, d)$  be a partially ordered complete GMS. Suppose that  $T, S : X \rightarrow 2^X$  are set-valued mappings and satisfies the following conditions:

- (i)  $H(Ax, By) \leq \psi(M(Ax, By))$  for all  $A, B = T$  or  $S$ ;
- (ii)  $T$  and  $S$  be a weakly increasing pair on  $X$  with respect to  $<_2$ ;
- (iii) there exists  $x_0 \in X$  such that  $Tx_0 <_2 \{x_0\}$  and  $STx_0 <_2 \{x_0\}$ ;
- (iv)  $T$  or  $S$  is order closed.

Then  $T, S$  have common fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$  converges to the common fixed point of  $T, S$ .

*Proof.* Define the sequence  $x_n$  in  $X$  by  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$  for all  $n \in N_0$ . If  $x_n = x_{n+1}$  for some  $n \in N_0$ , then  $x^* = x_n$  is a common fixed point for  $T, S$ . Using that the pair of set-valued mappings  $T$  and  $S$  is weakly increasing and by define

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases}$$

It can be easily shown that the sequence  $x_n$  is non increasing with respect to  $\leq$  i.e; and

$$\alpha_*(Tx_0, \{x_0\}) \geq 1 \Rightarrow \exists x_1 \in Tx_0, \text{ such that } \alpha(x_1, x_0) \geq 1 \Rightarrow x_1 \leq x_0.$$

Now since  $T$  and  $S$  are weakly increasing with respect to  $<_2$ , we have  $Sx_1 <_2 Tx_0$ . Thus there exist some  $x_2 \in Sx_1$  such that  $x_2 \leq x_1$ . Again since  $T$  and  $S$  are weakly increasing with respect to  $<_2$ , we have  $Tx_2 \leq_2 Sx_1$ . Thus there exist some  $x_3 \in Tx_2$  such that  $x_3 \leq x_2$ . Continue this process, we will get a non increasing sequence  $\{x_n\}_{n=1}^{\infty}$  which satisfies  $x_{2n+1} \in Tx_{2n}$  and  $x_{2n+2} \in Sx_{2n+1}$ ,  $n = 0, 1, 2, 3, \dots$ . We get

$$x_0 \geq x_1 \geq x_2 \geq \dots \geq x_{2n} \geq x_{2n+1} \geq x_{2n+2} \geq \dots$$

In particular  $x_{n+k}, x_n$  are comparable for all  $k \in N$ ,  $\alpha(x_{n+k}, x_n) \geq 1$  for all  $n \in N_0$  and by (4) we have  $\lim_{n \rightarrow \infty} d(x_{n+k}, x_n) = 0$ . Following the proof of Theorem 3.2, we know that  $\{x_n\}$  is a Cauchy sequence in the partially ordered complete GMS  $(X, \leq, d)$ . There exists  $x^* \in X$  such that  $\lim_{n \rightarrow +\infty} d(x_n, x^*) = 0$ . In the case, suppose that, for example,  $T$  is a order closed multi-valued mappings then we have that  $\lim_{n \rightarrow +\infty} d(Tx_n, Tx^*) = 0$ , which (taking  $n$  even) implies that  $x^* \in Tx^*$ . The proof is similar when  $S$  is a order closed set-valued mappings. Then  $x^*$  is a common fixed point of  $T, S$ .  $\square$

**Corollary 4.3.** Let  $(X, \leq, d)$  be a partially ordered complete GMS. Suppose that  $T : X \rightarrow 2^X$  is set-valued mapping and satisfies the following conditions:

- (i)  $H(Tx, Ty) \leq \psi(M(Tx, Ty))$ ;
- (ii)  $T$  and  $i_x$  be a weakly increasing pair on  $X$  with respect to  $<_1$  or  $<_2$ ;
- (iii) there exists  $x_0 \in X$  such that  $\{x_0\} <_1 Tx_0$  and  $\{x_0\} <_1 T^2x_0$ , or
- (iii)\*  $Tx_0 <_2 \{x_0\}$  and  $T^2x_0 <_2 \{x_0\}$ ;
- (iv)  $T$  is order closed.

Then  $T$  has fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{n+1} \in Tx_n$  converges to the fixed point of  $T$ .

Now, we prove the following result for self-maps.

**Corollary 4.4.** Let  $(X, \leq, d)$  be a partially ordered complete GMS. Suppose that  $T, S : X \rightarrow X$  are self-mappings and satisfies the following conditions:

- (i)  $d(Ax, By) \leq \psi(M(Ax, By))$  for all  $A, B = T$  or  $S$ ;
- (ii)  $T$  and  $S$  be a weakly increasing pair on  $X$  with respect to  $<_1$  or  $<_2$ ;
- (iii) there exists  $x_0 \in X$  such that  $x_0 <_1 Tx_0$  and  $x_0 <_1 STx_0$  or
- (iii)\*  $Tx_0 <_2 x_0$  and  $STx_0 <_2 x_0$ ;
- (iv)  $T$  or  $S$  is continuous.

Then  $T, S$  have common fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{2n+1} = Tx_{2n}$  and  $x_{2n+2} = Sx_{2n+1}$  converges to the common fixed point of  $T, S$ .

**Corollary 4.5.** Let  $(X, \leq, d)$  be a partially ordered complete GMS. Suppose that  $T : X \rightarrow X$  is self-mapping and satisfies the following conditions:

- (i)  $d(Tx, Ty) \leq \psi(M(Tx, Ty))$ ;
- (ii)  $T$  and  $i_x$  be a weakly increasing pair on  $X$  with respect to  $<_1$  or  $<_2$ ;
- (iii) there exists  $x_0 \in X$  such that  $x_0 <_1 Tx_0$  and  $x_0 <_1 T^2x_0$  or  
(iii)\*  $Tx_0 <_2 x_0$  and  $T^2x_0 <_2 x_0$ ;
- (iv)  $T$  is continuous.

Then  $T$  has fixed point  $x^* \in X$ . Further, for each  $x_0 \in X$ , the iterated sequence  $\{x_n\}$  with  $x_{n+1} = Tx_n$  converges to the fixed point of  $T$ .

## 5. Coupled Fixed Point Theorem

Recall that a function  $\eta : R_+ \rightarrow R_+$  is said to be super-additive if  $\eta(s) + \eta(t) \leq \eta(s + t)$  for all  $s, t \in R_+$ .

It is well-known that every nondecreasing, convex function  $\eta : R_+ \rightarrow R_+$  with  $\eta(0) = 0$  is super-additive (cf. [4]).

**Definition 5.1** ([12]). Let  $F : X \times X \rightarrow X$  be a mapping, where  $(X, d)$  is a metric space. We say that  $(x, y) \in X \times X$  is a coupled fixed point of  $F$  if

$$x = F(x, y), \quad y = F(y, x).$$

Note that if  $(x, y)$  is a coupled fixed point of  $F$  then  $(y, x)$  are coupled fixed points of  $F$  too. Our results are based on the following simple lemma.

**Lemma 5.1** ([18]). Let  $F : X \times X \rightarrow X$  be a given mapping. Define the mapping  $T_F : X \times X \rightarrow X \times X$  by  $T_F(x, y) = (F(x, y), F(y, x))$  for all  $(x, y) \in X \times X$ . Then,  $(x, y)$  is a coupled fixed point of  $F$  if and only if  $(x, y)$  is a fixed point of  $T_F$ .

**Theorem 5.2.** Let  $(X, d)$  be a complete GMS and  $F : X \times X \rightarrow X$  be a given continuous mapping. Assume there are exist nondecreasing functions  $\psi_i : [0, +\infty) \rightarrow [0, +\infty)$ ,  $i = 1, 2$ , such that  $\psi = \psi_1 + \psi_2$  is convex,  $\psi(0) = 0$ ,  $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$  for all  $t > 0$ , a function  $\alpha : X^2 \times X^2 \rightarrow [0, +\infty)$  and satisfies the following conditions:

- (i) for all  $(x, y), (u, v) \in X \times X$ ,

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi_1(d(x, u)) + \psi_2(d(y, v));$$

- (ii) if for all  $(x, y), (u, v) \in X \times X$ ,

$$\alpha((x, y), (u, v)) \geq 1 \Rightarrow \alpha(T_F(x, y), T_F(u, v)) \geq 1;$$

- (iii) there exists  $(x_0, y_0) \in X \times X$  such that

$$\alpha((x_0, y_0), T_F(x_0, y_0)) \geq 1 \text{ and } \alpha((x_0, y_0), T_F^2(x_0, y_0)) \geq 1; \text{ or}$$

- (iii)\* there exists  $(x_0, y_0) \in X \times X$  such that

$$\alpha(T_F(x_0, y_0), (x_0, y_0)) \geq 1 \text{ and } \alpha(T_F^2(x_0, y_0), (x_0, y_0)) \geq 1.$$

Then,  $F$  has a coupled fixed point, that is, there exists  $(x^*, y^*) \in X \times X$  such that  $x^* = F(x^*, y^*)$  and  $y^* = F(y^*, x^*)$ .

*Proof.* The idea consists in transporting the problem to the complete GMS  $(Y, \delta)$ , where  $Y = X \times X$  and  $\delta((x, y), (u, v)) = d(x, u) + d(y, v)$ , for all  $(x, y), (u, v) \in X \times X$ . From condition (i), we have

$$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq \psi_1(d(x, u)) + \psi_2(d(y, v)) \quad (26)$$

and

$$\alpha((v, u), (y, x))d(F(v, u), F(y, x)) \leq \psi_1(d(v, y)) + \psi_2(d(u, x)) \quad (27)$$

for all  $x, y, u, v \in X$ . Adding (26) to (27), we get (note that  $\psi$  is super-additive)

$$\begin{aligned} \beta(\xi, \eta)\delta(T_F\xi, T_F\eta) &\leq \psi_1(d(\xi_1, \eta_1)) + \psi_2(d(\xi_2, \eta_2)) + \psi_1(d(\eta_2, \xi_2)) + \psi_2(d(\eta_1, \xi_1)) \\ &\leq \psi_1(d(\xi_1, \eta_1) + d(\eta_2, \xi_2)) + \psi_2(d(\xi_2, \eta_2) + d(\eta_1, \xi_1)) \\ &= \psi(d(\xi_1, \eta_1) + d(\eta_2, \xi_2)) \\ &= \psi(\delta(\xi, \eta)) \end{aligned} \quad (28)$$

for all  $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in Y$ , where  $\beta : Y \times Y \rightarrow [0, +\infty)$  is the function defined by

$$\beta((\xi_1, \xi_2), (\eta_1, \eta_2)) = \min\{\alpha((\xi_1, \xi_2), (\eta_1, \eta_2)), \alpha((\eta_2, \eta_1), (\xi_2, \xi_1))\} \quad (29)$$

and  $T_F : Y \rightarrow Y$  is given by Lemma 5.1. Let  $\{(x_n, y_n)\}$  be a sequence in  $Y = X \times X$  such that  $\beta((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1$  and  $(x_n, y_n) \rightarrow (x, y)$  as  $n \rightarrow +\infty$ . Then  $T_F$  is continuous and  $\beta$ - $(\psi_1 + \psi_2)$ -contractive mapping. Let  $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in Y, \beta(\xi, \eta) \geq 1$ . Using condition (ii), we obtain immediately that  $\beta(T_F\xi, T_F\eta) \geq 1$ . Then  $T_F$  is  $\beta$ -admissible. Moreover, from condition (iii), we know that there exists  $(x_0, y_0) \in Y$  such that  $\beta((x_0, y_0), T_F(x_0, y_0)) \geq 1$  and  $\beta((x_0, y_0), T_F^2(x_0, y_0)) \geq 1$ . All the hypotheses of Corollary 3.4 are satisfied, and so we deduce the existence of a fixed point of  $T_F$  that gives us the existence of a coupled fixed point of  $F$ .  $\square$

## 6. Application to Nonlinear Integral Equation

In this section, we prove the existence for certain nonlinear integral equations and result for a fractional-order integral equation.

$$x(t) = p(t) + f(t, x(t), x(t)) \int_0^t k(t, s)g(s, x(s), x(s))ds, \quad t \in [0, T], \quad (30)$$

where  $T > 0$  and  $p : [0, T] \rightarrow \mathbf{R}$ . We suppose that the following conditions are satisfied.

- (i) The function  $f, g : [0, T] \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous and  $k : [0, T] \times [0, T] \rightarrow [0, +\infty)$  is a function such that  $k(t, \cdot) \in L^1([0, T])$  for all  $t \in [0, T]$ .
- (ii) There exists an upper semi-continuous function  $\psi_i : [0, +\infty) \rightarrow [0, +\infty), i = 1, 2$ , are nondecreasing functions such that  $\psi = \psi_1 + \psi_2$  is convex,  $\psi(0) = 0$ , and  $\lim_{n \rightarrow \infty} \psi^n(t) = 0$  for each  $t > 0$ , suppose that for  $x \geq u$  and  $y \geq v$ , we have

$$0 \leq g(t, x, y) - g(t, u, v) \leq \frac{1}{F_0}(\psi_1(x - u) + \psi_2(y - v)), \quad (31)$$

where

$$F_0 = \max\{|f(t, 0, 0)| : t \in [0, T]\}.$$

- (iii) for every  $s \in [0, T]$ , we have  $\left\| \int_0^T k(t, s)ds \right\|_{\infty} < 1$ .



**Theorem 6.1.** Consider nonlinear integral equations (30) with  $g \in C([0, T] \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$  is  $C^1$  and nondecreasing in the third variables. Then the fractional integral equation (30) with the assumptions (i)-(iii) has at least one solution  $x^* \in C([0, T], \mathbf{R})$ .

*Proof.* Let  $X = C([0, T], \mathbf{R})$  is partially ordered if we define the following order relation in  $X$ :

$$x, y \in X, \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad \text{for all } t \in [0, T].$$

It is well-known that  $(X, d)$  is a complete metric space with the metric

$$d(x, y) = \sup_{t \in [0, T]} |x(t) - y(t)|, \quad x, y \in C([0, T], \mathbf{R}).$$

Suppose  $\{x_n\}$  is a nondecreasing sequence in  $X$  that converges to  $x \in X$ . Then for every  $t \in [0, T]$ , the sequence of the real numbers

$$x_1(t) \leq x_2(t) \leq \dots \leq x_n(t) \leq \dots,$$

converges to  $x(t)$ . Therefore, for all  $t \in I$  and  $n \in N$ , we have  $x_n(t) \leq x(t)$ . Hence  $x_n \leq x$ , for all  $n \in N$ . Also,  $X \times X$  is a partially ordered set if we define the following order relation in  $X \times X$ :

$$(x, y) \leq_r (u, v) \Leftrightarrow x(t) \leq u(t) \text{ and } y(t) \leq v(t), \quad \text{for all } t \in [0, T],$$

for all  $(x, y), (u, v) \in X \times X$ . For any  $x, y \in X$ ,  $\max\{x(t), u(t)\}$  for all  $t \in [0, T]$  is in  $X$  and is the upper bound of  $x, u$ . Therefore, for every  $(x, y)$  and  $(u, v) \in X \times X$ ,  $\max\{x(t), u(t)\}, \max\{y(t), v(t)\}$ , in  $X \times X$  for all  $t \in [0, T]$  is comparable to  $(x, y)$  and  $(u, v)$ .

Define  $F : X \times X \rightarrow X$  by

$$F(x, y)(t) = p(t) + f(t, x(t), y(t)) \int_0^t k(t, s)g(s, x(s), y(s))ds, \quad \text{for all } t \in [0, T], \tag{32}$$

Since  $f$  is nondecreasing in the second and third of its variables then  $F$  is nondecreasing in each of its variables.

Now, for  $x \geq u, y \geq v$ , that is,  $x(t) \geq u(t), y(t) \geq v(t)$  for all  $t \in [0, T]$ . We have

$$\begin{aligned} d(F(x, y), F(u, v)) &= \sup_{t \in [0, T]} |F(x, y)(t) - F(u, v)(t)| \\ &= \sup_{t \in [0, T]} \left| \left\{ f(t, x(t), y(t)) \int_0^t k(t, s)g(s, x(s), y(s))ds \right. \right. \\ &\quad \left. \left. - f(t, u(t), v(t)) \int_0^t k(t, s)g(s, u(s), v(s))ds \right\} \right| \\ &\leq \sup_{t \in [0, T]} \left| \left\{ F_0 \left( \int_0^t k(t, s)g(s, x(s), y(s))ds - \int_0^t k(t, s)g(s, u(s), v(s))ds \right) \right\} \right| \\ &= \sup_{t \in [0, T]} \left| \left\{ F_0 \int_0^t k(t, s)(g(s, x(s), y(s)) - g(s, u(s), v(s)))ds \right\} \right| \\ &\leq \sup_{t \in [0, T]} \left| \left\{ F_0 \left( \int_0^t k(t, s) \frac{\psi_1(x - u) + \psi_2(y - v)}{F_0} ds \right) \right\} \right| \\ &= \sup_{t \in [0, T]} \left| \left\{ F_0 \times \frac{\psi_1(x - u) + \psi_2(y - v)}{F_0} \times \int_0^t k(t, s)ds \right\} \right| \end{aligned}$$



$$\begin{aligned}
&\leq \sup_{t \in [0, T]} \left\| \left\{ (\psi_1(x-u) + \psi_2(y-v)) \int_0^t k(t,s) ds \right\} \right\| \\
&= (\psi_1(x-u) + \psi_2(y-v)) \sup_{t \in [0, T]} \left\| \int_0^t k(t,s) ds \right\| \\
&\leq (\psi_1(x-u) + \psi_2(y-v)) \left\| \int_0^t k(t,s) ds \right\|_{\infty} \\
&\leq \psi_1(d(x,u)) + \psi_2(d(y,v)). \tag{33}
\end{aligned}$$

Thus  $F$  satisfies the condition of Theorem 5.2. Now, let  $(x^*, y^*)$  be a coupled lower solution of certain nonlinear integral equations problem (30) then we have  $x^* \leq F(x^*, y^*)$  and  $y^* \leq F(y^*, x^*)$ . Then, Theorem 6.1 gives that  $F$  has a unique coupled fixed point  $(x^*, y^*)$  with  $x^* = y^*$ . Then  $x^*(t)$  is the solution of certain nonlinear integral equations (30).  $\square$

An existence result for a fractional integral equation

$$x(t) = \frac{f(t, x(t), x(t))}{\Gamma(\alpha)} \int_0^t \frac{h'(s)g(s, x(s), x(s))}{(h(t) - h(s))^{1-\alpha}} ds, \quad t \in [0, T], \tag{34}$$

where  $T > 0$ ,  $\alpha \in (0, 1)$ ,  $h : [0, T] \rightarrow \mathbf{R}$  and  $\Gamma$  is the Euler gamma function given by  $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ . We suppose that the following conditions are satisfied.

(iv) The function  $h : [0, T] \rightarrow \mathbf{R}$  is  $C^1$  and nondecreasing.

(v) The function  $g : [0, T] \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous and there exists a nondecreasing function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that

$$|g(t, x(t), y(t))| \leq \omega(|(x(t), y(t))|) \quad (t, x(t), y(t)) \in [0, T] \times \mathbf{R} \times \mathbf{R}.$$

(vi) There exists  $r_0 > 0$  such that

$$(\psi(r_0) + F_0)\omega(r_0)(g(T) - g(0))^\alpha \leq r_0\Gamma(\alpha + 1) \quad \text{and} \quad \frac{\omega(r_0)}{\Gamma(\alpha + 1)} \times (g(T) - g(0))^\alpha \leq 1. \tag{35}$$

**Corollary 6.2** ([15]). Consider fractional-order integral equation (30) with  $g \in C([0, T] \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$  is  $C^1$  and nondecreasing in the third variables. Suppose that for  $x \geq u$  and  $y \geq v$ , we have

$$0 \leq g(t, x, y) - g(t, u, v) \leq \frac{\Gamma(\alpha + 1)}{F_0(h(t) - h(s))^\alpha} (\psi_1(x-u) + \psi_2(y-v)), \tag{36}$$

Then the fractional-order integral equation (30) with the assumptions (i)-(iii) has at least one solution  $x^* \in C([0, T], \mathbf{R})$ .

*Proof.* Let  $X = C([0, T], \mathbf{R})$  is partially ordered if we define in Theorem 6.1,

$$k(t, s) = \frac{h'(s)(h(t) - h(s))^{\alpha-1}}{\Gamma(\alpha)}$$

for all  $s, t \in [0, T]$ .  $\square$

We consider the set of all closed bounded real continuous function on  $[0, 1]$ , say  $X = CB([0, 1], \mathbf{R})$ , endowed with the metric  $d : X \times X \rightarrow \mathbf{R}$  given by

$$d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|, \quad \text{for all } x, y \in X.$$

Clearly,  $(X, d)$  is a complete metric space, which can be equipped with the graph  $G(X, E)$  with  $V = X$  and  $E \subset X \times X$  given by

$$(x, y) \in E \Leftrightarrow x(t) \leq y(t) \text{ for all } t \in [0, 1].$$

Thus,  $(G, d)$  is regular (see in [16]).

**Corollary 6.3** ([16]). *Let  $\Lambda : X \rightarrow X$  be the integral operator defined by*

$$\Lambda(x)(t) = p(t) + \int_0^t \frac{(t-u)^{\alpha-1}}{g(u, y(u))} g(u, x(u)) du, \quad t \in [0, 1], \alpha \in (0, 1). \quad (37)$$

*Suppose that the following conditions hold:*

- (a) *there exists  $x_0 \in X$  such that  $(x_0, \Lambda(x_0)) \in E$ ;*
- (b)  *$g(u, \cdot) : \mathbf{R} \rightarrow \mathbf{R}$  is increasing, for every  $u \in [0, 1]$ ;*
- (c) *for  $x, y \in X$  with  $(x, y) \in E$ , we have*

$$0 \leq g(u, y(u)) - g(u, x(u)) \leq \frac{\Gamma(\alpha+1)}{2} \ln(1 + |x(u) - y(u)|), \quad \text{for all } u \in [0, 1]. \quad (38)$$

*Then  $\Lambda$  has a fixed point.*

*Proof.* We define in Theorem 6.1,

$$k(t, s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$$

for all  $s, t \in [0, 1]$ . □

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