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Research Article

Method of Reduction of Order for Solving Singularly Perturbed Delay Differential Equations

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Abstract. In this paper, we have presented and illustrated the method of reduction of order for solving singularly perturbed delay differential equations. The given second order singularly perturbed delay differential equation is replaced by a pair of first order problems. These are in turn solved by initial value solvers. The integration of these initial value problems goes in the opposite direction. The applicability of this method is demonstrated by solving some model problems and the numerical results are compared with the exact solution. From the tables and figures, it is observed that the present method produces satisfactory results.

Keywords. Singular perturbations; Delay differential equations; Reduction of order

MSC. 65L11; 65Q10

1. Introduction

Singularly perturbed delay differential equation is a differential equation in which the highest order derivative is multiplied by a small parameter and involving a delay term. This type of equation arises frequently in the mathematical modelling of various practical phenomena for example: in the modelling of the human pupil-light reflex; model of HIV infection; the study of bi-stable devices in digital electronics; variational problem in control theory; first exit time problem in modelling of activation of neuronal variability; immune response; evolutionary biology;

dynamics of networks of two identical amplifier; mathematical ecology; population dynamics; the modelling of biological oscillator and in a variety of models for physiological process.For a detailed theory and analytical discussion on delay differential equations having boundary layer one may refer the popular books by Bellman and Cooke [2], and El'sgol'ts and Norkin [6]. Lange and Miura [7] are the first to discuss the behavior of the analytical solution of singularly perturbed differential difference equations. Doolan et al. [4] have described the uniform methods for solving various classes of these problems. Books by the authors: O'Malley [9], Bender and Orszag [3], Nayfeh [8], and Driver [5] remain the best source for the researchers in this area of singular perturbation theory. Reddy and Chakravarthy [10] have presented an easy and efficient initial value numerical method for solving singularly perturbed two point boundary value problems. Recently, Adilaxmi, Bhargavi and Reddy [1] have presented an Initial Value Technique using Exponentially Fitted Non-Standard Finite Difference Method for Singularly Perturbed Differential-Difference Equations. In this paper, we have presented and illustrated the method of reduction of order for solving singularly perturbed delay differential equations. The given second order singularly perturbed delay differential equation is replaced by a pair of first order problems. These are in turn solved by initial value solvers. The integration of these initial value problems goes in opposite direction. The applicability of this method is demonstrated by solving some model problems and the numerical results are compared with exact solution. From the tables and figures it is observed that the present method produces satisfactory results.

2. Method of Reduction of Order for Singularly Perturbed Delay Differential Equations

We consider a singularly perturbed delay differential equation of the form

$$-\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-1)=f(x), \quad x\in[-1,2] \tag{2.1}$$

with boundary conditions

$$y(x)=\phi(x), \;\; x\in[-1,0] \;\text{ and }\; y(2)=\beta, \tag{2.2}$$

where $0<\varepsilon\ll 1$ is a small positive parameter and ϕ, β are known function and constants. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions in the interval $[-1,2]$. Furthermore, we assume that $a(x)\geq M>0$ throughout the interval $[-1,2]$, where M is positive constant. This assumption merely implies that the boundary layer will be in the neighbourhood of $x=2$.

The method of reduction of order consists of the following steps:

Step 1: First we obtain the reduced problem by setting $\varepsilon=0$ in eq. (2.1) and solve it for the solution with the approximate boundary condition. Let $y_0(x)$ be the solution of the reduced problem of eq. (2.1) and eq. (2.2), i.e.:

$$a(x)y_0'(x)+b(x)y_0(x-1)=f(x), \quad x\in[0,1] \tag{2.3}$$

with

$$y_0(x) = \phi(x), \quad x \in [-1,0]. \tag{2.4}$$

By using the classical fourth order Runge-Kutta method we

$y_0(x)$ when $x \in [0,1]$ by $y_0(x) = \gamma$ for $x \in [-1,0]$.

Now, we solve

$$a(x)y_0'(x) + b(x)y_0(x-1) = f(x), \quad x \in [0,1], \tag{2.5}$$

$$y_0(x) = \gamma, \quad x \in [-1,0] \tag{2.6}$$

using classical fourth order Runge-Kutta method to get $y_0(x)$ when $x \in [0,2]$.

Step 2: Now, we setup the two first order differential equations equivalent to Eq. (2.1) as follows:

$$z'(x) - a'(x)y(x) + b(x)y(x-1) = f(x) \tag{2.7}$$

and

$$-\varepsilon y'(x) + a(x)y(x) = z(x), \quad x \in [0,2]. \tag{2.8}$$

Step 3: We setup the initial conditions as follows:

Using $y_0(x)$ and $y_0(1)$, the solution of the reduced problem, in eq. (2.8) we have

$$z(0) = -\varepsilon y_0'(0) + a(0)y_0(0), \tag{2.9}$$

where

$$y'(0) = \frac{[f(0) - b(0)\phi(-1)]}{a(0)}.$$

From $a(x)y_0'(x) = f(x) - b(x)\phi(x-1)$, we get:

$$z(1) = -\varepsilon y_0'(1) + a(1)y_0(1). \tag{2.10}$$

This will be the initial condition for eq. (2.7) and $y(2) = \beta$ will be the initial condition for eq. (2.8) when $x \in [1,2]$.

Step 4: The pair of initial values problems are as follows:

$$-\varepsilon y'(x) + a(x)y(x) = z(x), \quad \text{with } y(2) = \beta,\ x \in [1,2], \tag{2.11}$$

$$-\varepsilon y'(x) + a(x)y(x) = z(x), \quad \text{with } y(1) = \gamma,\ x \in [0,1]. \tag{2.12}$$

Thus in a manner of speaking, we have replaced the original boundary value problem given by (2.1)-(2.2) by a pair of initial value problems. The integration of these initial value problems goes in opposite direction, and the second problem (2.12) is solved only if the solution of the first one (2.11) is known. We solve these initial value problems (2.11) and (2.12) to obtain the solution over the interval $[0,2]$. There now exist a number of efficient methods for the solution of initial value problems. In order to solve the initial value problems in our numerical experimentation, we make use of classical fourth order Runge-Kutta method. In fact, any standard analytical or numerical methods can be used.

3. Numerical Examples

Example 1. We consider the singularly perturbed delay differential equation

$$-\varepsilon y''(x)+3y'(x)-y(x-1)=0, \quad x\in[-1,2] \tag{3.1}$$

with boundary conditions $y(x)=1$ for $x\in[-1,0]$ and $y(2)=2$.

The exact solution of the problem is given by

$$y(x)=\begin{cases}1+c_1\left[e^{\left(\frac{3x}{\varepsilon}\right)}-1\right]+\frac{x}{3}, & 0\le x\le 1,\\ c_2+\frac{x}{3}+\frac{(x-1)^2}{18}+\frac{\varepsilon x}{27}-\frac{c_1x}{3}-\frac{c_1x}{3}e^{\left(\frac{3(x-1)}{\varepsilon}\right)}+e^{\left(\frac{3(x-2)}{\varepsilon}\right)}\left[\frac{23}{18}-\frac{2\varepsilon}{27}-c_2+\frac{2c_1}{3}+\frac{2c_1}{3}e^{\left(\frac{3}{\varepsilon}\right)}\right], & 1\le x\le 2,\end{cases}$$

where

$$c_1=e^{\left(-\frac{6}{\varepsilon}\right)}\left[\frac{\frac{4\varepsilon}{9}-\frac{\varepsilon^2}{27}-3}{3-4e^{\left(-\frac{6}{\varepsilon}\right)}+\frac{2\varepsilon}{3}\left[e^{\left(-\frac{3}{\varepsilon}\right)}-e^{\left(-\frac{6}{\varepsilon}\right)}\right]}\right],$$

$$c_2=\frac{1-\frac{22}{18}e^{\left(-\frac{3}{\varepsilon}\right)}+\frac{2\varepsilon}{27}e^{\left(-\frac{3}{\varepsilon}\right)}-\frac{\varepsilon}{27}}{1-e^{\left(-\frac{3}{\varepsilon}\right)}}+\frac{c_1e^{\left(\frac{3}{\varepsilon}\right)}\left[1-e^{\left(-\frac{3}{\varepsilon}\right)}-\frac{2}{3}e^{\left(-\frac{6}{\varepsilon}\right)}\right]}{1-e^{\left(-\frac{3}{\varepsilon}\right)}}.$$

From Step 1 the reduced problem is

$$3y_o'(x)-y_o(x-1)=0, \quad y_o(0)=1,$$

$$y_o'(x)=\frac{y_o(x-1)}{3}, \quad \text{for } x\in[0,1] \text{ with } y_o(0)=1.$$

The solution of this problem is

$$y_o(x)=\frac{x}{3}+1,$$

$$3y_o'(x)=y_o(x-1),$$

$$y_o'(x)=\frac{\left(\frac{x-1}{3}+1\right)}{3}, \quad \text{for } x\in[1,2] \text{ with } y_o(1)=\frac{4}{3}.$$

The solution of this problem is

$$y_o(x)=\frac{x^2}{18}+\frac{2x}{9}+\frac{19}{18}.$$

Now

$$y_o(x)=\frac{x}{3}+1, \quad x\in[0,1],$$

$$y_o(x)=\frac{x^2}{18}+\frac{2x}{9}+\frac{19}{18}, \quad x\in[1,2].$$

From Step 2, the two first order equation equivalent to eq. (3.1)

$$-\varepsilon y'(x)+3y(x)=z(x),$$

$$z'(x)=-\varepsilon y''(x)+3y'(x), \quad \text{where } z'(x)=y(x-1).$$

Now from Step 3, we have

$$z(0)=-\varepsilon y_o'(0)+3y_o(0) \text{ i.e., } z(0)=3-\frac{\varepsilon}{3}$$

resulting the solution

$$z(x) = x + 3 - \frac{\varepsilon}{3}, \quad \text{for } x \in [0,1],$$

$$z(1) = -\varepsilon y_o'(1) + 3y_o(1) \quad \text{i.e., } z(1) = 4 - \frac{\varepsilon}{3}$$

resulting the solution

$$z(x) = \frac{x^2}{6} - \frac{x}{3} + x + \frac{19}{6} - \frac{\varepsilon}{3}, \quad \text{for } x \in [1,2].$$

Hence the pair of initial value problems are:

$$-\varepsilon y'(x) + 3y(x) = z(x), \quad \text{with } y(2) = 2 \text{ for } x \in [1,2]$$

and

$$-\varepsilon y'(x) + 3y(x) = z(x), \quad \text{with } y(1) = \frac{4}{3} \text{ for } x \in [0,1].$$

The numerical results are presented in Table 1 for this example and the solution is plotted in Figure 1 for $\varepsilon = 2^{-8}$, $N = 23$.

Table 1. Numerical results for Example 1 for $\varepsilon = 2^{-8}$, $N = 23$

x	Present solution	Exact solution	Error
0	1.0000	1.0000	0.0000
0.01	1.0033	1.0034	0.0001
0.02	1.0067	1.0077	0.0010
0.03	1.0100	1.0102	0.0002
0.04	1.0133	1.0136	0.0003
0.05	1.0177	1.0182	0.0005
0.06	1.6886	1.6897	0.0011
0.07	1.6941	1.6932	0.0009
0.08	1.7044	1.7066	0.0022
0.09	1.7053	1.7076	0.0023
1.00	1.7069	1.7099	0.0030
1.20	1.7110	1.7120	0.0010
1.30	1.7154	1.7156	0.0002
1.40	1.7261	1.7265	0.0004
1.50	1.7282	1.7285	0.0004
1.60	1.7429	1.7439	0.0010
1.70	1.7629	1.7635	0.0006
1.80	1.8001	1.8005	0.0004
1.90	1.9708	1.9710	0.0002
2.00	2.0000	2.0000	0.0000

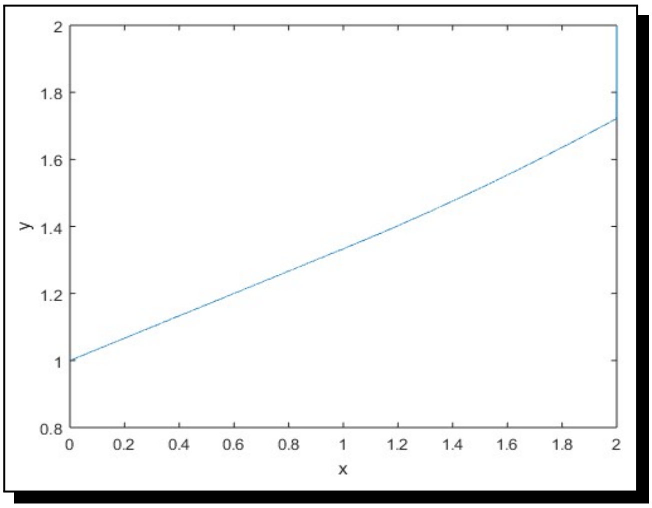

Figure 1. Numerical solution for Example 1 with $\varepsilon = 2^{-8}$, $N = 23$

Example 2. We consider the singularly perturbed delay differential equation

$$-\varepsilon y''(x) + 2y'(x) - 5y(x-1) = 0, \quad x \in [-1,2] \tag{3.2}$$

with boundary conditions $y(x) = 1$ for $x \in [-1,0]$ and $y(2) = 2$.

From Step 1 the reduced problem is

$$2y_o'(x) - 5y_o(x-1) = 0, \quad y_o(0) = 1,$$

$$y_o'(x) = \frac{5y_o(x-1)}{2}, \quad \text{for } x \in [0,1] \text{ with } y_o(0) = 1.$$

The solution of this problem is

$$y_o(x) = \frac{5x}{2} + 1,$$

$$2y_o'(x) = 5y_o(x-1),$$

$$y_o'(x) = \frac{25(x-1)}{4} + \frac{5}{2}, \quad \text{for } x \in [1,2] \text{ with } y_o(1) = \frac{7}{2}.$$

The solution of this problem is

$$y_o(x) = \frac{25}{4}\left(\frac{x^2}{2} - x\right) + \frac{5x}{2} + \frac{33}{8}.$$

Now

$$y_o(x) = \frac{5x}{2} + 1, \quad x \in [0,1],$$

$$y_o(x) = \frac{25}{4}\left(\frac{x^2}{2} - x\right) + \frac{5x}{2} + \frac{33}{8}, \quad x \in [1,2].$$

From Step 2, the two first order equation equivalent to eq. (3.2)

$$-\varepsilon y'(x) + 2y(x) = z(x),$$

$$z'(x) = -\varepsilon y''(x) + 2y'(x), \quad \text{where } z'(x) = 5y(x-1).$$

Now, from Step 3, we have

$$z(0) = -\varepsilon y_o'(0) + 2y_o(0) \quad \text{i.e., } z(0) = \frac{4-5\varepsilon}{2}$$

resulting the solution

$$z(x) = 5x + \frac{4-5\varepsilon}{2},$$

$$z(1) = -\varepsilon y_o'(1) + 2y_o(1) \quad \text{i.e., } z(1) = 5 + \frac{4-5\varepsilon}{2}$$

resulting the solution

$$z(x) = \frac{25}{2}\left(\frac{x^2}{2} - x\right) + 5x + \frac{33-10\varepsilon}{4}.$$

Hence the pair of initial value problems are:

$$-\varepsilon y'(x) + 2y(x) = z(x), \quad \text{with } y(2) = 2 \text{ for } x \in [1,2]$$

and

$$-\varepsilon y'(x) + 2y(x) = z(x), \quad \text{with } y(1) = \frac{7}{2} \text{ for } x \in [0,1].$$

The numerical results are presented in Table 2 for this example and the solution is plotted in Figure 2 for $\varepsilon = 2^{-8}$, $N = 23$.

Table 2. Numerical results for Example 2 for $\varepsilon = 2^{-8}$, $N = 23$

x	Present solution	Exact solution	Error
0	1.0000	1.0000	0.0000
0.01	1.0049	1.0050	0.0001
0.02	1.0099	1.0100	0.0001
0.03	1.0399	1.0496	0.0097
0.04	1.0799	1.0810	0.0011
0.05	1.2049	1.2052	0.0003
0.06	1.3249	1.3259	0.0010
0.07	1.3889	1.3898	0.0009
0.08	1.4299	1.4315	0.0016
0.09	1.4645	1.4648	0.0002
1.00	1.4749	1.4752	0.0003
1.20	7.7020	7.7025	0.0005
1.30	7.8880	7.8890	0.0010
1.40	7.9665	7.9668	0.0003
1.50	8.2707	8.2709	0.0002
1.60	8.4675	8.4685	0.0010
1.70	8.6674	8.6685	0.0011
1.80	8.7355	8.7359	0.0004
1.90	9.0241	9.0243	0.0002
2.00	2.0000	2.0000	0.0000

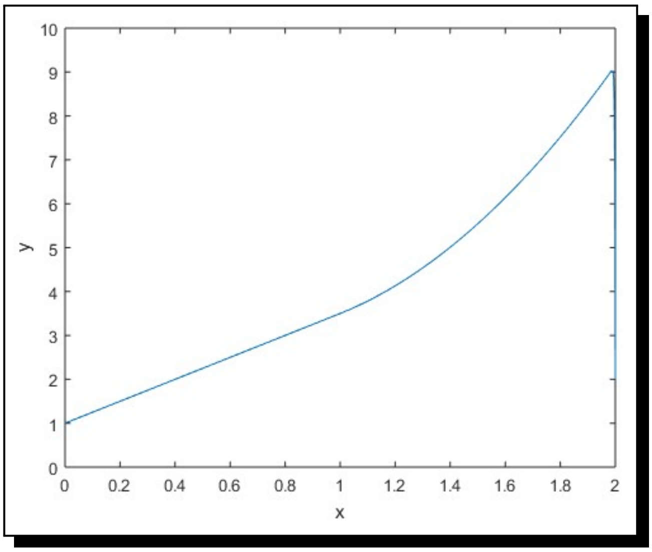

Figure 2. Numerical solution for Example 2 with $\varepsilon = 2^{-8}$, $N = 23$

4. Discussion and Conclusion

We have presented and illustrated the method of reduction of order for solving singularly perturbed delay differential equations. The given second order singularly perturbed delay differential equation is replaced by a pair of first order problems. These are in turn solved by initial value solvers. The integration of these initial value problems goes in opposite direction. To solve the initial value problems we used the classical fourth order Runge-Kutta method. In fact any standard analytical or numerical method can be used. The applicability of this method is demonstrated by solving some model problems and the numerical results are compared with exact solution. It is observed from the tables and figures that present method agrees with exact solution very well,

Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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