Common Fixed Point Results for Contractive Mapping in Complex Valued $A_b$-metric Space

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Abstract. In this article, we prove common fixed point results for two self mappings in complex valued $A_b$-metric space. Our results extend and generalize the common fixed point result of Singh and Singh [15].

Keywords. $A_b$-metric space; Complex valued metric space; Complex valued $b$-metric space; Complex valued $A_b$-metric space; Common fixed point

MSC. 47H10; 54H25; 37C25; 55M20

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1. Introduction

The concept of complex valued metric space was introduced by Azam et al. [6], which is the generalization of the classical metric space and proved some fixed point results for a pair of mappings for contractive condition satisfying a rational expression.

Subsequently, many authors have obtained fixed point and common fixed points of set mappings in complex valued metric spaces (see for instance [1,3,4,7–9,14,17–19,21]).

In 2013, Rao et al. [13] introduced the concept of complex valued $b$-metric space, which was general than the well known complex valued metric space. After that, many authors have generalize and extend the results in complex valued $b$-metric spaces (see for instance [10–13]).

In 2016, Ughade et al. [20], introduce the notation of $A_b$-metric space and proved some fixed point theorems under contraction and expansion type condition.
Recently, in 2019, Singh and Singh [15] introduced the concept of complex valued $A_b$-metric space and proved fixed point theorem, and also in 2020, he is proved common fixed point for two self mappings in rational expression and complex valued $A_b$-metric spaces, which is generalization of the results giving by Mukheimer [11].

In this paper, we describe and extend common fixed point theorem in complex valued $A_b$-metric space. Our results generalized the results of Singh and Singh [15].

### 2. Basic Concept and Mathematical Preliminary

In this section, we recall some properties of $A$-metric space, $A_b$-metric space, complex valued metric space, complex valued $b$-metric space and complex valued $A_b$-metric space.

**Definition 2.1 ([2]).** Let $X$ be a nonempty set. A function $A : X^n \rightarrow [0, \infty)$ is called an $A$-metric on $X$ if for any $x_i, a \in X, i = 1, 2, 3, \ldots, n$, the following conditions hold:

(A1) $A(x_1, x_2, x_3, \ldots, x_{(n-1)}, x_n) \geq 0$,

(A2) $A(x_1, x_2, x_3, \ldots, x_{(n-1)}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$,

(A3) $A(x_1, x_2, x_3, \ldots, x_{(n-1)}, x_n) \leq A(x_1, x_1, x_1, \ldots, (x_1)_{(n-1)}, a) + A(x_2, x_2, x_2, \ldots, (x_2)_{(n-1)}, a) + A(x_3, x_3, x_3, \ldots, (x_3)_{(n-1)}, a) + \cdots + A(x_{n-1}, x_{n-1}, x_{n-1}, \ldots, (x_{n-1})_{(n-1)}, a) + A(x_n, x_n, x_n, \ldots, (x_n)_{(n-1)}, a)$.

The pair $(X, A)$ is called an $A$-metric space.

**Definition 2.2 ([20]).** Let $X$ be a nonempty set and $b \geq 1$ be a given number. A function $A : X^n \rightarrow [0, \infty)$ is called an $A_b$-metric on $X$ if for any $x_i, a \in X, i = 1, 2, 3, \cdots, n$, the following conditions hold:

(A$_b$1) $A(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \geq 0$,

(A$_b$2) $A(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = 0$ if and only if $x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$,

(A$_b$3) $A(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) \leq b \left[ A(x_1, x_1, x_1, \ldots, (x_1)_{(n-1)}, a) + A(x_2, x_2, x_2, \ldots, (x_2)_{(n-1)}, a) + A(x_3, x_3, x_3, \ldots, (x_3)_{(n-1)}, a) + \cdots + A(x_{n-1}, x_{n-1}, x_{n-1}, \ldots, (x_{n-1})_{(n-1)}, a) + A(x_n, x_n, x_n, \ldots, (x_n)_{(n-1)}, a) \right]$.

The pair $(X, A)$ is called an $A_b$-metric space.

**Remark 2.3.** $A_b$-metric space is more general than $A$-metric space. Moreover, $A$-metric space is a special case of $A_b$-metric space with $b = 1$. 
Example 2.4. Let $X = [1, +\infty)$. Define $A_b : X^n \to [0, \infty)$ by

$$A_b(x_1, x_2, x_3, \cdots, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2$$

for all $x_i \in X$, $i = 1, 2, 3, \cdots, n$.

Then $(X, A_b)$ is an $A_b$-metric space with $b = 2 > 1$.

The concept of complex valued metric space was initiated by Azam et al. [6]. Let $C$ be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\preceq$ on $C$ as follows:

$z_1 \preceq z_2$ if and only if $\text{Re}(z_1) \leq \text{Re}(z_2)$ and $\text{Im}(z_1) \leq \text{Im}(z_2)$.

It follows that $z_1 \preceq z_2$ if one of the following conditions are satisfied:

$(C_1) \quad \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,

$(C_2) \quad \text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$,

$(C_3) \quad \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$,

$(C_4) \quad \text{Re}(z_1) < \text{Re}(z_2)$ and $\text{Im}(z_1) < \text{Im}(z_2)$.

Particularly, we write $z_1 \preceq z_2$ if $z_1 \neq z_2$ and one of $[C_2], [C_3]$ and $[C_4]$ is satisfied and we write $z_1 \preceq z_2$ if only $[C_4]$ is satisfied. The following statements hold:

1. If $a, b \in R$ with $a \leq b$, then $az \preceq bz$ for all $0 \preceq z \in C$.
2. If $z_1 \preceq z_2$, then $az_1 \preceq az_2$ for all $0 \leq a \in R$.
3. If $0 \preceq z_1 \preceq z_2$, then $|z_1| \leq |z_2|$.
4. If $0 < z_1 < z_2$, then $|z_1| < |z_2|$.
5. If $z_1 \preceq z_2$ and $z_2 < z_3$, then $z_1 < z_3$.

Definition 2.5 ([6]). Let $X$ be a nonempty set. A function $d : X \times X \to C$ is called a complex valued metric on $X$ if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair $(X, d)$ is called a complex valued metric space.

Definition 2.6 ([11]). Let $X$ be a nonempty set and let $s \geq 1$. A function $d : X \times X \to C$ is called a complex valued $b$-metric on $X$ if for all $x, y, z \in X$, the following conditions are satisfied:

(i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \preceq s[d(x, z) + d(z, y)]$.

The pair $(X, d)$ is called a complex valued $b$-metric space.

Definition 2.7 ([15]). Let $X$ be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $A : X^n \to C$ satisfies for all $x_i, a \in X$, $i = 1, 2, 3, \cdots, n$:

(CA$_b$1) $0 \preceq A(x_1, x_2, x_3, \cdots, x_n)$,
where \( \alpha \) valued \( A \)

**(CA\(_b\)) 2** \( A(x_1, x_2, x_3, \cdots, x_n) = 0 \Leftrightarrow x_1 = x_2 = x_3 = \cdots = x_n, \)

**(CA\(_b\)) 3** \( A(x_1, x_2, x_3, \cdots, x_n) \geq b \left[ A(x_1, x_1, x_1, \cdots, (x_1)_{(n-1)}, a) + A(x_2, x_2, x_2, \cdots, (x_2)_{(n-1)}, a) + \cdots + A(x_n, x_n, x_n, \cdots, (x_n)_{(n-1)}, a) \right]. \)

Then \( A \) is called a complex valued \( A_b \)-metric on \( X \) and the pair \((X, A)\) is called a complex valued \( A_b \)-metric space.

**Example 2.8 ([15])**. Let \( X = \mathbb{R} \) and \( A : X^n \to C \) be such that

\[
A(x_1, x_2, x_3, \cdots, x_n) = (\alpha + i\beta)A_+(x_1, x_2, x_3, \cdots, x_n),
\]

where \( \alpha, \beta \geq 0 \) are constants and \( A_+ \) is an \( A_b \)-metric on \( X \). Then \( A \) is a complex valued \( A_b \)-metric on \( X \). As a particular case, we have the following example of complex valued \( A_b \)-metric on \( X \). The mapping \( A : X^n \to C \) defined by \( A(x_1, x_2, x_3, \cdots, x_n) = (1 + i) \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2 \) is a complex valued \( A_b \)-metric on \( X = \mathbb{R} \) with \( b = 2 \).

**Definition 2.9 ([15])**. A complex valued \( A_b \)-metric space \((X, A)\) is said to be symmetric if

\[
A(x_1, x_1, x_1, \cdots, (x_1)_{(n-1)}, x_2) = A(x_2, x_2, x_2, \cdots, (x_2)_{(n-1)}, x_1),
\]

for all \( x_1, x_2 \in X \).

**Definition 2.10 ([15])**. Let \((X, A)\) be a complex valued \( A_b \)-metric space.

(i) A sequence \( \{x_p\} \) in \( X \) is said to be complex valued \( A_b \)-convergent to \( x \) if for every \( a \in C \) with \( 0 < a \), there exists \( k \in \mathbb{N} \) such that \( A(x_p, x_p, \cdots, x_p, x) < a \) or \( A(x, x, \cdots, x, x_p) < a \) for all \( p \geq k \) and is denoted by \( \lim_{p \to \infty} x_p = x \) or \( x_p \to x \) as \( p \to \infty \).

(ii) A sequence \( \{x_p\} \) in \( X \) is called complex valued \( A_b \)-Cauchy if for every \( a \in C \) with \( 0 < a \), there exists \( k \in \mathbb{N} \) such that \( A(x_p, x_p, \cdots, x_p, x_p) < a \) for each \( p, q \geq k \).

(iii) If every complex valued \( A_b \)-Cauchy sequence is complex valued \( A_b \)-convergent in \( X \), then \((X, A)\) is said to be complex valued \( A_b \)-complete.

**Lemma 2.11 ([15])**. Let \((X, A)\) be a complex valued \( A_b \)-metric space and let \( \{x_p\} \) be a sequence in \( X \). Then \( \{x_p\} \) is complex valued \( A_b \)-convergent to \( x \) if and only if \( |A(x_p, x_p, \cdots, x_p, x)| \to 0 \) as \( p \to \infty \) or \( |A(x, x, \cdots, x, x_p)| \to 0 \) as \( p \to \infty \).

**Lemma 2.12**. Let \((X, A)\) be a complex valued \( A_b \)-metric space and let \( \{x_p\} \) be a sequence in \( X \). Then \( \{x_p\} \) is complex valued \( A_b \)-Cauchy sequence if and only if \( |A(x_p, x_p, \cdots, x_p, x_q)| \to 0 \) as \( p, q \to \infty \).

**Lemma 2.13**. Let \((X, A)\) be a complex valued \( A_b \)-metric space. Then

\[
A(x, x, \cdots, x, y) \geq bA(y, y, \cdots, y, x).
\]

for all \( x, y \in X \).
3. Main Results

**Theorem 3.1.** Let \( (X,A) \) be a complete complex valued \( A_b \)-metric space and \( f,g : X \rightarrow X \) be any two mapping satisfying

\[
A(fx,fx,\cdots,fx,gy) \preceq \alpha A(x,x,\cdots,y).
\] (3.1)

for all \( x,y \in X \) where \( \alpha \in [0,\frac{1}{b^2}] \). Then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point and let \( \{x_k\} \) in \( X \) be defined as

\[
x_{2k+1} = fx_{2k} = f^{2k+1}x_0,
\]

\[
x_{2k+2} = gx_{2k+1} = g^{2k+2}x_0,
\]

for \( k = 0,1,2,3,\cdots. \)

Then, we show that the sequence \( \{x_k\} \) is complex valued \( A_b \)-Cauchy.

From (3.1), we have

\[
A(x_{2k+1},x_{2k+1},\cdots,x_{2k+1},x_{2k+2}) \preceq A(fx_{2k},fx_{2k},\cdots,fx_{2k},gy_{2k+1})
\]

\[
\preceq \alpha A(x_{2k},x_{2k},\cdots,x_{2k},y_{2k+1})
\]

\[
\vdots
\]

\[
\preceq \alpha^k A(x_0,x_0,\cdots,x_0,y_1).
\] (3.2)

Using \([CA_b3]\) and (3.2), for \( k, l \in \mathbb{N} \) with \( k < l \), we have

\[
A(x_k,x_k,\cdots,x_k,x_l) \preceq (n-1) bA(x_k,x_k,\cdots,x_k,x_{k+1}) + b^2 A(x_{k+1},x_{k+1},\cdots,x_{k+1},x_l)
\]

\[
\preceq (n-1) bA(x_k,x_k,\cdots,x_k,x_{k+1}) + b^2 A(x_{k+1},x_{k+1},\cdots,x_{k+1},x_{k+2}) + \cdots
\]

\[
+ b^{2l-k-1} A(x_{l-1},x_{l-1},\cdots,x_{l-1},x_l)
\]

\[
\preceq (n-1) bA^k + b^2 A^k + b^4 A^{k+2} + \cdots + b^{2l-k-1} A^{l-1} A(x_0,\cdots,x_{k+1})
\]

\[
\preceq (n-1) bA^k + b^2 A^k + b^4 A^{k+2} + \cdots + b^{2l-k-1} A^{l-1} A(x_0,\cdots,x_{k+1})
\]

\[
\preceq (n-1) bA^k
\]

Thus, we obtain

\[
|A(x_k,x_k,\cdots,x_k,x_l)| \leq \frac{(n-1)(ba)^k}{1-b^2A} |A(x_0,x_0,\cdots,x_0,x_1)|.
\]

Since \( \alpha \in [0,\frac{1}{b^2}] \) where \( b > 1 \), taking limit as \( k,l \to \infty \), we have

\[
|A(x_0,x_0,\cdots,x_0,x_1)| \leq \frac{(n-1)(ba)^k}{1-b^2A} |A(x_0,x_0,\cdots,x_0,x_1)| \to 0.
\]

Therefore, \( |A(x_0,x_0,\cdots,x_0,x_1)| \to 0 \) as \( k,l \to \infty \).

So, by Lemma 2.11 (\( x_k \)) is a complex valued \( A_b \)-Cauchy sequence. Since \( (X,A) \) is complete, there exist \( u \in X \) such that the sequence \( \{x_k\} \) is complex valued \( A_b \)-convergent to \( u \).

Now, we show that \( u \) is fixed point of \( f \). We have

\[
A(fu,fu,\cdots,fu,u) \preceq (n-1) bA(fu,fu,\cdots,fu,x_{2k+2}) + bA(u,u,\cdots,u,x_{2k+2})
\]

\[
= (n-1) bA(fu,fu,\cdots,fu,gy_{2k+1}) + bA(u,u,\cdots,u,x_{2k+2})
\]
Theorem 3.3. Let \( (X, A) \) be a complete complex valued \( A_b \)-metric space and \( f, g : X \to X \) be any two mapping for some positive constant \( k \)
\[
A(f^{2k+1}x, f^{2k+1}x, \ldots, f^{2k+1}x, g^{2k+2}y) \lesssim aA(x, x, \ldots, x, y),
\]
for all \( x, y \in X \) where \( a \in (0, \frac{1}{b^2}) \), then \( f \) and \( g \) have a unique common fixed point in \( X \).

From Theorem 3.1 that \( f^{2k+1}x \) has a unique fixed point \( u \) in \( X \). But \( f^{2k+1}(fu) = f(f^{2k+1}u) = fu \). So, \( fu \) is also fixed point of \( f^{2k+1} \). Hence \( fu = u \) is a fixed point of \( f \). Since the fixed point of \( f \) is also fixed point of \( f^{2k+1} \), the fixed point of \( f \) is unique. Similarly it can be established that \( gu = u \). Then \( fu = u = gu \). Thus \( u \) is common fixed point of \( f \) and \( g \).

**Corollary 3.2.** Let \( (X, A) \) be a complete complex valued \( A_b \)-metric space and \( f, g : X \to X \) be any two mapping satisfying the following condition
\[
A(fx, fx, \ldots, fx, gy) \lesssim a[A(x, x, \ldots, x, fx) + A(y, y, \ldots, y, gy)], \tag{3.4}
\]
for all \( x, y \in X \) and \( a \in \left(0, \frac{1}{2(n-1)b^2}\right) \). Then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point and let us define a sequence \( (x_k) \) in \( X \) as
\[
x_{2k+1} = fx_{2k} = f^{2k+1}x_0, \quad x_{2k+2} = gx_{2k+1} = g^{2k+2}x_0,
\]
for \( k = 0, 1, 2, 3, \ldots \).

Then, we show that the sequence \( (x_k) \) is complex valued \( A_b \)-Cauchy sequence.
From (3.4), we have
\[
A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})
= A(f x_{2k}, f x_{2k}, \ldots, f x_{2k}, g y_{2k+1})
\leq a[A(x_{2k}, x_{2k}, \ldots, x_{2k}, f x_{2k}) + A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, g x_{2k+1})]
= a[A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1}) + A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})]
\leq \frac{a}{1-a}[A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})]
\]
or
\[
\Rightarrow |A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})| \leq \frac{a}{1-a}|A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})|. \tag{3.5}
\]
Similarly, using the symmetry of X, we get
\[
|A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2k+3})| \leq \frac{a}{1-a}|A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})|. \tag{3.6}
\]
From (3.5) and (3.6), we have
\[
|A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})| \leq h|A(x_{2k-1}, x_{2k-1}, \ldots, x_{2k-1}, x_{2k})|, \tag{3.7}
\]
for all \(k \in \mathbb{N}\), where \(h = \frac{a}{1-a} < 1\).

By repeatedly applying (3.7), we get
\[
|A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})| \leq h^{2k}|A(x_0, x_0, \ldots, x_0, x_1)|. \tag{3.8}
\]

Using (CA\(\alpha\)3) and (3.8), we have for \(k, l \in \mathbb{N}\) with \(k < l\) we get
\[
|A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2l})|
\leq (n-1)b|A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})| + b|A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2l})|
\leq (n-1)b|A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})| + (n-1)b^2|A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})|
+ b^3|A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2l})|
\leq (n-1)b|A(x_{2k}, x_{2k}, \ldots, x_{2k}, x_{2k+1})| + (n-1)b^2|A(x_{2k+1}, x_{2k+1}, \ldots, x_{2k+1}, x_{2k+2})|
+ (n-1)b^3|A(x_{2k+2}, x_{2k+2}, \ldots, x_{2k+2}, x_{2k+3})| + \cdots
+ (n-1)b^{2k-2l-1}|A(x_{2l-2}, x_{2l-2}, \ldots, x_{2l-2}, x_{2l-1})|
+ b^{2l-2k-1}|A(x_{2l-1}, x_{2l-1}, \ldots, x_{2l-1}, x_{2l})|
\leq [(n-1)b a^{2k} + (n-1)b a^{2k+1} + \cdots + (n-1)b^{2k-2l-1} a^{2l-2}]
+ (n-1)b^{2l-2k-1} a^{2l-1}|A(x_0, x_0, \ldots, x_0, x_1)|
= (n-1)[(b a)^{2k} + (b a)^{2k+1} + \cdots + (b a)^{2l-2} + (b a)^{2l-1}]|A(x_0, x_0, \ldots, x_0, x_1)|
= (n-1)[(b a)^{2k} + (b a)^{2k+1} + \cdots]|A(x_0, x_0, \ldots, x_0, x_1)|
\leq \frac{(n-1)(b a)^{2k}}{1-b a}|A(x_0, x_0, \ldots, x_0, x_1)| \to 0 \text{ as } k, l \to \infty \text{ (by Lemma 2.1)}. \tag{3.9}
\]
Hence the sequence \(x_{2k}\) is complex valued \(A_b\)-Cauchy in \(X\). Since \((X, A)\) is a complete, there exists \(x^* \in X\) such that \(\lim_{k \to \infty} x_{2k} = x^*\).

We show that \(x^*\) is a fixed point of \(f\).
\[
A(f x^*, f x^*, \ldots, f x^*, x^*) \leq (n-1)b A(f x^*, f x^*, \ldots, f x^*, f x_{2k+1}) + b^2 A(f x^{2k}, f x^{2k}, \ldots, f x^{2k}, x^*).
\]
A(x^*, x^*, \cdots, x^*)
\preceq (n-1)bA(x^*, x^*, \cdots, x^*, f x_{2k+1}) + b^2A(x_{2k+1}, f x_{2k+1}, \cdots, f x_{2k+1}, x^*)
\preceq (n-1)bA(x^*, x^*, \cdots, x^*, f x^*) + A(x_{2k}, f x_{2k}, \cdots, f x_{2k}, f x_{2k+1})
+ b^2A(x_{2k+1}, f x_{2k+1}, \cdots, f x_{2k+1}, x^*)
\preceq (n-1)bA(x^*, x^*, \cdots, x^*, f x^*) + (n-1)bA(x_{2k}, f x_{2k}, \cdots, f x_{2k}, f x_{2k+1})
+ b^2A(x_{2k+1}, f x_{2k+1}, \cdots, f x_{2k+1}, x^*)
\preceq (n-1)bA(x^*, x^*, \cdots, x^*, f x^*) + (n-1)bA(x_{2k}, f x_{2k}, \cdots, f x_{2k}, f x_{2k+1})
+ b^2A(x_{2k+1}, f x_{2k+1}, \cdots, f x_{2k+1}, x^*)
\Rightarrow |A(x^*, x^*, \cdots, x^*, f x^*)| \leq (n-1)b^2 a|A(x^*, x^*, \cdots, f x^*, x^*)|
+ (n-1)bA(x_{2k}, f x_{2k}, \cdots, f x_{2k}, f x_{2k+1})|
+ b^2|A(x_{2k+1}, f x_{2k+1}, \cdots, f x_{2k+1}, x^*)|
\leq \frac{1}{1 - (n-1)b^2 a}|(n-1)bA(x_{2k}, f x_{2k}, \cdots, f x_{2k}, f x_{2k+1})|
+ b^2|A(x_{2k+1}, f x_{2k+1}, \cdots, f x_{2k+1}, x^*)| \to 0, \text{ as } k \to \infty.
\Rightarrow |A(x^*, x^*, \cdots, x^*, f x^*)| = 0.
\Rightarrow x^* = x^*.
Thus, x^* is a fixed point of f.
Hence, x^* is common fixed point of f and g.

Now, we show that the uniqueness of the common fixed point of f and g.
Let us assume that y^* \in X is another common fixed point of f and g. Then we have
A(x^*, x^*, \cdots, x^*, y^*) \preceq A(x^*, x^*, \cdots, f x^*, g y^*)
\preceq a[ A(x^*, x^*, \cdots, x^*, f x^*) + A(y^*, y^*, \cdots, y^*, g y^*) ]
\preceq a[ A(x^*, x^*, \cdots, x^*, x^*) + A(y^*, y^*, \cdots, y^*, y^*) ]
\preceq 0.
Hence
|A(x^*, x^*, \cdots, x^*, y^*)| \leq 0.
\Rightarrow x^* = y^*.
Thus x^* is the unique common fixed point of f and g. This completes the proof of the theorem.

**Theorem 3.4.** Let (X, d) be a complete complex valued A-b-metric space and let f, g : X \to X be any two mappings satisfying the following condition

\[ A(f x, f x, \cdots, f x, g y) \preceq a[ A(x, x, \cdots, x, g y) + A(y, y, \cdots, y, f x) ], \quad (3.10) \]

for all x, y \in X and a \in \left[ 0, \frac{1}{b^2(n-1)+1} \right], then f and g have a unique common fixed point in X.
Proof. Let \( x_0 \in X \) be an arbitrary point and let us define a sequence \( \{x_{2n}\} \) in \( X \) as

\[
x_{2n+1} = f x_{2n} = f^{2n+1} x_0,
\]

\[
x_{2n+2} = g x_{2n+1} = g^{2n+2} x_0,
\]

for \( n = 0, 1, 2, 3, \ldots \).

Put \( x = x_{2n-1}, y = x_{2n} \) in (3.10) we have

\[
A(x_{2n}, x_{2n}, \cdots, x_{2n}, x_{2n+1})
\]

\[
= A(f x_{2n-1}, f x_{2n-1}, \cdots, f x_{2n-1}, g x_{2n})
\]

\[
\lesssim a[A(x_{2n-1}, x_{2n-1}, \cdots, x_{2n-1}, g x_{2n}) + A(f x_{2n}, f x_{2n}, \cdots, f x_{2n}, f x_{2n-1})]
\]

\[
= a[A(x_{2n-1}, x_{2n-1}, \cdots, x_{2n-1}, x_{2n+1}) + A(x_{2n}, x_{2n}, \cdots, x_{2n}, x_{2n})]
\]

\[
= aA(x_{2n-1}, x_{2n-1}, \cdots, x_{2n-1}, x_{2n+1})
\]

\[
\lesssim (n - 1) a b A(x_{2n-1}, x_{2n-1}, \cdots, x_{2n-1}, x_{2n}) + a b^2 A(x_{2n+1}, x_{2n+1}, \cdots, x_{2n+1}, x_{2n}).
\]

Therefore

\[
|A(x_{2n}, x_{2n}, \cdots, x_{2n}, x_{2n+1})| \leq (n - 1) a b |A(x_{2n-1}, x_{2n-1}, \cdots, x_{2n-1}, x_{2n})| + a b^2 |A(x_{2n+1}, x_{2n+1}, \cdots, x_{2n+1}, x_{2n})|
\]

\[
\leq \frac{(2n - 1) a b}{1 - a b^2} |A(x_{2n-1}, x_{2n-1}, \cdots, x_{2n-1}, x_{2n})|.
\] (3.11)

If we put \( x_{2n}, x_{2n}, \cdots, x_{2n+1} = A_{2n} \) and \( x_{2n-1}, x_{2n-1}, \cdots, x_{2n} = A_{2n-1} \).

Then, from (3.11), we have

\[
|A_{2n}| \leq \frac{(2n - 1) a b}{1 - a b^2} |A_{2n-1}|
\]

\[
\Rightarrow |A_{2n}| \leq k |A_{2n-1}|,
\] (3.12)

where \( \frac{(2n-1)a b}{1-a b^2} < 1 \).

Repeating this process, we get

\[
|A(x_{2n}, x_{2n}, \cdots, x_{2n+1})| \leq k |A(x_{2n-1}, x_{2n-1}, \cdots, x_{2n})|
\]

\[
\leq k^2 |A(x_{2n-2}, x_{2n-2}, \cdots, x_{2n-1})|
\]

\[
\vdots
\]

\[
\leq k^{2n} |A(x_0, x_0, \cdots, x_1)|,
\] (3.13)

for all \( n \geq 1 \).

Now

\[
a < \frac{1}{b^2((2n-1)b+1)} \Rightarrow a b^2 < \frac{1}{(2n-1)b+1}
\]

\[
\Rightarrow 1 - a b^2 > 1 - \frac{1}{(2n-1)b+1}
\]

\[
\Rightarrow \frac{(2n-1)b}{(2n-1)b+1} > 0.
\]
Also, we have
\[ \frac{1}{b^3((2n-1)+b^2)} \Rightarrow a b^3(2n-1) + a b^2 < 1 \]
\[ \Rightarrow a b^3(2n-1) < 1 - a b^2 \]
\[ \Rightarrow \frac{a b^3(2n-1)}{1 - a b^2} < 1 \]
\[ \Rightarrow a(2n-1)b < 1 \]
\[ \Rightarrow \frac{1}{b^2} < 1 \]
\[ \Rightarrow k < 1. \]

Using (CAₙ)₃ and (3.13), we have for all \( n, m \in \mathbb{N} \), with \( n < m \)
\[ A(f^{2n}x₀, f^{2n}x₀, \ldots, f^{2n}x₀) \]
\[ \leq b(n-1)|A(f^{2n}x₀, \ldots, f^{2n}x₀, f^{2n+1}x₀)| + |A(f^{2n}x₀, \ldots, f^{2n}x₀, f^{2n+1}x₀)| \]
\[ \leq b(n-1)|A(f^{2n}x₀, \ldots, f^{2n}x₀, f^{2n+1}x₀)| + b^2|A(f^{2n+1}x₀, \ldots, f^{2n+1}x₀, f^{2n}x₀)| \]
\[ \leq b(n-1)|A(f^{2n}x₀, \ldots, f^{2n}x₀, f^{2n+1}x₀)| + b^3(n-1)|A(f^{2n+1}x₀, \ldots, f^{2n+1}x₀, f^{2n+2}x₀)| \]
\[ + b^4|A(f^{2n+2}x₀, \ldots, f^{2n+2}x₀, f^{2n}x₀)| \]
\[ \leq b(n-1)|A(f^{2n}x₀, \ldots, f^{2n}x₀, f^{2n+1}x₀)| + b^2|A(f^{2n+1}x₀, \ldots, f^{2n+1}x₀, f^{2n+2}x₀)| + \ldots \]
\[ + b^{2m-1}|A(f^{2n-m}x₀, \ldots, f^{2m-1}x₀, f^{2m}x₀)| \]
\[ \leq (n-1)b[k^2 + b^2k^2 + \ldots + b^{2m-1}k^{2m-1}]|A(x₀, x₀, \ldots, x₀, x₁)| \]
\[ = (n-1)b k^{2n} (1 + b^2k + (b^2k)^2 + \ldots + (b^2k)^{2m-1})|A(x₀, x₀, \ldots, x₀, x₁)| \]
\[ \leq \frac{(n-1)b k^{2n}}{1 - b^2k} |A(x₀, x₀, \ldots, x₀, x₁)| \to 0, \quad as \ n, m \to \infty. \]

Hence \( x_{2n} \) is complex valued \( A_b \)-Cauchy sequence in \( X \). Since \( X \) is complex, there exists \( v \in X \) such that \( \lim_{n \to \infty} x_{2n} = v \). We show that \( v \) is fixed point of \( f \).

We have
\[ A(f^2v, f^2v, \ldots, f^2v, v) \]
\[ \leq (n-1)baA(f^2v, v, \ldots, v, f^{2n+1}x₀) + bA(v, v, \ldots, v, f^{2n+1}x₀) \]
\[ \leq [(n-1)ba + b]A(v, v, \ldots, v, f^{2n+1}x₀) + (n-1)baA(f^{2n}x₀, f^{2n}x₀, \ldots, f^{2n}x₀, f^{2n}x₀, f^{2n}x₀) \]
\[ \Rightarrow |A(f^2v, \ldots, f^2v, v)| \leq \frac{1}{1-(n-1)ba} [(n-1)ba + b]A(v, v, \ldots, v, f^{2n+1}x₀) \]
\[ \Rightarrow |A(f^2v, \ldots, f^2v, v)| = 0. \]
\[ \Rightarrow f^2v = v. \]
Therefore, \( v \) is a fixed point of \( f \). Similarly, we can show that, \( v \) is a fixed point of \( g \) i.e. \( gv = v \). Thus \( fv = v = gv \).

Hence \( v \in X \) is common fixed point of \( f \) and \( g \).

Now, we show that the common fixed point of \( f \) and \( g \) are unique.

Let \( w \in X \) be another common fixed point of \( f \) and \( g \). Then we have

\[
A(v, v, \ldots, v, w) = A(fv, fv, \ldots, fv, w)
\]

\[
\geq a[A(v, v, \ldots, v, w) + A(w, w, \ldots, w, w)]
\]

\[
= a[A(v, v, \ldots, v, w) + A(w, w, \ldots, w, v)]
\]

\[
\geq a[A(v, v, \ldots, v, w) + bA(v, v, \ldots, v, w)]
\]

\[
\geq a(1 + b)A(v, v, \ldots, v, w)
\]

\[
\Rightarrow |A(v, v, \ldots, v, w)| \leq a(1 + b)|A(v, v, \ldots, v, w)|.
\]

But

\[
a < \frac{1}{b^2(2n - 1)b + 1}
\]

\[
< \frac{1}{b^2(b + 1)}
\]

\[
\Rightarrow a(b + 1) < \frac{1}{b^2} < 1.
\]

Therefore, we must have

\[
|A(v, v, \ldots, v, w)| = 0 \Rightarrow v = w.
\]

Hence \( v \) is the unique common fixed point of \( f \) and \( g \). This completes the proof of the theorem.

\[\square\]

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**Competing Interests**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

**References**


Common Fixed Point Results for Contractive Mapping in Complex Valued...: S. K. Tiwari and M. Gauratra


Common Fixed Point Results for Contractive Mapping in Complex Valued


