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Research Article

A Fitted Second Order Special Finite Difference Method for Singularly Perturbed Differential-Difference Equations Exhibiting Dual Layers

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Abstract. In this paper, a singularly perturbed differential-difference equation boundary value problem having boundary layer at both the end is examined. To solve such type of problems, a fitted special finite difference scheme is used. The differential-difference equation is replaced by an asymptotically equivalent singular perturbation problem using the Taylor's series expansion and afterwards fitted special finite difference scheme is applied. To demonstrate the applicability of this method, three numerical examples are solved and numerical results are presented which are in agreement with the available/exact results.

Keywords. Differential-difference equations; Boundary layer; Dual layer; Exponentially fitted finite difference methods

MSC. 65L11; 65Q20

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1. Introduction

A singularly perturbed delay differential equation is a differential equation in which the highest order derivative is multiplied by a small parameter and involving at least one delay term. Actually, the singularly perturbed problems are generally the first approximation of the considered physical model. Hence in such cases, more realistic model should include some of the fast and future states of the system and hence, a real system should be modelled by differential equations with delay or advance parameter. Such type of equation arises frequently in the mathematical modelling of various practical phenomena for example, in the modelling of the human pupil-light reflex, Model of HIV infection, the study of bi-stable devices in digital electronics, variational problem in control theory; first exit time problem in modelling of activation of neuronal variability, immune response, evolutionary biology, dynamics of networks of two identical amplifier, mathematical ecology, population dynamics, the modelling of biological oscillator and in a variety of models for physiological process.

Lange and Miura [8,9], published a series of papers for solving these problems. First started with problems of the form:

$$\varepsilon^2 y''(x) + q(x)y(x) + \alpha(x)y'(x-1) + \beta(x)y(x-1) = \psi(x)$$

with conditions

$$y(x) = \phi(x), \quad -1 \le x \le 0,\ y(1) = \gamma$$

where $q(x)$, $\alpha(x)$, $\beta(x)$, $\phi(x)$, $\psi(x)$ are assumed to be continuously differentiable in $[0,l]$, and γ; $l > 1$ are constants independent of ε and $q(x) > 0$. Modified WKB method together with the matched asymptotic expansions is used to study the above problem. Chakravarthy and Reddy [16], presented an initial value approach for the solution of singularly perturbed problems. Rao and Chakravarthy [13,14], constructed a scheme for solving partial differential difference equations. Salama and Al-Amery [18], has given an asymptotic method for solving differential difference equations. Reddy et al. [17], have presented a new scheme for solving singularly perturbed differential-difference equations. Kanth and Murali [20], described a simple scheme of a non-linear differential-difference equations. Adilaxmi et al. [1], presented an initial value technique using exponentially fitted non-standard finite difference method for singularly perturbed differential-difference equations. Kadalbajoo and Sharma [6,7], has given numerical treatment of boundary value problems for second order singularly perturbed delay differential equations. Pakdemirli [12], has described application of the perturbation iteration method to boundary layer type problems. Reddy and Awoke [15], has described solution of singularly perturbed differential-difference equations via fitted method. For the more theory of perturbation problems, one may refer books: Bellman and Cooke [2], Driver [3], Elsgolts and Norkin [4], Hale [5], Nayfeh [10], O'Malley [11] and Van Dyke [19].

In this paper, a singularly perturbed differential-difference equation boundary value problem having boundary layer at both the end is examined. To solve such type of problems, a fitted special finite difference scheme is used. The differential-difference equation is replaced by an asymptotically equivalent singular perturbation problem using the Taylor's series expansion and afterwards fitted special finite difference scheme is applied. To demonstrate the applicability of this method, three numerical examples are solved and numerical results are presented which are in agreement with the available/exact results.

2. Description of the Fitted Method

Consider the differential-difference equation

$$\varepsilon y''(x) + a(x)y(x-\delta) + c(x)y(x) + b(x)y(x+\eta) = f(x), \quad 0 < x < 1 \tag{1}$$

with the boundary conditions

$$y(x) = \alpha(x), \quad -\delta \le x \le 0 \tag{2}$$

and

$$y(x) = \beta(x), \quad 1 \le x \le 1+\eta, \tag{3}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$ is the small delay parameter, $0 < \eta = O(\varepsilon)$ is the small advanced parameter, $a(x)$, $b(x)$, $c(x)$, $f(x)$, $\alpha(x)$ and $\beta(x)$ are sufficiently differentiable in $(0,1)$. If $a(x)+b(x)+c(x) \le 0$ on the interval $[0,1]$, then the solution of eq. (1) exhibits boundary layers at both ends of the interval $[0,1]$, whereas it exhibits oscillatory behaviour $a(x)+b(x)+c(x) > 0$.

Using Taylor series expansion, in the neighbourhood of x

$$y(x-\delta) \approx y(x) - \delta y'(x), \tag{4}$$

$$y\left(x+\eta\right) \approx y(x) + \eta y'(x). \tag{5}$$

Substitute eq. (4) and eq. (5) into eq. (1), we get singularly perturbed ordinary differential equation

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \tag{6}$$

with the boundary conditions

$$y(0) = \alpha(0) = \varphi_0, \tag{7}$$

$$y(1) = \beta(1) = \gamma_1, \tag{8}$$

where $p(x) = b(x)\eta - a(x)\delta$, $q(x) = a(x)+b(x)+c(x)$, φ_0 and γ_1 are constants.

Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$, the transformation from eq. (1) to eq. (6) is admitted. For more details on the validity of this transformation one can refer Elśgoltś and Norkin [4].

Now, we divide the interval $[0,1]$ into n equal parts with constant mesh length h.

Let $0 = x_0, x_1, \ldots, x_n = 1$ be the mesh points, then we have $x_i = ih$, $i = 0,1,2,\ldots,n$. We choose N such that $x_N = \frac{1}{2}$. Since the problem exhibits two boundary layers across the interval, we divide the interval $[0,1]$ into two sub intervals $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$. Clearly, in the interval $\left[0,\frac{1}{2}\right]$ the boundary layer will be at the left end i.e. at $x = 0$, and in the interval $\left[\frac{1}{2},1\right]$ the boundary layer will be at right end i.e. at $x = 1$.

2.1 Problem with left end boundary layer in $\left[0,\frac{1}{2}\right]$

The idea given by Van Veldhuizen to the boundary value problem eq. (6) by considering the fitted special finite difference scheme for eq. (6) as follows:

$$\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + p_i\left(\frac{y_{i+1}-y_i}{h} - \frac{h}{2}y_i''\right) + q_i\left(\frac{y_{i+1}+y_{i-1}}{2}\right) = f_i \tag{9}$$

for $i = 1,2,\ldots,N-1$.

Now, introduce a fitting factor σ in the above scheme, eq. (9) as fallows:

$$\sigma\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + p_i\left(\frac{y_{i+1}-y_i}{h} - \frac{h}{2}y_i''\right) + q_i\left(\frac{y_{i+1}+y_{i-1}}{2}\right) = f_i, \tag{10}$$

$$\sigma\varepsilon\left(\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right) + p_i\left(\frac{y_{i+1}-y_i}{h} - \frac{h}{2}\left(\frac{f_i - p_i y_i' - q_i y_i}{\varepsilon}\right)\right) + q_i\left(\frac{y_{i+1}+y_{i-1}}{2}\right) = f_i. \tag{11}$$

Substituting $y_i' = \frac{y_{i+1}-y_i}{h}$ in the above eq. (11) and simplifying, we get

$$\left(\frac{\varepsilon\sigma}{h^2} + \frac{q_i}{2}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma}{h^2} + \frac{p_i}{h} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon} - \frac{h}{2\varepsilon}p_i q_{i+\frac{1}{2}}\right)y_i + \left(\frac{\varepsilon\sigma}{h^2} + \frac{p_i}{h} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon} + \frac{q_i}{2}\right)y_{i+1}$$

$$= f_i + \frac{h}{2\varepsilon} p_i f_{i+\frac{1}{2}}. \tag{12}$$

From the theory of singular perturbation, it is known that the solution of eq. (6) to eq. (8) is of the form (O'Malley [11])

$$y(x) = y_0(x) + \frac{p(0)}{p(x)}(\phi_0 - y_0(0))e^{-\int_0^x (\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)})dx} + O(\varepsilon), \tag{13}$$

where $y_0(x)$ is the solution of $p(x)y'(x) + q(x)y(x) = f(x)$, $y(1) = \beta(1) = \gamma_1$. By taking the Taylor's series expansions of $p(x)$ and $q(x)$ about the point 0 and restricting to their first terms, we get

$$y(x) = y_0(x) + (\phi_0 - y_0(0))e^{-(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)})x} + O(\varepsilon). \tag{14}$$

At $x = x_i$, the fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of (12) converges uniformly to the solution of eq. (6) to eq. (8) and which is equivalent to the solution of eq. (1) to eq. (3). Now multiplying eq. (12) by hand taking limit $h \to 0$, using eq. (14), we get

$$\sigma = \frac{\rho(p(0) + \frac{\rho}{2}p^2(0))(1 - e^{-(\frac{p^2(0)-\varepsilon q(0)}{p(0)})\rho}}{e^{(\frac{p^2(0)-\varepsilon q(0)}{p(0)})\rho} - 2 + e^{-(\frac{p^2(0)-\varepsilon q(0)}{p(0)})\rho}}, \tag{15}$$

where $\rho = \frac{h}{\varepsilon}$. Arranging eq. (12) in three term recurrence relation, we get system of equations of $N-1$ equations.

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, \dots, N-1, \tag{16}$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} + \frac{q_i}{2}, F_i = 2\frac{\varepsilon\sigma}{h^2} + \frac{p_i}{h} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon} - \frac{h}{2\varepsilon} p_i q_{i+\frac{1}{2}}, G_i = \frac{\varepsilon\sigma}{h^2} + \frac{p_i}{h} + \frac{p_i p_{i+\frac{1}{2}}}{2\varepsilon} + \frac{q_i}{2}, H_i = f_i + \frac{h}{2\varepsilon} p_i f_{i+\frac{1}{2}}.$$

2.2 Problem with right end boundary layer in $[\frac{1}{2}, 1]$

For the right end boundary layer in eq. (6), we consider the special second order finite difference scheme as

$$\varepsilon\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + p_i\left(\frac{y_i - y_{i-1}}{h} + \frac{h}{2}y_i''\right) + q_i\left(\frac{y_{i+1} + y_{i-1}}{2}\right) = f_i \tag{17}$$

for $i = N+1, N+2, \dots, n-1$.

Now, introduce a fitting factor $\sigma 1$ in the above scheme, en. (17) as follows:

$$\varepsilon\sigma 1\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + p_i\left(\frac{y_i - y_{i-1}}{h} + \frac{h}{2}y_i''\right) + q_i\left(\frac{y_{i+1} + y_{i-1}}{2}\right) = f_i, \tag{18}$$

$$\varepsilon\sigma 1\left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}\right) + p_i\left(\frac{y_i - y_{i-1}}{h} + \frac{h}{2}\left(\frac{f_i - p_i y_i' - q_i y_i}{\varepsilon}\right)\right) + q_i\left(\frac{y_{i+1} + y_{i-1}}{2}\right) = f_i. \tag{19}$$

Substituting $y_i' = \frac{y_i - y_{i-1}}{h}$ in the above eq. (19) and simplifying, we get

$$\left(\frac{\varepsilon\sigma 1}{h^2} - \frac{p_i}{h} + \frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon} + \frac{q_i}{2}\right)y_{i-1} - \left(\frac{2\varepsilon\sigma 1}{h^2} - \frac{p_i}{h} + \frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon} + \frac{h}{2\varepsilon}p_i q_{i-\frac{1}{2}}\right)y_i + \left(\frac{\varepsilon\sigma 1}{h^2} + \frac{q_i}{2}\right)y_{i+1}$$

$$= f_i - \frac{h}{2\varepsilon} p_i f_{i-\frac{1}{2}}. \tag{20}$$

Again from the theory of singular perturbation, it is known that the solution of eq. (6) to eq. (8) is of the form (O'Malley [11])

$$y(x) = y_0(x) + \frac{p(1)}{p(x)}\left(\gamma_1 - y_0(1)\right)e^{-\int_1^x (\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)})dx} + O(\varepsilon), \tag{21}$$

where $y_0(x)$ is the solution of $p(x)y'(x)+q(x)y(x)=f(x)$, $y(0)=\alpha(0)=\varphi_0$. By taking the Taylor's series expansions of $p(x)$ and $q(x)$ about the point 1 and restricting to their first terms, we get

$$y(x)=y_0(x)+\left(\gamma_1-y_0(1)\right)e^{-\left(\frac{p(1)}{\varepsilon}-\frac{q(1)}{p(1)}\right)(x-1)}+O(\varepsilon). \tag{22}$$

At $x=x_i$, the fitting factor $\sigma 1(\rho)$ is to be determined in such a way that the solution of (20) converges uniformly to the solution of eq. (6) to eq. (8) and which is equivalent to the solution of eq. (1) to eq. (3). Now multiplying eq. (20) by h and taking limit $h\to 0$, using eq. (22), we get

$$\sigma 1=\frac{\rho\left(p(1)-\frac{\rho}{2}p^2(1)\right)\left(-1+e^{\left(\frac{p^2(1)-\varepsilon q(1)}{p(1)}\right)\rho}\right)}{e^{\left(\frac{\rho^2(1)-\varepsilon q(1)}{p(1)}\right)\rho}-2+e^{-\left(\frac{p^2(1)-\varepsilon q(1)}{p(1)}\right)\rho}}, \tag{23}$$

where $\rho=\frac{h}{\varepsilon}$. Arranging eq. (20) in three term recurrence relation, we get system of equations of $n-N-1$ equations.

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i,\quad i=N+1,N+2,\dots,n-1. \tag{24}$$

where

$$E_i=\frac{\varepsilon\sigma 1}{h^2}-\frac{p_i}{h}+\frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon}+\frac{q_i}{2},\ F_i=\frac{2\varepsilon\sigma 1}{h^2}-\frac{p_i}{h}+\frac{p_i p_{i-\frac{1}{2}}}{2\varepsilon}+\frac{h}{2\varepsilon}p_i q_{i-\frac{1}{2}},\ G_i=\frac{\varepsilon\sigma 1}{h^2}+\frac{q_i}{2},\ H_i=f_i-\frac{h}{2\varepsilon}p_i f_{i-\frac{1}{2}}.$$

We have a system of $n-2$ equations from both left and right end boundary layer problem with $n+1$ unknowns. From the given boundary conditions, eq. (7) and eq. (8), we get two equations i.e.

$$y(0)=\alpha(0)=\varphi_0,\qquad y(1)=\beta(1)=\gamma_1.$$

We need one more equation to solve for the unknowns $(y_0,y_1,\dots,y_n)$. For this, we consider the eq. (6) at $\varepsilon=0$ and the point $x=x_N$, we get

$$p(x_N)y'(x_N)+q(x_N)y(x_N)=f(x_N).$$

Using second order central finite difference formula, we get

$$\frac{p_N}{2h}y_{N-1}-q_N y_N+\left(-\frac{p_N}{2h}\right)y_{N+1}=-f_N. \tag{25}$$

With this eq. (25), we now have $n+1$ equations to solve for the unknowns $(y_0,y_1,\dots,y_n)$. Using invariant embedding algorithm also known as Thomas algorithm, we get the solution.

3. Numerical Experiments

In this section, three numerical examples are presented and observed that the solutions obtained from this method are well agreed with the available/exact solutions.

The exact solution of the differential-difference equation

$$\varepsilon y''(x)+a(x)y(x-\delta)+c(x)y(x)+b(x)y(x+\eta)=f(x),\quad 0<x<1$$

with the boundary conditions $y(x)=\alpha(x)$, $-\delta\le x\le 0$ and $y(x)=\beta(x)$, $1\le x\le 1+\eta$ with constant coefficients (i.e. $a(x)=a$, $b(x)=b$, $c(x)=c$, $f(x)=f$, $\alpha(x)=\alpha$, $\beta(x)=\beta$ are constants) is given by Lange and Miura [9]

$$y(x)=\frac{\begin{pmatrix}\{(1-a-b-c)\exp(m_2)-1\}\exp(m_1x)\\-\{(1-a-b-c)\exp(m_1)-1\}\exp(m_2x)\end{pmatrix}}{(a+b+c)(\exp(m_1)-\exp(m_2))}+\frac{1}{(a+b+c)} \tag{26}$$

where

$$m_1 = \frac{\left[(a\delta - b\eta) + \sqrt{(b\eta - a\delta)^2 - 4\varepsilon(a+b+c)}\right]}{2\varepsilon}, \quad m_2 = \frac{\left[(a\delta - b\eta) - \sqrt{(b\eta - a\delta)^2 - 4\varepsilon(a+b+c)}\right]}{2\varepsilon}.$$

Example 1. Consider the differential-difference equation having dual boundary layer

$$\varepsilon y''(x) - 2y(x-\delta) - y(x) - 2y(x+\eta) = 1, \quad 0 < x < 1$$

with boundary conditions $y(0) = 1$ and $y(1) = 0$. The exact solution is given by eq. (26). Results are shown in Table 1 and 2 and the layer behaviour in Figure 1 and 2 for different values of δ and η.

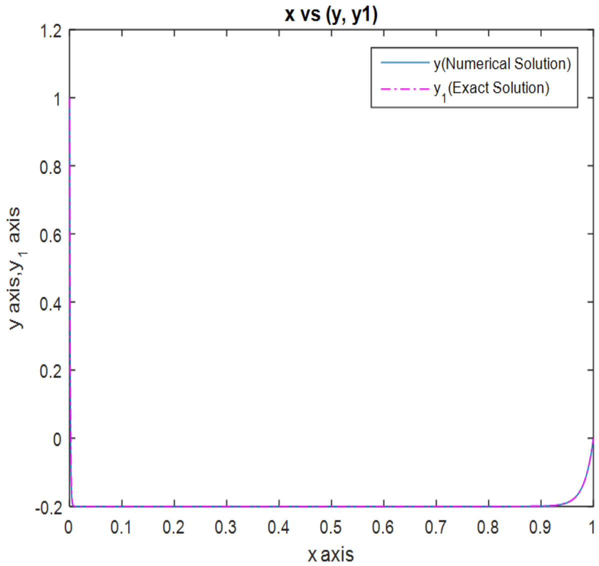

Figure 1. $h = 0.001$, $\varepsilon = 0.0001$, $\delta = 0.07$ and $\eta = 0.03$

x	Numerical solution	Exact solution	Solution by [7]
0	1	1	1
0.001	0.26926685	0.30868006	0.42984378
0.002	-0.01649051	0.01562950	0.13058599
0.003	-0.12823756	-0.10859463	-0.02648538
0.004	-0.17193689	-0.16125326	-0.10892741
0.005	-0.18902576	-0.18357525	-0.15219874
0.100	-0.20000000	-0.20000000	-0.20000000
0.300	-0.20000000	-0.20000000	-0.20000000
0.500	-0.19999999	-0.19999999	-0.19999999
0.700	-0.19999999	-0.19999999	-0.19999999
0.900	-0.19942061	-0.19940988	-0.19931853
0.995	-0.05067700	-0.05053989	-0.04946047
0.996	-0.04169036	-0.04157408	-0.04065940
0.997	-0.03216288	-0.03207043	-0.03134379
0.998	-0.02206201	-0.02199667	-0.02148356
0.999	-0.01135324	-0.01131861	-0.01104686
1	0	0	0

Table 1

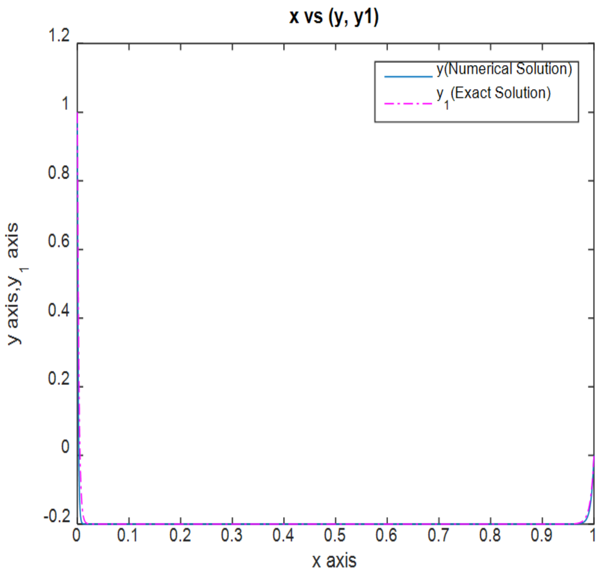

Figure 2. $h = 0.001$, $\varepsilon = 0.0001$, $\delta = 0.07$ and 0.06

x	Numerical solution	Exact solution	Solution by [7]
0	1	1	1
0.001	0.42375148	0.64990779	0.66882623
0.002	0.12422159	0.40195270	0.42904918
0.003	-0.03147191	0.22633691	0.25544535
0.004	-0.11240029	0.10195588	0.12975239
0.005	-0.15446629	0.01386221	0.03874794
0.100	-0.20000000	-0.19999999	-0.19999999
0.300	-0.20000000	-0.20000000	-0.20000000
0.500	-0.20000000	-0.20000000	-0.20000000
0.700	-0.20000000	-0.20000000	-0.20000000
0.900	-0.19999999	-0.19999989	-0.19999984
0.995	-0.11871564	-0.10311037	-0.10098645
0.996	-0.10267787	-0.08799746	-0.08603757
0.997	-0.08347578	-0.07052724	-0.06883173
0.998	-0.06048503	-0.05033201	-0.04902819
0.999	-0.03295811	-0.02698671	-0.02623475
1	0	0	0

Table 2

Example 2. Consider the differential-difference equation having dual boundary layer

$$\varepsilon y''(x) + 0.25y(x-\delta) - y(x) + 0.25y(x+\eta) = 1, \quad 0 < x < 1$$

with boundary conditions $y(0) = 1$ and $y(1) = 0$. The exact solution is given by eq. (26). Results are shown in Table 3 and 4 and the layer behaviour in Figure 3 and 4 for different values of δ and η.

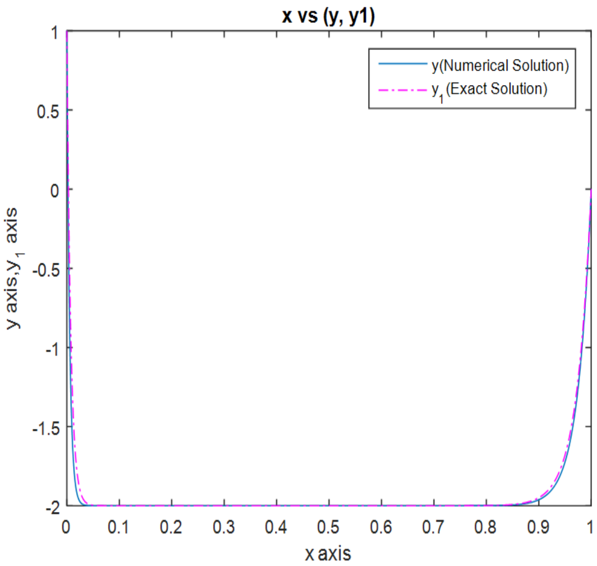

Figure 3. $h = 0.001$, $\varepsilon = 0.0001$, $\delta = 0.03$ and $\eta = .07$

x	Numerical solution	Exact solution	Solution by [7]
0	1	1	1
0.001	0.48061198	0.61695060	0.63026479
0.002	0.05114527	0.28281015	0.30609763
0.003	-0.30396814	-0.00866619	0.02188247
0.004	-0.59760101	-0.26292593	-0.22730456
0.005	-0.84039742	-0.48472099	-0.44578053
0.100	-1.99999998	-1.99999649	-1.99999417
0.300	-1.99999999	-1.99999999	-1.99999999
0.500	-1.99999999	-1.99999977	-1.99999997
0.700	-1.99998602	-1.99996594	-1.99996177
0.900	-1.96176269	-1.94854804	-1.94652875
0.995	-0.35902161	-0.33448484	-0.33127601
0.996	-0.29278472	-0.27239333	-0.26973108
0.997	-0.22387423	-0.20798701	-0.20591628
0.998	-0.15218222	-0.14117958	-0.13974790
0.999	-0.07759641	-0.07188153	-0.07113914
1	0	1	0

Table 3

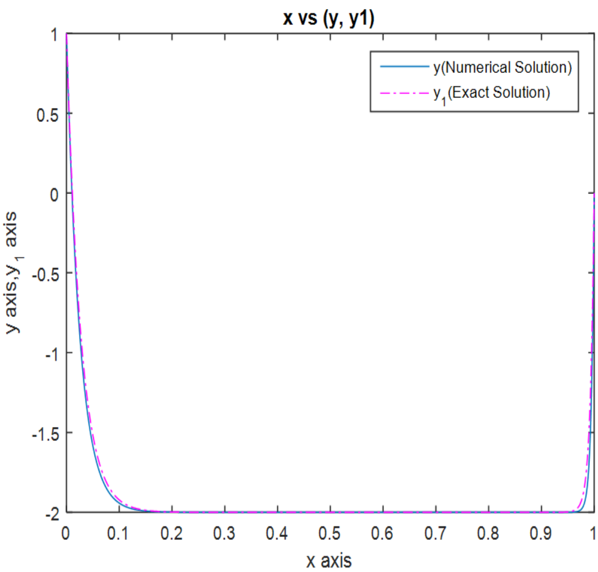

Figure 4. $h = 0.001$, $\varepsilon = 0.0001$, $\delta = 0.07$ and $\eta = 0.03$

x	Numerical solution	Exact solution	Solution by [7]
0	1	1	1
0.001	0.88360538	0.89217770	0.89105458
0.002	0.77172666	0.78823061	0.78606553
0.003	0.66418864	0.68801947	0.68488917
0.004	0.56082290	0.59140999	0.58738705
0.005	0.46146757	0.49827273	0.49342573
0.100	-1.94264404	-1.92282207	-1.92576223
0.300	-1.99997903	-1.99994892	-1.99995453
0.500	-1.99999999	-1.99999996	-1.99999997
0.700	-1.99999999	-1.99999999	-1.99999999
0.900	-1.99999998	-1.99999766	-1.99999848
0.995	-1.22693161	-0.98981399	-1.01843802
0.996	-1.06506734	-0.84195062	-0.86827690
0.997	-0.86931209	-0.67244413	-0.69514386
0.998	-0.63256981	-0.47812656	-0.49552461
0.999	-0.34625867	-0.25536626	-0.26536725
1	0	0	0

Table 4

Example 3. Consider the differential-difference equation having dual boundary layer

$$\varepsilon y''(x) - y(x-\delta) - y(x) - 3y(x+\eta) = 1, \quad 0 < x < 1$$

with boundary conditions $y(0) = 1$ and $y(1) = 0$. The exact solution is given by eq. (26). Results are shown in Table 5 and 6 and the layer behaviour in Figure 5 and 6 for different values of δ and η.

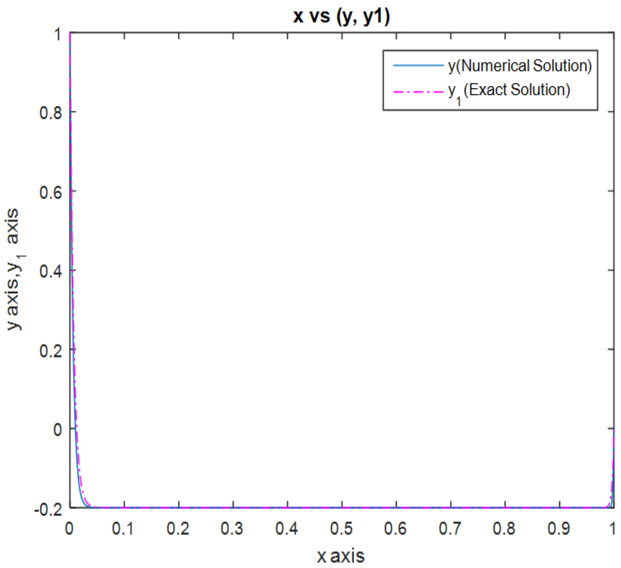

Figure 5. $h = 0.001$, $\varepsilon = 0.0001$, $\delta = 0.07$ and $\eta = 0.03$

x	Numerical solution	Exact solution
0	1	1
0.001	0.80225133	0.83807971
0.002	0.63708978	0.69800791
0.003	0.49914529	0.57683650
0.004	0.38393275	0.47201518
0.005	0.28770615	0.38133777
0.100	-0.19999998	-0.19999939
0.300	-0.20000000	-0.20000000
0.500	-0.19999999	-0.20000000
0.700	-0.20000000	-0.20000000
0.900	-0.20000000	-0.20000000
0.995	-0.19241104	-0.16435629
0.996	-0.18540004	-0.14967401
0.997	-0.17191198	-0.12894384
0.998	-0.14596306	-0.09967454
0.999	-0.09604141	-0.05834870
1	0	0

Table 5

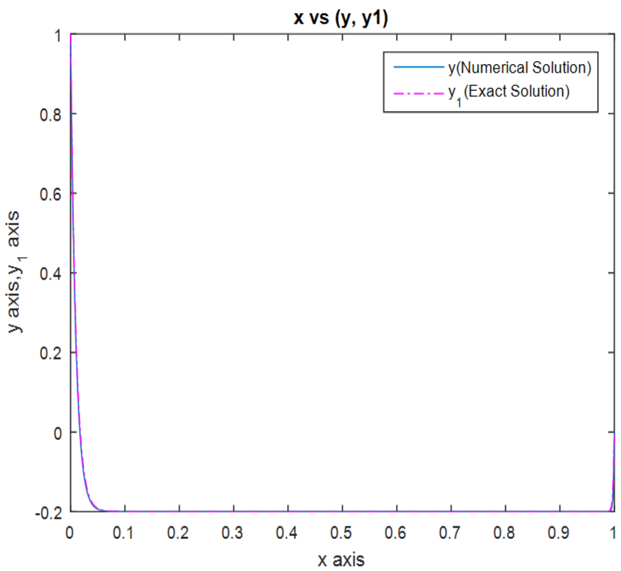

Figure 6. $h = 0.001$, $\varepsilon = 0.0001$, $\delta = 0.08$ and $\eta = 0.04$

x	Numerical solution	Exact solution
0	1	1
0.001	0.88103175	0.88580490
0.002	0.77385804	0.78247690
0.003	0.67730955	0.68898186
0.004	0.59033290	0.60438405
0.005	0.51197914	0.52783679
0.100	-0.19996493	-0.19994552
0.300	-0.19999999	-0.19999999
0.500	-0.20000000	-0.20000000
0.700	-0.20000000	-0.20000000
0.900	-0.20000000	-0.20000000
0.995	-0.19202193	-0.18358300
0.996	-0.18480417	-0.17293294
0.997	-0.17105652	-0.15537396
0.998	-0.14487138	-0.12642411
0.999	-0.09499655	-0.07869386
1	0	0

Table 6

4. Discussion and Conclusions

We have presented the method for solving singularly perturbed differential-difference equations with dual boundary layer. Firstly, differential-difference equation converted into ordinary differential equation by using Taylor's expansion, afterwards we applied exponentially fitted special finite difference scheme and obtained the value of fitting factor from the theory of singular perturbations. We have implemented this method on three standard examples. Numerical, exact results and layer behaviour are presented in their respective figures and tables for different values of the parameters. It can be observed that our numerical solutions approximate the exact solutions very well which shows the efficiency of this method.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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