Journal of Informatics and Mathematical Sciences
Vol. 11, Nos. 3-4, pp. 301–311, 2019
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v11i3-4.1265

http://www.rgnpublications.com

Research Article

The Analysis of Bifurcation Solutions for the Camassa-Holm Equation

Hussein K. Kadhim* and Mudhir A. Abdul Hussain

Department of Mathematics, University of Basrah, Basrah, Iraq
***Corresponding author:** khashanhussein@gmail.com

Abstract. This paper studies the Camassa-Holm equation's bifurcation solutions by using the local method of Lyapunov-Schmidt. The Camassa-Holm equation has been studied with the formula ODE. We have found the key function corresponding to the functional related to this equation. The bifurcation analysis of this function has been investigated by the boundary singularities. We have found the parametric equation of the bifurcation set (caustic) with the geometric description of this caustic. Also, the critical points' bifurcation spreading has been found.

Keywords. Camassa-Holm equation; Bifurcation solutions; Boundary Singularities; Caustic

MSC. 34K18; 34K10

Received: June 25, 2019 **Accepted:** July 18, 2019

1. Introduction

The nonlinear problems which occur in mathematics and physics may be formed in the form of operator equation,

$$f(x,\lambda) = b, \quad x \in O,\ b \in Y,\ \lambda \in \mathbb{R}^n \tag{1}$$

in which f is a smooth Fredholm map whose index is zero and X, Y are *Banach's* spaces and $O \subseteq X$ is open. The method of reduction for these problems to the finite dimensional equation,

$$\Theta(\xi,\lambda) = \beta, \quad \xi \in M,\ \beta \in N \tag{2}$$

may be used, where M and N are smooth finite dimensional manifolds. Lyapunov-Schmidt method can reduce equation (1) to equation (2) in which equation (2) has all the analytical and topological features of equation (1) (bifurcation diagram, multiplicity, etc.), as such information can be found in [8], [9], [10], [13]. Singularities of smooth maps play an important part in the investigation of bifurcation solutions of BVPs. One can find a good review of these studies in [5]. In initial years, the study of singularities of smooth maps and its applications to the BVPs took an important character in the works of Sapronov and his group. For example, in [12] Shvyreva studied the boundary singularities of the function

$$\tilde{W}(\eta,\gamma)=\eta_1^4+(c\eta_1+\eta_2)^2-2\varepsilon_1\eta_1^2+2\varepsilon_2\eta_1^2\eta_2+2\varepsilon_3\eta_1\eta_2+2\varepsilon_4\eta_1+2\varepsilon_5\eta_2,$$

where $\eta=(\eta_1,\eta_2)$, $\gamma=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5)$, $\eta_1,\eta_2\geq 0$, and considered the functional,

$$V(u,\lambda)=\int_0^\pi\left(\frac{(u')^2}{2}+\lambda(\cos(u(x))-1)\right)dx,$$

with the extra condition and in [1] A. Hussain has studied the following problem, $\frac{d^4u}{dx^4}+\alpha\frac{d^2u}{dx^2}+\beta u+u^2=0$, $u(0)=u(1)=u''(0)=u''(1)=0$, with the extra condition $u(x_1)\geq 0$, $u(x_2)\geq 0$, $x_1,x_2\in[0,1]$. Our study differs from previous studies in the assumption of a boundary value problem that was not previously studied in the same manner as our paper, and which we will talk about it in Section 3.

Lyapunov-Schmidt method supposes that $f:\Omega\subset E\to F$ is a smooth nonlinear Fredholm map of index zero. The map f has variational property, when there is a functional $V:\Omega\subset E\to\mathbb{R}$ such that $f=grad_H V$ or equivalently, $\frac{\partial V}{\partial x}(x,\lambda)h=\langle f(x,\lambda),h\rangle_H$, $\forall\ x\in\Omega$, $h\in E$ where $\langle\cdot,\cdot\rangle_H$ is the scalar inner product in Hilbert space H. Also, it assumes that $E\subset F\subset H$. The solutions of equation $f(x,\lambda)=0$ are the own critical points of functional $V(x,\lambda)$. The finite dimensional reduction method (Lyapunov-Schmidt method) can reduce the problem, $V(x,\lambda)\to extr$, $x\in E$, $\lambda\in\mathbb{R}^n$ into equivalent problem $W(\xi,\lambda)\to extr$, $\xi\in\mathbb{R}^n$, where $W(\xi,\lambda)$ is called key function. If we let $N=span\{e_1,\ldots,e_n\}$ is a subspace of Banach space E, where $\{e_1,\ldots,e_n\}$ is an orthonormal set in H, then the key function $W(\xi,\lambda)$ may be defined by the form of $W(\xi,\lambda)=\inf\limits_{x:\langle x,e_i\rangle=\xi_i}V(x,\lambda)$, $\xi=(\xi_1,\ldots,\xi_n)$. The function W possesses every the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, etc.) [9]. The study of bifurcating solutions of functional V is tantamount to the study of bifurcating solutions of key function. If f possesses a variational property, then the equation $\Theta(\xi,\lambda)=grad\,W(\xi,\lambda)=0$ is called bifurcating equation.

Definition 1.1 ([4])**.** The set of every λ for which the function $f(x,\lambda)$ possesses degenerate critical points is called bifurcation set (Caustic) and denoted by Σ.

The paper is structured as follows. In Section 2, we introduce the sense of boundary singularities of Fredholm functional. In Section 3, we investigate bifurcation solutions of the Camassa-Holm equation with boundary conditions. Then, we present our outcomes. In Section 4, we conclude our study.

2. Boundary Singularities of Fredholm Functional [11]

To investigate the behavior of a Fredholm functional in a neighborhood of an angular singular point, one uses the reduction to an analogous extremes problem

$$W(x)\to extr$$

where

$$x \in C,\ C = \{x = (x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+t}, \ldots, x_n)^\top \in \mathbb{R}^n :$$
$$\exists \text{ integers } t,\ m \ni 0 \le t \le n-m \text{ and } x_m \ge 0,\ x_{m+1} \ge 0, \ldots,\ x_{m+t} \ge 0\}$$

We say that a point $a \in C$ is conditionally critical for a smooth function W in $\mathbb{R}^n$ if $grad\, W(a)$ is orthogonal to the least face of C containing a and the set $\{(x_m, x_{m+1}, \ldots, x_{m+t})^\top \in \mathbb{R}^n : x_m \ge 0,\ x_{m+1} \ge 0, \ldots,\ x_{m+t} \ge 0\}$ is called m-hedral angle.

The multiplicity of the conditionally critical point$a$ (and is denoted by $\overline{\mu}$) is the dimension of the quotient algebra denotes the set,

$$\hat{Q} = \frac{\prod_a(\mathbb{R}^n)}{I},$$

where $\prod_a(\mathbb{R}^n)$ is the ring of germs of smooth functions on $\mathbb{R}^n$ at the point $a$and

$$I := \begin{cases} \left(\frac{\partial W}{\partial x_1}, \ldots, x_m \frac{\partial W}{\partial x_m}, \ldots, x_{m+t} \frac{\partial W}{\partial x_{m+t}}, \ldots, \frac{\partial W}{\partial x_n}\right), & \text{for all } m > 1, \\ \left(x_m \frac{\partial W}{\partial x_m}, \ldots, x_{m+t} \frac{\partial W}{\partial x_{m+t}}, \ldots, \frac{\partial W}{\partial x_n}\right), & \text{for } m = 1 \end{cases}$$

is the angular Jacobi ideal in $\prod_a(\mathbb{R}^n)$. The multiplicity $\overline{\mu}$ of a conditionally critical point a is equal to the sum of multiplicities $\mu + \mu_0$, where μ is the (usual) multiplicity of W on $\mathbb{R}^n$, while μ_0 is the (usual) multiplicity of the restriction $W|\partial C$ (where ∂C is the boundary of the set C).

Then, we reduce the space of $W(x)$, $x \in \mathbb{R}^n$ to the space C as follows: Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in H. By Lyapunov-Schmidt method, one can write any element $z \in E$ as the form, $z = u + v$ where $u = \sum_{i=1}^{n} x_i e_i$, $v \perp e_i$, $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

If we consider there is a condition on E (domain of functional V) is as following, let z to be in E where z fulfills the following condition:

$$\langle z, e_s \rangle \ge 0, \tag{3}$$

for some $1 \le s \le n$, so we get $x_s \ge 0$. Also, we require that the coordinates $\{x_s\}_{\exists\ s \ni 1 \le s \le n}$ are equal the coordinates of m-hedral angle. From above we conclude the domain of $W(x)$, $x \in \mathbb{R}^n$ may reduce to the space C. If a critical point is usual, then spreadings of bifurcating extremes (bif-spreadings) are represented by the row $(c_0, c_1, \ldots, c_n)$, where c_i is the number of critical points of the Morse index i. If we are dealing with an angular(or boundary) critical point, then bif-spreadings are represented by the following matrix of order $(m+t+1) \times (n+1)$:

$$\begin{pmatrix} c_0^1 & c_1^1 & \cdots & c_n^1 \\ c_0^2 & c_1^2 & \cdots & c_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_0^{m+t} & c_1^{m+t} & \cdots & c_n^{m+t} \\ c_0 & c_1 & \cdots & c_n \end{pmatrix}.$$

Here c_i^j is the number of the angular critical points of index i (for $j = 1, 2, \ldots, m+t$), while c_i is the number of usual (situated inside C) critical points of index i.In this paper, we let$(n = m = 2, t = 0, s = 2)$ for the function in Section 3.1.

3. The Camassa-Holm Equation's Bifurcation Solutions

In 1993, Camassa and Holm used Hamiltonian method to derive a new formula for a completely integrable shallow water wave equation,

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{4}$$

where u is the fluid speed in the x trend (or equivalently the height of the waters free surface above a flat bottom), k is a constant related to the critical depthless water wave velocity, and subscripts indicate partial derivatives. This equation keeps higher order terms (the right hand of) (4) in a little amplitude expansion of incompressible Euler's conditions for unidirectional movement of wave at the free surface under the effect of gravity. Now, equation (4) is called equation of Camassa-Holm (CH). In recently years, CH equation has been generalized to the coming general Camassa-Holm (GCH) equation,

$$u_t + 2ku_x - u_{xxt} + \frac{1}{2}[f(u)]_x = 2u_x u_{xx} + uu_{xxx}, \tag{5}$$

where $f(u)$ a function of u and $[f(u)]_x$ is the derivative of f with respect to x. In[6,7], the authors investigated traveling wave solutions' the bifurcations for the general Camassa-Holm equation (4) and corresponding traveling wave system with $f(u) = \alpha u^2 + \beta\, u^3$. Note that we can obtain equation (4) from equation (5) by assuming $\alpha = 3$ and $\beta = 0$ in the function $f(u) = \alpha u^2 + \beta u^3$.

Suppose that $u(x,t) = z(y)$, $y = x - \alpha t$, where α the wave speed. The equation (4) can be converted to the coming ODE with variable $z(y)$,

$$\alpha z'' + \beta z + c32z^2 - \left(\frac{1}{2}(z')^2 + zz''\right) = 0, \tag{6}$$

where α, β are parameters and $' = \frac{d}{dy}$.

In this section, we investigate the equation (6)'s bifurcation solutions with the coming boundary conditions which satisfies the equation (6),

$$z(0) = z(1) = 0,$$

where $z = z(y)$, $y \in [0,1]$.

Assume that $f : E \to M$ is a nonlinear Fredholm operator whose index equal zero from Banach space E to Banach space M, where $E = C^2([0,1],\mathbb{R})$ is the space of every continuous functions that have derivative of order at most two, $M = C^0([0,1],\mathbb{R})$ is the space of every continuous function and f are defined by the operator equation,

$$f(z,\lambda) = \alpha z'' + \beta z + \frac{3}{2}z^2 - \left(\frac{1}{2}(z')^2 + zz''\right) = 0, \tag{7}$$

where $\lambda = (\alpha, \beta)$. Every solution of the equation (6) (1-periodic solution) is a solution of the operator equation (7). Since, the operator f possesses variational property, then there exists functional V such that,

$$f(z,\lambda) = grad_H V(z,\lambda),$$

where

$$V(z,\lambda) = \frac{1}{2}\int_0^1 (-\alpha(z')^2 + \beta z^2 + z^3 + z(z')^2)dx,$$

where z fulfills the condition (3) (when $n = 2$, $s = 2$) and $grad_H V$ denotes the gradient of V.

In this case every solution of equation (7) is functional V's a critical point.

The purpose of the paper is to find the solution areas of the equation (6) where each bifurcating solution of the equation (6) equals a critical point of functional V and each critical point of functional V coincides a critical point of the key function of functional V. Therefor, in subsections below, we shall investigate a function's the extremes bifurcation in which is its extremes bifurcation's study tantamount investigating the key function's the extremes bifurcation of functional V (i.e. the study of functional V's bifurcating solutions is tantamount to the study of bifurcating solutions of this function). Hence, the study of the equation (6)'s bifurcating solutions is equivalent to the study of bifurcating solutions of this function.

3.1 Singularities of the Function of Codimension Three

In this section, we consider the following function [2] defined by

$$W(s,\rho)=\frac{x_1^3}{3}+x_1x_2^2+\lambda_1x_1^2+\lambda_2x_2^2, \tag{8}$$

where $s=(x_1,x_2)$, $x_2\ge 0$, $\rho=(\lambda_1,\lambda_2)$ and λ_1, λ_2 are parameters.

Function (8) has codimension three at the origin, hence it has multiplicity four at the original. The main purpose is to find geometrical description (bifurcation diagram) of the caustic of function (8)and then to determine the spreading of the critical points of this function. Since, the germ(the principle part) of function (8) is

$$W_0=\frac{x_1^3}{3}+x_1x_2^2.$$

So, from section 2 we have $I=\left(\frac{\partial W_0}{\partial x_1},x_2\,\frac{\partial W_0}{\partial x_2}\right)=(x_1^2+x_2^2,2x_1x_2^2)=(x_1^2+x_2^2,x_1x_2^2)$, and $\overline{\mu}=6$ where $\mu=4$ and $\mu_0=2$. Since multiplicity $\overline{\mu}$ is equal to the number of critical points [3], hence the number of critical points of function (8) is six, two points lie on the boundary $y=0$ and four points lie in the interior, so the caustic of function (8) is the union of three sets,

$$\Sigma=\Sigma_{1,0}^{int}\cup\Sigma_{1,0}^{ext}\cup\Sigma_{1,1}$$

where $\Sigma_{1,0}^{int}$ and $\Sigma_{1,0}^{ext}$ are the subsets (components) of the caustic corresponding to the degeneration of boundary singularities along the boundary and along the normal, respectively, while $\Sigma_{1,1}$ is the component corresponding to the degeneration of interior (non-boundary) critical points.

3.2 Degeneration Along the Boundary (Internal Degeneration)

The coming lemma gives the equation which represents the set $\Sigma_{1,0}^{int}$.

Lemma 3.1. *The parametric equation which represents the set $\Sigma_{1,0}^{int}$ is given by the form*

$$\lambda_1^2=0.$$

Proof. To determine the set $\Sigma_{1,0}^{int}$, we consider boundary critical points of function (8) such that the second-order partial derivatives of this function with respect to x_1 vanishes at these points, i.e., the following relations are valid:

$$\frac{\partial W(x_1,0,\lambda_1,\lambda_2)}{\partial x_1}=\frac{\partial^2 W(x_1,0,\lambda_1,\lambda_2)}{\partial x_1^2}=0$$

or

$$x_1^2 + 2x_1\lambda_1 = 2x_1 + 2\lambda_1 = 0.$$

We may represent the above relations as the equations system,

$$x_1^2 + 2x_1\lambda_1 = 0, \tag{9}$$

$$2x_1 + 2\lambda_1 = 0. \tag{10}$$

From the equation (10) one get $x_1 = -\lambda_1$, and substituting the value of x_1 in the equation (9) we find,

$$\lambda_1^2 = 0$$

which represents the set $\Sigma_{1,0}^{int}$. □

3.3 Degeneration Along the Boundary (External Degeneration)

The coming lemma shows the equation which represents the set $\Sigma_{1,0}^{ext}$ is empty.

Lemma 3.2. *The parametric equation which represents the set $\Sigma_{1,0}^{ext}$ is empty.*

Proof. To determine the set $\Sigma_{1,0}^{ext}$ we consider boundary critical points of function (8) such that the first-order partial derivatives of this function with respect to x_2 vanishes at these points, i.e., the following relations are valid:

$$\frac{\partial W(x_1,0,\lambda_1,\lambda_2)}{\partial x_1} = \frac{\partial W(x_1,0,\lambda_1,\lambda_2)}{\partial x_2} = 0$$

or

$$x_1^2 + 2x_1\lambda_1 = 0.$$

Because the above equation is true for all values λ_1 andλ_2, so there is no graph that represents the set $\Sigma_{1,0}^{ext}$, hence it is empty. □

3.4 Degeneration of Interior (Non-Boundary)

The coming lemma gives the equation which represents the set $\Sigma_{1,1}$.

Lemma 3.3. *The parametric equation which represents the set $\Sigma_{1,1}$ is given by the equation,*

$$\lambda_2(-2\lambda_1 + \lambda_2) = 0.$$

Proof. To determine the set $\Sigma_{1,1}$, we consider the critical points of function (8) defined by the system,

$$\frac{\partial W(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_1} = \frac{\partial W(x_1,x_2,\lambda_1,\lambda_2)}{\partial x_2} = 0, \quad x_2 > 0$$

or

$$x_1^2 + 2x_1\lambda_1 + x_2^2 = 2x_1x_2 + 2x_2\lambda_2 = 0. \tag{11}$$

Then, make the determinate of Hessian matrix of function (8) equal to zero to get the equation,

$$4x_1^2 + 4x_1\lambda_1 + 4x_1\lambda_2 - 4x_2^2 + 4\lambda_1\lambda_2 = 0. \tag{12}$$

We can express (11) and (12) in the following system:

$$x_1^2 + 2x_1\lambda_1 + x_2^2 = 0, \tag{13}$$

$$x_2\,(x_1+\lambda_2)=0, \tag{14}$$

$$4x_1^2+4x_1\lambda_1+4x_1\lambda_2-4x_2^2+4\lambda_1\lambda_2=0. \tag{15}$$

From the equation (14), one get $x_2=0$ or $x_1=-\lambda_2$, but $x_2>0$ by the above assumption, hence we must substitute only $x_1=-\lambda_2$ in the equation (13) to get,

$$x_2^2-2\lambda_1\lambda_2+\lambda_2^2=0 \quad \text{or} \quad x_2^2=2\lambda_1\lambda_2-\lambda_2^2$$

and replacing the value of x_2^2 in the equation (15), we obtain, $-2\lambda_1\lambda_2+\lambda_2^2=0$ or

$$\lambda_2\,(-2\lambda_1+\lambda_2)=0,$$

which represents the set $\Sigma_{1,1}$. □

Theorem 3.1 below gives parametric equation of the bifurcating set (caustic) of function (8).

Theorem 3.1. *Parametric equation of bifurcation set (caustic) of function* (8) *is given by the equation,*

$$\lambda_1^2\lambda_2\,(-2\lambda_1+\lambda_2)=0.$$

Proof. Since, the caustic of function (8) consists of the union of three sets,

$$\Sigma=\Sigma_{1,0}^{int}\cup\Sigma_{1,0}^{ext}\cup\Sigma_{1,1},$$

so parametric equation of the caustic will be composed of the multiplication of all the left sides of the equations of the caustic components with equal to zero. Since, the equations of the caustic components had been found in the Lemmas 3.1, 3.2 and 3.3, hence the following equation,

$$\lambda_1^2\lambda_2\,(-2\lambda_1+\lambda_2)=0,$$

will represent the parametric equation of the bifurcating set(caustic) of function (8). □

Proposition 3.1 below gives existence's conditions for non-degenerate real critical points in$\lambda_1\lambda_2$-plane.

Proposition 3.1. *If the condition* $2\lambda_1<\lambda_2$; $(\lambda_1,\lambda_2<0)$ *or* $2\lambda_1>\lambda_2$; $(\lambda_1,\lambda_2>0)$ *satisfied, then function* (8) *has three non-degenerate real critical points(one interior point and two boundary points).*

Proof. The critical points of function (8) can be expressed as follows $P_1=\left(-\lambda_2,\sqrt{2\lambda_1\lambda_2-\lambda_2^2}\right)$, $P_2=\left(-\lambda_2,-\sqrt{2\lambda_1\lambda_2-\lambda_2^2}\right)$, $P_3=(-2\,\lambda_1,0)$ and $P_4=(0,0)$.

The two points P_1 and P_2 are non-degenerate real points when $2\lambda_1\lambda_2-\lambda_2^2>0$, or equivalently $[2\lambda_1<\lambda_2;\ (\lambda_1,\lambda_2<0)$ or $2\lambda_1>\lambda_2;\ (\lambda_1,\lambda_2>0)]$, otherwise they are complex points or degenerate points. Clearly, the two points P_3 and P_4 are non-degenerate real points for all $\lambda_1,\lambda_2\in\lambda_1\lambda_2$-*plane*. Now note that the point P_1 is lying in the interior of the domain of function (8), but the point P_2 is lying outside its domain. On the other hand, we find the points P_3 and P_4 are boundary points.This completes the proof. □

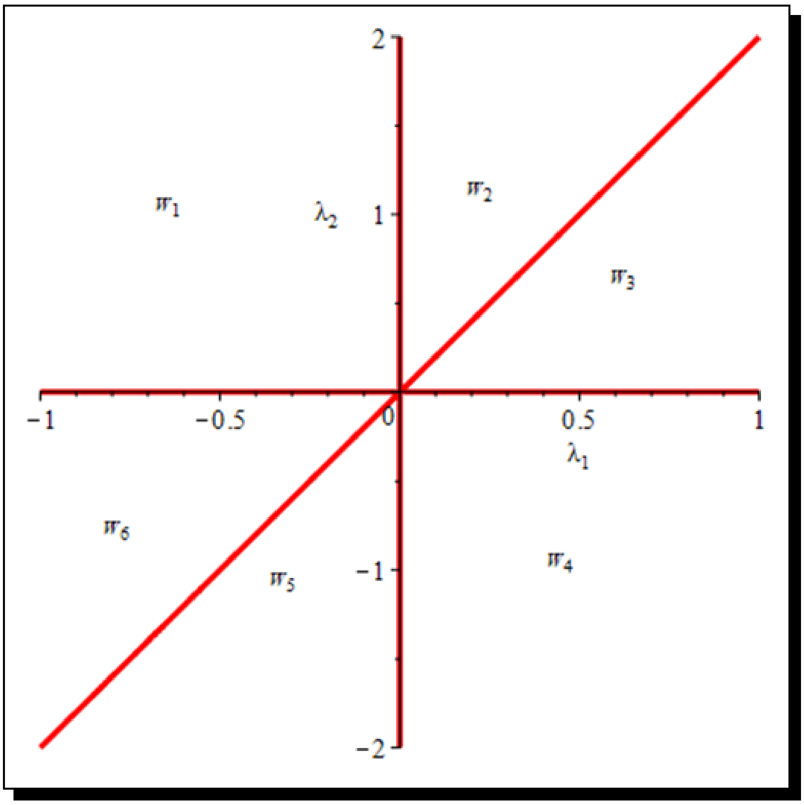

Figure 1. Describes the caustic of function (8) in $\lambda_1\lambda_2$-plane

Theorem 3.2. *The matrices of bif-spreadings of the critical points of function* (8) *are as follows:*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Proof. From the caustic equation which we had found in Theorem 3.1, we can find the geometric description of this equation as in Figure 1. This figure can decompose the plane of parameters into six regions W_i, $i = 1,2,3,4,5,6$; every region contains a fixed number of non-degenerate real critical points. These points are divided into internal and boundary points. The quality of the internal points can be determined using the second derivative test, while the quality of the boundary points can be determined using Morse's lemma [11]. Hence, the spreading of the critical points is as follows: if the parameters λ_1, λ_2 belong to

(1) W_3 or W_6, then have three critical points (one boundary saddle point, one boundary minimum point and one interior saddle point), or

(2) W_1 or W_2 or W_4 or W_5, then have two critical points (one boundary saddle point and one boundary minimum point).

From the above points, we get the matrices of bif-spreadings as is described in (16). □

In addition, parts (a) and (b) in Figure 2 show the locations of contour lines with respect to the boundary of the domain of function (8), the number and type of critical points corresponding for all region in the caustic of function (8), where showing of (a) corresponds the region W_3 or W_6 and (b) corresponds the region W_1 or W_2 or W_4 or W_5.

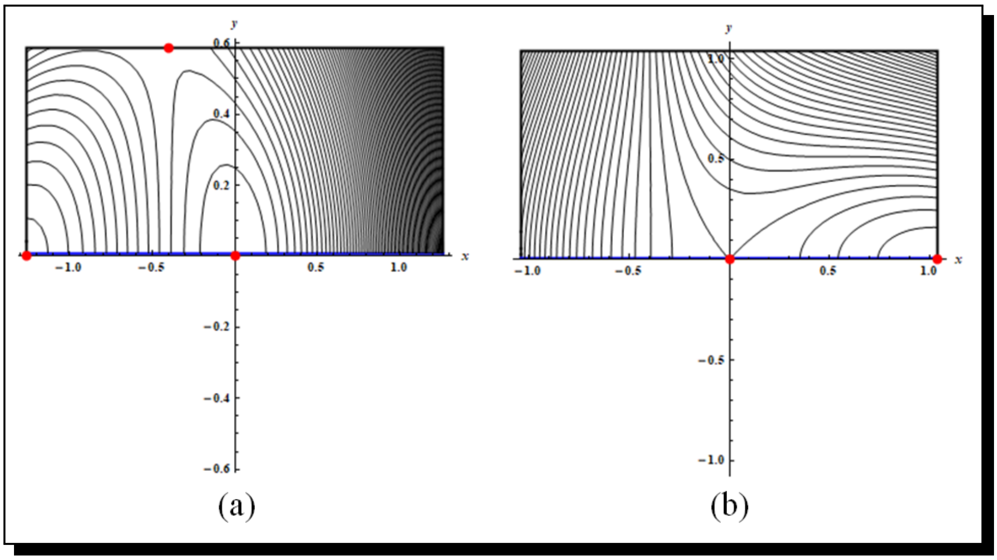

Figure 2

In the following theorem, we prove that investigating of functional V's extremes bifurcation is reduced to investigating of function (8)'s extremes bifurcation.

Theorem 3.3. *The normal form of the key function W_1 corresponding to the functional V is given by,*

$$W_1(y,\rho) = \frac{{x_1}^3}{3} + x_1{x_2}^2 + \lambda_1{x_1}^2 + \lambda_2{x_2}^2,$$

where $y = (x_1, x_2)$, $x_2 \geq 0$, $\rho = (\lambda_1, \lambda_2)$ and λ_1, λ_2 are the parameters.

Proof. By using the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (7) at the point $(0,\lambda)$ has the form,

$$A\ h = 0, \quad h \in Eh(0) = h(1) = 0,$$

where $A = \alpha\frac{d^2}{dx^2} + \alpha$.

The solution of the linearized equation which satisfies the initial conditions is given by $e_p(x) = c_p \sin(p\pi x)$, $p = 1,2,\ldots$ and the characteristic equation corresponding to this solution is

$$-\alpha(p\pi)^2 + \beta = 0.$$

This equation gives in *$\alpha\beta$-plane* characteristic lines ℓ_p. The characteristic lines ℓ_p consist of the points (α,β) for which the linearized equation has non-zero solutions [10]. The point of intersection of the characteristic lines in *$\alpha\beta$-plane* is bifurcation point, so the bifurcation point for the equation (6) is $(\alpha,\beta) = (0,0)$. Localized parameters α, β as following, $\alpha = 0 + \delta_1$, $\beta = 0 + \delta_2$, δ_1, δ_2 are small parameters, lead to bifurcation along the modes, $e_1(x) = c_1 \sin(\pi x)$, $e_2(x) = c_2 \sin(2\pi x)$. Since, $\|e_1\| = \|e_2\| = 1$ then we have $c_1 = c_2 = \sqrt{2}$.

Let $N = Ker(A) = span\{e_1, e_2\}$, then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E = N \oplus N^{\perp}, \ \ N^{\perp} = \left\{ v \in E : \int_0^1 v e_k\ dx = 0, k = 1,2 \right\}.$$

Similarly, the space M can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$M = N \oplus \tilde{N}^{\perp}, \quad \tilde{N}^{\perp} = \left\{ \omega \in M : \int_0^1 \omega e_k \, dx = 0, \ k = 1,2 \right\}.$$

There exists two projections $P : E \to N$ and $I - P : E \to N^{\perp}$ such that $Pz = \omega$and $(I - P)z = v$, (I is the identity operator). Hence every vector $z \in E$ can be written in the form,

$$z = \omega + v, \ \omega = x_1 e_1 + x_2 e_2 \in N, \ v \in N^{\perp}, \ x_i = \langle z, \ e_i \rangle .$$

Thus, by the implicit function theorem, there exists a smooth map $\Theta : N \to N^{\perp}$, such that

$$\tilde{W}(w,\gamma) = V(\Theta(w,\gamma),\gamma),$$

$$w = (x_1, x_2), \quad \gamma = (\delta_1, \delta_2),$$

and then the key function $\tilde{W}$ can be written in the the form,

$$\begin{aligned}\tilde{W}(w,\gamma) &= V(x_1 e_1 + x_2 e_2 + \Theta(x_1 e_1 + x_2 e_2, \gamma), \gamma) \\ &= W_2(w,\gamma) + o(|w|^3) + O(|w|^3)O(\gamma),\end{aligned}$$

where

$$W_2(w,\gamma) = \left(\frac{4}{3}\frac{\sqrt{2}}{\pi} + \frac{2}{3}\sqrt{2}\pi \right) x_1^3 + \left(\frac{16}{5}\frac{\sqrt{2}}{\pi} + \frac{24\sqrt{2}\pi}{5} \right) x_2^2 x_1 + \left(\frac{-1}{2}\alpha\pi^2 + \frac{\beta}{2} \right) x_1^2 + \left(-2\alpha\pi^2 + \frac{\beta}{2} \right) x_2^2.$$

Because z fulfills the condition (3) (when $n = s = 2$), we can get $x_2 \geq 0$.

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating for the function $\tilde{W}$ are completely determined by its principal part W_2. If, we replace x_1 and x_2 by

$$\sqrt[3]{\frac{\pi}{2\sqrt{2}(2+\pi^2)}}\, x_1 \quad \text{and} \quad \sqrt{\left(\sqrt[3]{\frac{2\sqrt{2}(2+\pi^2)}{\pi}} \right) \frac{5\pi}{8\sqrt{2}(2+3\pi^2)}}\, x_2$$

in the function W_2 respectively, then W_1 and W_2 are contact equivalence, since in this case they have the same germ (the same principle part),

$$W_0(x_1, x_2) = \frac{x_1^3}{3} + x_1 x_2^2$$

and the deformation. Therefore the caustic of the function W_2 coincides with the caustic of the function W_1.

Thus, the function W_1 has all the topological and analytical properties of functional V, so the study of bifurcation analysis of the equation (7) is equivalent to the study of bifurcation analysis of the function W_1. This shows that the study of bifurcation of extremes of the functional V is reduced to the study of bifurcation of extremes of the function (8). □

4. Conclusion

In this paper the solution areas for the equation (6) are found. Four regions are found, each region contains a fixed number and a fixed quality of solutions. Each solution represents a critical point of functional, which in turn corresponds to a critical point of the key function of functional. Furthermore, the geometrical description of the branching diagram (caustic) was found with spreading of the branching of the critical points. Studying the branching solutions for the equation (6) is an application for studying singularities of the function (8).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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