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Research Article

Birkhoff Center of a Quotient of Almost Distributive Fuzzy Lattices

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Abstract. The concept of a fuzzy congruence relation is introduced, and we also prove quotient isomorphism in Almost Distributive Fuzzy lattice.

Keywords. Almost distributive lattice; Almost distributive fuzzy lattice, fuzzy poset; Relatively complemented ADFL; Birkhoff center of an Almost distributive lattice; Birkhoff center of an almost distributive fuzzy lattice

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1. Introduction

The concept of an Almost distributive lattice was introduced by U. M. Swamy and G. C. Rao [7]. Swamy et al. [9] have extended the above concept for a general partial ordered set P and prove that $B(P)$ is relatively complemented distributive lattice in which the operations are least upper bound and greatest lower bound in P (provided that $B(P)$ is non-empty in Birkhoff center of an ADL).

The concept of Birkhoff center $B(R)$ of an ADL with maximal elements was introduced by Swamy and Ramesh [8] and prove that $B(R)$ is a relatively complemented Almost distributive lattice. The concept of a fuzzy set was first introduced by L. A. Zadah [10], and this concept was adapted by Goguen [4] and Sanchez [6] to define and study fuzzy relations. As a continuation of these studies, we define fuzzy relation, fuzzy poset and fuzzy lattice which enables us to define Birkhoff center of a quotient Almost distributive fuzzy lattice.

In this paper, we introduce the concept of quotient in an Almost distribution fuzzy lattice with maximal elements. Mainly, we obtain the equivalency of the quotient relation in ADL to the quotient relation in an ADFL with the property of Almost distributive lattice and Fuzzy partial order relation.

Throughout this paper, we consider only $ADFL_s$ which contain at least one maximal element. (R,A) denote an ADFL. An ADL $(R,\vee,\wedge,0)$ represented by R and $x\in(R,A)\Leftrightarrow x\in R$.

2. Preliminaries

Definition 2.1 ([5]). An algebra $(R,\vee,\wedge,0)$ of type $(2,2,0)$ is said to be an *Almost distributive lattice* (ADL) if it satisfies the following conditions:

(1) $a\vee 0=a$.

(2) $0\wedge a=0$.

(3) $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$.

(4) $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$.

(5) $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$.

(6) $(a\vee b)\wedge b=b$ for all $a,b,c\in R$.

The element 0 is called as usual zero element of R.

Definition 2.2 ([5]). Let X be a non-empty set. Fix $x_o\in X$.
For any $x,y\in X$, $x\wedge y=\begin{cases}x_o & \text{if } x=x_o\\ y & \text{if } x\neq x_o\end{cases}$ and $x\vee y=\begin{cases}y & \text{if } x=x_o\\ x & \text{if } x\neq x_o.\end{cases}$
Then $(X,\vee,\wedge,x_o)$ is an ADL with x_o as its zero element. This ADL is called a discrete ADL.

Example 2.3. Every distributive lattice with zero is an ADL.
For any a, b in an ADL R we say that a is less than or equal to b and write $a\leq b$, if $a\wedge b=a$. Then $\leq$ is a partial ordering on R.

Lemma 2.4 ([5]). *For any $a,b\in R$, we have*

(1) $a\wedge 0=0$ *and* $0\vee a=a$.

(2) $a\wedge a=a=a\vee a$.

(3) $(a\wedge b)\vee b=b$, $a\vee(b\wedge a)=a$ *and* $a\wedge(a\vee b)=a$.

(4) $a\wedge b=b\Leftrightarrow a\vee b=a$.

(5) $a\wedge b=a\Leftrightarrow a\vee b=b$.

(6) $a\wedge b\leq b$ *and* $a\leq a\vee b$.

(7) $a\wedge b=b\wedge a$, *whenever* $a\leq b$.

(8) $a\vee(b\vee a)=a\vee b$.

Theorem 2.5 ([5]). *For any $a,b \in R$, the following are equivalent to each other.*

(1) $(a \wedge b) \vee a = a$.

(2) $a \wedge (b \vee a) = a$.

(3) $(b \wedge a) \vee b = b$.

(4) $b \wedge (a \vee b) = b$.

(5) $a \wedge b = b \wedge a$.

(6) $a \vee b = b \vee a$.

(7) *The supremum of a and b exists and equal to $a \vee b$.*

(8) *There exists $x \in R$ such that $a \leq x$ and $b \leq x$.*

(9) *The infimum of a and b exists and equal to $a \wedge b$.*

Theorem 2.6 ([5]). *For any $a,b \in R$, we have*

(1) $(a \vee b) \wedge c = (b \vee a) \wedge c$.

(2) $\bigwedge$ *is associative in R.*

(3) $a \wedge b \wedge c = b \wedge a \wedge c$.

From the above theorem, it follows that for any $x \in R$ the set $\{a \wedge x \mid a \in R\}$ forms a bounded distributive lattice.

In particular, we have $((a \wedge b) \vee c) \wedge x = ((a \vee c) \wedge (b \vee c)) \wedge x$, for all $a,b,c,x \in R$.

An element $m \in R$ is said to be maximal if $m \leq x$ implies $m = x$.

Lemma 2.7. *Let R be an ADL with 0, and $m \in R$. Then the following are equivalent:*

(1) *m is a maximal element with respect to the partial ordering "$\leq$".*

(2) *$m \vee x = m$, for all $x \in R$.*

(3) *$m \wedge x = x$, for all $x \in R$.*

Definition 2.8 ([5]). Anon-empty subset I of R is said to be an ideal of R if it satisfies the following conditions.

(1) $a,b \in I \Rightarrow a \vee b \in I$.

(2) $a \in I$, $x \in R \Rightarrow a \wedge x \in I$.

Theorem 2.9 ([5]). *The following are equivalent for any ADL R.*

(1) *R is relatively complemented.*

(2) *Given $x,y \in R$, there exists $a \in R$ such that $x \wedge a = 0$ and $x \vee a = x \vee y$.*

(3) *For any $x \in R$, the interval $[0,x]$ is complemented.*

Theorem 2.10 ([5]). *A relatively complemented ADL R is associative.*

Definition 2.11 ([5]). An ADL $(R, \vee, \wedge)$ is said to be relatively complemented if every interval $[a,b]$, $a \leq b$ in R is a complemented lattice.

Definition 2.12 ([3]). Let X be a non-empty set, a function $A : X \times X \to [0,1]$ is said to be fuzzy partial order relation if it satisfies the following condition:

(1) $A(x,x) = 1$, for all $x \in X$, i.e., A is reflexive.

(2) $A(x,y) > 0$ and $A(y,x) > 0$ implies that $x = y$, i.e., A is antisymmetric.

(3) $A(x,z) \geq \sup_{y \in X} \min[A(x,y), A(y,z)] > 0$, i.e., A is transitive.

Definition 2.13 ([3]). Let (X,A) be a fuzzy poset. Then (X,A) is a fuzzy lattice if and only if $x \vee y$, and $x \wedge y$ exists for all $x, y \in X$.

Definition 2.14 ([8]). Given an ADL R with maximal element.
Define $B(R) = \{a \in R \mid a \wedge b = 0$, and $a \vee b$ is maximal for some $b \in R\}$. Then $B(R)$ is called the Brikhoff center of R.
Let $a \wedge b = 0$, $a \vee b$ is maximal. Then $b \wedge a = 0$, and $b \vee a$ is maximal, in this case a and b are called complements to each other.

Theorem 2.15. *For any $a \in R$, $a \in B(R)$ if and only if there exist two sub ADL$_s$ R_1 and R_2 of R with maximal elements and an isomorphism $f : R \to R_1 \times R_2$ such that $f(a) = (m_1, 0)$, where m_1 is a maximal element in R_1.*

Definition 2.16 ([2]). Let $(R, \vee, \wedge, 0)$ be an algebra of type $(2,2,0)$ and we call (R,A) is an *Almost Distributive Fuzzy Lattice* (ADFL) if the following condition satisfied:

(1) $A(a, a \vee 0) = A(a \vee 0, a) = 1$.

(2) $A(0, 0 \wedge a) = A(0 \wedge a, 0) = 1$.

(3) $A((a \vee b) \wedge c, (a \wedge c) \vee (b \wedge c)) = A((a \wedge c) \vee (b \wedge c), (a \vee b) \wedge c) = 1$.

(4) $A(a \wedge (b \vee c), (a \wedge b) \vee (a \wedge c)) = A((a \wedge b) \vee (a \wedge c), a \wedge (b \vee c)) = 1$.

(5) $A(a \vee (b \wedge c), (a \vee b) \wedge (a \vee c)) = A((a \vee b) \wedge (a \vee c), a \vee (b \wedge c)) = 1$.

(6) $A((a \vee b) \wedge b, b) = A(b, (a \vee b) \wedge b) = 1$, for all $a, b, c \in R$.

Definition 2.17 ([1]). Let (R,A) be an ADFL with maximal element.
Define $B_A(R) = \{a \in R \mid A(a \wedge b, 0) > 0$ and $A((a \vee b) \vee x, a \vee b) > 0$ for some $b \in R$, for all $x \in R\}$. Then, $B_A(R)$ is called the Birkhoff center of (R,A).

3. Birkhoff Center in a Quotient of Almost Distributive Fuzzy Lattices

Definition 3.1. Let (R,A) be an ADFL. Then the relation $\eta_A = \{(a,b) \in R \times R \mid A(b, a \wedge b) > 0$ and $A(a, b \wedge a) > 0\}$ is a fuzzy congruence relation on (R,A) and is the smallest such that $\frac{R}{\eta_A}$ is a fuzzy lattice.

Lemma 3.2. *Let (R,A) be an ADFL. Then the relation $\eta = \{(a,b) \in R \times R \mid a \wedge b = b$ and $b \wedge a = a\}$ is a congruence relation on R if and only if $\eta_A = \{(a,b) \in R \times R \mid A(b, a \wedge b) > 0$ and $A(a, b \wedge a) > 0\}$ is a fuzzy congruence relation on (R,A).*

Proof. Let (R,A) be an ADFL with 0 and $\eta = \{(a,b) \in R \times R \mid a \wedge b = b$ and $b \wedge a = a\}$ is a congruence relation on R.

Imply that $a \wedge b \leq b$ and $b \leq a \wedge b$, $b \wedge a \leq a$, and $a \leq b \wedge a$.

So that we have $A(b, a \wedge b) > 0$ and $A(a, b \wedge a) > 0$, for all $a,b \in R$.

Hence $\eta_A = \{(a,b) \in R \times R \mid A(b, a \wedge b) > 0$ and $A(a, b \wedge a) > 0\}$ is a fuzzy congruence relation.

On the other hand ,suppose $\eta_A = \{(a,b) \in R \times R \mid A(b, a \wedge b) > 0$ and $A(a, b \wedge a) > 0\}$ is a fuzzy congruence relation.

Imply that $A(b, a \wedge b) > 0$, and $A(a, b \wedge a) > 0$. But $a \wedge b \leq b$ and $b \wedge a \leq a$.

So that we have $A(a \wedge b, b) > 0$ and $A(b \wedge a, a) > 0$.

Hence $a \wedge b = b$ and $b \wedge a = a$ by anti symmetry property of A.

Thus $\eta = \{(a,b) \in R \times R \mid a \wedge b = b$ and $b \wedge a = a\}$ is a congruence relation on R. □

Definition 3.3. Let (R_1, A_1) and (R_2, A_2) be two $ADFL_s$. Then a mapping $f : (R_1, A_1) \to (R_2, A_2)$ is said to be a fuzzy lattice homomorphism. If it satisfy the following condition for any $x, y, 0 \in R_1$:

(1) $A_2(f(x \wedge y), f(x) \wedge f(y)) = A_2(f(x) \wedge f(y), f(x \wedge y)) = 1$.

(2) $A_2(f(x \vee y), f(x) \vee f(y)) = A_2(f(x) \vee f(y), f(x \vee y)) = 1$.

(3) $A_2(f(0), 0) > 0$.

Definition 3.4. Let $(R/\eta, A)$ be the quotient of an Almost distributive fuzzy lattice. Then $B_A(R/\eta) = \{a/\eta \in R/\eta \mid A(a/\eta \wedge b/\eta, 0) > 0$ and $A((a/\eta \vee b/\eta) \vee x/\eta, a/\eta \vee b/\eta) > 0$, for some $b \in R$ and for all $x \in R\}$. Then $B_A(R/\eta)$ is called Birkhoff center of a quotient in an Almost distributive fuzzy lattice.

Theorem 3.5. *Let (R,A) be an ADFL with 0. Then $B_A(R/\eta)$ is isomorphic to $B_A(R/\eta/B_A(R) \times B_A(R)$.*

Proof. Let $\frac{a}{\eta} \in B_A(R/\eta)$. Then there exist $b \in R$ such that

$$A\left(\frac{a}{\eta} \wedge \frac{b}{\eta}, 0\right) > 0 \quad \text{and} \quad A\left(\left(\frac{a}{\eta} \vee \frac{b}{\eta}\right) \vee \frac{x}{\eta}, \frac{a}{\eta} \vee \frac{b}{\eta}\right) > 0.$$

Therefore,

$$A(a \wedge b, 0) > 0 \quad \text{and} \quad A\left(\frac{(a \vee b)}{\eta} \vee \frac{x}{\eta}, \frac{a \vee b}{\eta}\right) > 0.$$

Now, for any $x \in R$, we have $A\left(\frac{x}{\eta}, \frac{(a \vee b) \wedge x}{\eta}\right) > 0$.

Then $((a \vee b) \wedge x, x) \in \eta_A$ and hence $A(x, (a \vee b) \wedge x) > 0$.

Therefore, $a \in B_A(R)$.

Consider the map $f : B_A(R) \to B_A(R/\eta)$ defined by $A\left(f(a), \frac{a}{\eta}\right) = A\left(\frac{a}{\eta}, f(a)\right) = 1$ for any $a \in B_A(R)$.

Let $a, b \in B_A(R)$. Then $\frac{a}{\eta}, \frac{b}{\eta} \in B_A(R/\eta)$.

(1) $A(f(a \wedge b), f(a) \wedge f(b)) = A\left(\frac{a \wedge b}{\eta}, \frac{a}{\eta} \wedge \frac{b}{\eta}\right) = A\left(\frac{a \wedge b}{\eta}, \frac{a \wedge b}{\eta}\right) = 1$.

Hence $A(f(a \wedge b), f(a) \wedge f(b)) = 1$. Similarly, $A(f(a) \wedge f(b), f(a \wedge b)) = 1$.

Therefore, $A(f(a \wedge b), f(a) \wedge f(b)) = A(f(a) \wedge f(b), f(a \wedge b)) = 1$.

(2) $A(f(a\vee b),f(a)\vee f(b))=A\left(\frac{a\vee b}{\eta},\frac{a}{\eta}\vee\frac{b}{\eta}\right)=A\left(\frac{a\vee b}{\eta},\frac{a\vee b}{\eta}\right)=1$.
Hence $A(f(a\vee b),f(a)\vee f(b))=1$. Similarly, $A(f(a)\vee f(b),f(a\vee b))=1$.
Therefore, $A(f(a\vee b),f(a)\vee f(b))=A(f(a)\vee f(b),f(a\vee b))=1$.

(3) Let $0\in B_A(R)$. Then $f(0)=\frac{0}{\eta}\in B_A(R/\eta)$.
So that $A\left(f(0),\frac{0}{\eta}\right)=A\left(\frac{0}{\eta},\frac{0}{\eta}\right)=1$. Imply that $A(f(0),\frac{0}{\eta})>0$.
Therefore, f is a fuzzy lattice homomorphism.
Let $a,b\in B_A(R)$. Then write $w=a\vee b$,

$$A\left(f(w),\frac{a\vee b}{\eta}\right)=A\left(\frac{w}{\eta},\frac{a\vee b}{\eta}\right)=A\left(\frac{a\vee b}{\eta},\frac{a\vee b}{\eta}\right)=1.$$

Hence $A\left(f(w),\frac{a\vee b}{\eta}\right)>0$. Similarly, $A\left(\frac{a\vee b}{\eta},f(w)\right)>0$.
Therefore, $f(w)=\frac{a\vee b}{\eta}$. So that f is epimorphism.
Let $x,y\in B_A(R)$. Then

$$\begin{aligned}A(x,y)&=A((a\vee b)\wedge x,y)\\&=A((a\wedge x)\vee(b\wedge x),y)\\&=A((a\wedge y)\vee(b\wedge y),y)\\&\qquad\text{(replacing } x \text{ by } y)\\&=A((a\vee b)\wedge y,y)\\&=A(y,y)=1>0.\end{aligned}$$

Hence $A(x,y)>0$. Similarly, $A(y,x)>o$. So that we have $x=y$.
Therefore, f is monomorphism.
For $a\in B_A(R)$, there exist $b\in R$ such that $A(a\wedge b,0)>0$, then

$$A\left(f(a\wedge b),\frac{0}{\eta}\right)=A\left(f(0),\frac{0}{\eta}\right)=A\left(\frac{0}{\eta},\frac{0}{\eta}\right)=1.$$

Hence

$$A\left(f(a\wedge b),\frac{0}{\eta}\right)=1.$$

Similarly,

$$A\left(\frac{0}{\eta},f(a\wedge b)\right)=1.$$

Therefore,

$$A\left(f(a\wedge b),\frac{0}{\eta}\right)=A\left(\frac{0}{\eta},f(a\wedge b)\right)=1.$$

So that $0\in\ker f$ and f is monomorphism. We have $\ker f$ is zero.
Hence $\ker f=\eta\cap(B_A(R)\times B_A(R))$.
By fundamental theorem of homomorphism $B_A(R)/\eta/B_A(R)\times B_A(R)\cong B_A(R/\eta)$. □

Definition 3.6. An ideal I of an ADFL (R,A) is complemented if and only if $I=(a]_A$ for some $a\in R$.

Theorem 3.7. *An ideal I of (R,A) is complemented if and only if $I=(a]_A$ for some $a \in B_A(R)$.*

Proof. Let I be an ideal of (R,A). Suppose that I is complemented.
Then there exist an ideal I' of (R,A) such that $I \cap I' \subseteq (0]_A$ and $R \subseteq I \vee I'$.
Choose a maximal element m of (R,A). Then $A(m,a\vee b)>0$ for some $a \in I$ and $b \in I'$.
Since $I \wedge I' \subseteq (0]_A$, $A(b\wedge a,0)>0$, and we have $A((a\vee b)\vee x, a\vee b)>0$.
Therefore, $a \in B_A(R)$.
Now, since $a \in I$, we have $(a]_A \subseteq I$.
Also, $x \in I \Rightarrow b \wedge x \in I \wedge I' \subseteq (0]_A$

$$
\begin{aligned}
&\Rightarrow\quad A(b\wedge x,0)>0\\
&\Rightarrow\quad A(x,m\wedge x)=A(m\wedge x,x)=1\\
&\Rightarrow\quad A(x,(a\vee b)\wedge x)=A((a\vee b)\wedge x,x)=1\\
&\qquad =A(x,(a\wedge x)\vee(b\wedge x))\\
&\qquad =A((a\wedge x)\vee(b\wedge x),x)=1\\
&\qquad =A(x,(a\wedge x)\vee 0)\\
&\qquad =A((a\wedge x)\vee 0,x)=1,\\
&\qquad\qquad \text{since } b\wedge x=0,\ b\in I',\ x\in I\\
&\qquad =A(x,a\wedge x)=A(a\wedge x,x)=1\\
&\Rightarrow\quad x\in(a]_A.
\end{aligned}
$$

Hence $I \subseteq (a]_A$. So that $I=(a]_A$. Similarly, $I'=(a]_A$.
Conversely, suppose that $I=(a]_A$, for some $a \in B_A(R)$.
Then, there exist $b \in R$ such that $A(a\wedge b,0)>0$ and $A((a\vee b)\vee x, a\vee b)>0$.
Now, $(a]_A \cap (b]_A=(a\wedge b]_A \subseteq (0]_A$ and $R \subseteq (a]_A \vee (b]_A=(a\vee b]_A$.
Therefore, I is complemented. □

4. Conclusion

In this paper we introduce the concept of Birkhoff center in a Quotient of Almost Distributive Fuzzy Lattice and fuzzy congruence relation in an ADFL. We also include homomorphism of a quotient and complimented ideals in an ADFL. Finally basic theorems and corollaries have been proved.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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