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Research Article

A New Approach for the Solution of Fuzzy Initial Value Problems Through Runge-Kutta Method

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Abstract. In this paper we propose a new approach for the solution of second order fuzzy initial value problem without converting to a system of linear fuzzy differential equations using Runge Kutta Method of fourth order under H-differentiability especially increasing length of support. Numerical examples are provided to show the stability and convergence of the proposed method with error control.

Keywords. Generalized H-differentiability; Fuzzy derivatives; Fuzzy differential equations; Runge-Kutta method formula

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1. Introduction

The combination of fuzzy differential equations and fuzzy analysis is a new area of research in Mathematics. In current scenario the uncertainty is discussed based on fuzzy concepts. Fuzzy initial value problems could not be solved exactly and so we can adopt many approximations method to find the solutions. Chang and Zadeh [6] were the first persons who introduced the fuzzy derivative. Then, based on the extension principle Dubois and Prade [7] presented the fuzzy derivative concept. In 1987, Kandel and Byatt [11] introduced the fuzzy differential equations, and then Puri and Ralesu [13] made Hukuhara derivative. Following a generalized

H-differentiability of fuzzy functions was introduced by Bede and Gal [3, 4]. Ma *et al*. [12] solved the fuzzy differential equations in numerical method. Ghanaie *et al*. [1] expressed the generalized Runge-Kutta approximation method of order two and analyze its error and gave nuclear decay equation to show its efficiency under generalized differentiability.

Based on Seikkala derivative of fuzzy process, Jayakumar *et al*. [9, 10] proposed numerical algorithms for solving fuzzy differential equations using the Runge-Kutta method of order $N = 5$ is studied in detail by a complete error analysis and is illustrated by solving some linear and nonlinear Fuzzy Cauchy Problems. Rabiei *et al*. [8] developed two step method that tells to less number of stages leads to less number of function evaluation for solving second-order *fuzzy differential equations* (FDEs) by the *Fuzzy Improved Runge-Kutta Nystrom* (FIRKN) method based on the generalized concept of higher-order fuzzy differentiability. Jayakumar *et al*. [10] studied a numerical method to get approximate fuzzy solution using partition of fuzzy interval and generalization of Hukuhara difference of fuzzy differential equation with fully fuzzy initial values. The rest of the paper is organized as follows: In Section 2, the basic concepts on fuzzy sets and its derivatives to be used in the paper. In Section 3, we introduce second order fuzzy Runge-Kutta method formula and followed by the existence and uniqueness of the solution and convergence of the proposed method are presented in detail. We give some numerical examples on Runge-Kutta method formula in Section 4. In the last we present our conclusion.

2. Preliminaries

Definition 2.1. A fuzzy set $\tilde{a}$ defined on the set of real numbers R is said to be a fuzzy number if its membership function $\tilde{a} : R[0,1]$ has the following:

(i) $\tilde{a}$ is convex, i.e. $\tilde{a}\{\lambda x_1 + (1-\lambda)x_2\} \geq \min\{\tilde{a}(x_1), \tilde{a}(x_2)\}$, for all $x_1, x_2 \in R$ and $\lambda \in [0,1]$,

(ii) $\tilde{a}$ is normal i.e. there exists an $x \in R$ such that $\tilde{a}(x) = 1$,

(iii) $\tilde{a}$ is piecewise continuous.

Definition 2.2. A triangular fuzzy number $\tilde{A}$ is a fuzzy number specified by $\tilde{A} = (a_1, a_2, a_3)$ with their membership function

$$\tilde{A}(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3, \\ 0, & x \geq a_3. \end{cases}$$

Definition 2.3 (Arithmetic operations on fuzzy numbers)**.** The fuzzy number $\tilde{a} \in [0,1]$ can also be represented as a pair $\tilde{a} = (\underline{a}(r), \overline{a}(r))$ for $0 \leq r \leq 1$ which satisfies

(i) $\underline{a}(\alpha)$ is a bounded monotonic increasing left continuous function.

(ii) $\overline{a}(\alpha)$ is a bounded monotonic decreasing left continuous function.

(iii) $\underline{a}(\alpha) \leq \overline{a}(\alpha)$, $0 \leq \alpha \leq 1$

Let $D : E \times E \to R_+ U\{0\}$ then $D(u,v) = \sup_{r\in[0,1]} \max\{|\underline{u}^r, \underline{v}^r|, |\overline{u}^r, \overline{v}^r|\}$ be the Hausdroff distance between two fuzzy numbers where $[u]^r = [\underline{u}^r, \overline{u}^r], [v]^r = [\underline{v}^r, \overline{v}^r]$.

Let E be the space of fuzzy numbers. It is easy to see that D_H is a metric in E and has the following properties:

(i) $D_H(a+c,b+c) = D_H(a,b)$, $\forall\ a,b,c \in R$,

(ii) $D_H(ka,kb) = |k|D_H(a,b)$, $\forall\ k \in E, a,b \in R$,

(iii) $D_H(a+b,c+d) \le D_H(a,c)+D_H(b,d)$, $\forall\ a,b,c,d \in R$ and (D_H,R) is a complete metric space.

Theorem 2.1 ([13]). (i) *If we denote $\tilde{0} = \chi\{0\}$ then $\tilde{0} \in E$ is natural element with respect to $+$, i.e. $\tilde{a}+\tilde{0} = \tilde{0}+\tilde{a} = \tilde{a}$, for all $\tilde{a} \in E$.*

(ii) *With respect to $\tilde{0}$, none of $\tilde{a} \in E\backslash R$, has opposite in E.*

(iii) *For any $a,b \in R$ with $a,b \ge 0$ or $a,b \le 0$ and any $a \in E$, we have $(a+b)\cdot u = a\cdot u + b\cdot u$. For general $a,b \in R$, the above property does not hold.*

(iv) *For any $\tilde{a},\tilde{b} \in E$ and any $\lambda \in R$, we have $\lambda\cdot(\tilde{a}+\tilde{b}) = \lambda\cdot\tilde{a}+\lambda\cdot\tilde{b}$.*

(v) *For any $\tilde{a} \in E$ and any $\lambda,\mu \in R$, we have $\lambda\cdot(\mu\cdot\tilde{a}) = (\lambda\cdot\mu)\cdot\tilde{a}$.*

This arithmetic operations are used in this Fuzzy Initial Value Problems.

2.1 Fuzzy Initial Value Problems

Consider the nth order fuzzy initial value differential equation is given by

$$\tilde{y}^n(t) = \tilde{f}\left(t,\tilde{y}(t),\tilde{y}'(t),\dots,\tilde{y}^{(n-1)}(t)\right),$$
$$\tilde{y}(t_0) = \tilde{x}_0,\dots,\tilde{y}^{(n-1)}(t_0) = \tilde{x}_0,$$

where y is a function of t, $f(t,y(t),y'(t),\dots,y^{(n-1)}(t))$ is a fuzzy function of the crisp variable t and the fuzzy variable $y(t),y'(t),\dots$ is the fuzzy derivative of y, where $\tilde{y}_0 \in E$; we denote by E the family of all the fuzzy numbers of R, where the function $F:[t_0,T]XE\times EX\dots XE \to R$ is a continuous fuzzy function with fuzzy initial values $\tilde{y}_0^{j-1} = x_0 j = 1,2,\dots,m$ is a fuzzy numbers.

We denote the fuzzy function y by $y = [\underline{y},\overline{y}]$ it means that the r-level set of $y(t)$for $t \in [t_0,T]$

$$[\tilde{y}(t_0)]_\alpha = [\underline{\tilde{y}}(t_0;\alpha),\overline{\tilde{y}}(t_0;\alpha)]\alpha \in [0,1].$$

By using the Zadeh's extension principle, we have the membership function

$$\begin{aligned}[\tilde{f}(t,y)]^\alpha &= \tilde{f}(t,[y]^\alpha)\\ &= \tilde{f}(t,[\underline{y_\alpha}\overline{y_\alpha}])\\ &= (\min\tilde{f}(t,[\underline{y_\alpha},\overline{y_\alpha}],\max\tilde{f}(t,[\underline{y_\alpha},\overline{y_\alpha}]).\end{aligned}$$

Definition 2.4. Let be $u,v \in R$. If there exists $w \in R$ such that $u = v\oplus w$, then w is called the H-difference of u and v and is denoted by $u\Theta v$.

Definition 2.5 (Generalized Fuzzy Derivative)**.** Let $F:(a,b)\to E$ and $t_0 \in (a,b)$. We say that F is generalized differentiable at t_0 (Bede-Gal differentiability) if there exists an element $F'(t_0) \in R$ such that

(i) For $h > 0$ sufficiently small $\exists\ F(t_0+h)\Theta F(t_0), F(t_0)\Theta F(t_0-h)$, and the limits satisfy $\lim\limits_{h\to 0}\frac{F(t_0+h)\Theta f(t_0)}{h} = \lim\limits_{h\to 0}\frac{F(t_0)\Theta F(t_0-h)}{h} = F'(t_0)$

(ii) For $h>0$ sufficiently small $\exists\ F(t_0)\Theta F(t_0+h), F(t_0-h)\Theta F(t_0)$, and the limits satisfy $\lim\limits_{h\to 0}\frac{F(t_0)\Theta F(t_0+h)}{(-h)}=\lim\limits_{h\to 0}\frac{F(t_0-h)\Theta F(t_0)}{(-h)}=F'(t_0)$, h and $(-h)$ at denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$, respectively.

Definition 2.6. Let $F':(a,b)\to E$ and $t_0\in(a,b)$. We say that F' is strongly generalized differentiable at $t_0$if there exists an element $F''(t_0)\in E$ such that

(i) For $h>0$ sufficiently small $\exists\ F'(t_0+h)\Theta F'(t_0), F'(t_0)\Theta F'(t_0-h)$, and the limits satisfy

$$\lim_{h\to 0}\frac{F'(t_0+h)\Theta F'(t_0)}{h}=\lim_{h\to 0}\frac{F'(t_0)\Theta F'(t_0-h)}{h}=F''(t_0)$$

(ii) For $h>0$ sufficiently small $\exists\ F'(t_0)\Theta F'(t_0+h), F'(t_0-h)\Theta F'(t_0)$,and the limits satisfy

$$\lim_{h\to 0}\frac{F'(t_0)\Theta F'(t_0+h)}{(-h)}=\lim_{h\to 0}\frac{F'(t_0-h)\Theta F'(t_0)}{(-h)}F''(t_0)$$

All the limits are taken in the metric space (E,D), and at the end points of $t_0\in(a,b)$ and we consider only one-sided derivatives.

Remark 2.1. A function that is generalized H-differentiability as in cases (i) and (ii) of Definition 2.6, will be referred as (i) — differentiable or as (ii) — differentiable, respectively

3. Fourth Order RK-Method

In solving a second order fuzzy ordinary differential equations of the form

$$\tilde{y}''=\tilde{f}(t,\tilde{y},\tilde{y}') \text{ subject to } \tilde{y}(t_0)=\tilde{y}_0, \tilde{y}'(t_0)=\tilde{y}'_0, \tag{3.1}$$

where $\tilde{y}(t_0)$ and $\tilde{y}'(t_0)$ are triangular fuzzy numbers.

An approximate solutions of eq. (3.1) are

$$\tilde{f}[t_n]^\alpha=[\underline{f}(t_n;\alpha),\overline{f}(t_n;\alpha)]. \tag{3.2}$$

By using Fourth order Runge-Kutta method for approximate solution is calculated as follows

$$\underline{f}(t_{n+1};\alpha)=\underline{f}(t_n;\alpha)+h[\underline{f}'(t_n;\alpha)+\sum_{j=1}^{3}C_jP_{j,1}(t_n,\underline{f}(t_n;\alpha),\underline{f}'(t_n;\alpha))]$$

$$\overline{f}(t_{n+1};\alpha)=\overline{f}(t_n;\alpha)+h[\overline{f}'(t_n;\alpha)+\sum_{j=1}^{3}C_jP_{j,2}(t_n,\overline{f}(t_n;\alpha),\overline{f}'(t_n;\alpha))]$$

where C_j are constants and $P_{j,1}$, $P_{j,2}$ for $j=1,2,3,4$ are follows:

$$P_{1,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))=\min\{(t_n,\tilde{x},\tilde{y})|\tilde{x}\in(\underline{f}(t_n;\alpha),\overline{f}(t_n;\alpha)) \text{ and } \tilde{y}\in(\underline{f}'(t_n;\alpha),\overline{f}'(t_n;\alpha))\}$$

$$P_{1,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))=\max\{(t_n,\tilde{x},\tilde{y})|\tilde{x}\in(\underline{f}(t_n;\alpha),\overline{f}(t_n;\alpha)) \text{ and } \tilde{y}\in(\underline{f}'(t_n;\alpha),\overline{f}'(t_n;\alpha))\}$$

$$P_{2,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))=\min\{\left(t_n+\frac{h}{2},\tilde{x},\tilde{y}\right)|$$
$$\tilde{x}\in(Q_{1,1}(t_n,f(t_n;\alpha),\tilde{f}'(t_n;\alpha))),(Q_{1,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))),$$
$$\tilde{y}\in(R_{1,1}(t_n,f(t_n;\alpha),\tilde{f}'(t_n;\alpha))),(R_{1,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha)))$$

$$P_{2,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))=\max\{\left(t_n+\frac{h}{2},\tilde{x},\tilde{y}\right)|$$

$$\tilde{x} \in (Q_{1,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (Q_{1,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))),$$
$$\tilde{y} \in (R_{1,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (R_{1,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)))$$

$$P_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \min\{\left(t_n + \frac{h}{2}, \tilde{x}, \tilde{y}\right) |$$
$$\tilde{x} \in (Q_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (Q_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))),$$
$$\tilde{y} \in (R_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (R_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)))$$

$$P_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \max\{\left(t_n + \frac{h}{2}, \tilde{x}, \tilde{y}\right) |$$
$$\tilde{x} \in (Q_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (Q_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))),$$
$$\tilde{y} \in (R_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (R_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)))$$

$$P_{4,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \min\{(t_n + h, \tilde{x}, \tilde{y})|$$
$$\tilde{x} \in (Q_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (Q_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))),$$
$$\tilde{y} \in (R_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (R_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)))\}$$

$$P_{4,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \max\{(t_n + h, \tilde{x}, \tilde{y})|$$
$$\tilde{x} \in (Q_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (Q_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))),$$
$$\tilde{y} \in (R_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))), (R_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)))\}$$

where

$$Q_{1,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \overline{f}(t_n;\alpha) + \frac{h}{2}\overline{f}'(t_n;\alpha) + \frac{h}{8}K_{1,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$Q_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \underline{f}(t_n;\alpha) + \frac{h}{2}\underline{f}'(t_n;\alpha) + \frac{h}{8}K_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$Q_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \overline{f}(t_n;\alpha) + \frac{h}{2}\overline{f}'(t_n;\alpha) + \frac{h}{8}K_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$Q_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \underline{f}(t_n;\alpha) + \frac{h}{2}\underline{f}'(t_n;\alpha) + \frac{h}{8}K_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$Q_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \overline{f}(t_n;\alpha) + \frac{h}{2}\overline{f}'(t_n;\alpha) + \frac{h}{8}K_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$R_{1,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \underline{f}'(t_n;\alpha) + \frac{h}{2}I_{1,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$R_{1,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \overline{f}'(t_n;\alpha) + \frac{h}{2}I_{1,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$R_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \underline{f}'(t_n;\alpha) + \frac{h}{2}I_{2,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$R_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \overline{y}'(t_n;\alpha) + \frac{h}{2}I_{2,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$R_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \underline{f}'(t_n;\alpha) + hI_{3,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$R_{3,2}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha)) = \overline{f}'(t_n;\alpha) + hI_{3,2}(t_n, f(t_n;\alpha), f'(t_n;\alpha))$$

Putting the initial values y_0, y_0' and RK method of order $N = 4$, we compute

$$\underline{f}'(t_{n+1};\alpha) = \underline{f}'(t_n;\alpha)) + \frac{h}{6}P_{1,1}(t_n, \tilde{f}(t_n;\alpha), \tilde{f}'(t_n;\alpha))$$

$$+2P_{2,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))+2P_{3,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))$$
$$+P_{4,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))$$
$$\overline{f}'(t_{n+1};\alpha)=\overline{f}'(t_n;\alpha))+\frac{h}{6}P_{1,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))$$
$$+2P_{2,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))+2P_{3,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))$$
$$+P_{4,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))$$
$$\underline{f}(t_{n+1};\alpha)=\underline{f}(t_n;\alpha))+h\Big[\tilde{f}'(t_n;\alpha)+\frac{1}{6}\Big[P_{1,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))$$
$$+P_{2,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))+P_{3,1}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))\Big]\Big]$$
$$\overline{f}(t_{n+1};\alpha)=\overline{f}(t_n;\alpha))+h\Big[\tilde{f}'(t_n;\alpha)+\frac{1}{6}\Big[P_{1,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))$$
$$+P_{2,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))+P_{3,2}(t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha))\Big]\Big]$$

The approximate series solutions at $t_n, 0\le n\le N$ are denoted by $[\tilde{f}(t_n;\alpha)]_\alpha=[\underline{f}(t_n;\alpha),\overline{f}(t_n;\alpha)]$ and the general solutions is

Calculated by grid points $a=t_0\le t_1\le t_2\le\cdots t_n=b$ and $h=\frac{(b-a)}{N}=t_{n+1}-t_n$

$$\underline{f}(t_{n+1};\alpha)=\underline{f}(t_n;\alpha)+\frac{1}{6}F[t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha)]$$
$$\overline{f}(t_{n+1};\alpha)=\overline{f}(t_n;\alpha)+\frac{1}{6}F[t_n,\tilde{f}(t_n;\alpha),\tilde{f}'(t_n;\alpha)]$$

The following lemmas will be applied to show the convergences of theses approximate i.e.

$$\lim_{h\to 0}\underline{f}(t,\alpha)=\underline{F}(t,\alpha) \text{ and } \lim_{h\to 0}\overline{f}(t,\alpha)=\overline{F}(t,\alpha).$$

In this paper, we discuss the accuracy and efficiency of the proposed method for solving Runge-Kutta method of order 4 involving generalized H-differentiability. Numerical examples are also provided to show the efficiency of the proposed method.

3.1 Convergence of the Numerical Method

Lemma 3.1. *Let the sequence of numbers* $\{w_n\}_{n=0}^N$ *satisfy* $|w_{n+1}|\le A|w_n|+B$, $0\le n\le N-1$ *for some given positive constants* A *and* B, *then* $|w_n|\le A^n|w_0|+B\dfrac{A^n-1}{A-1}$, $0\le n\le N-1$.

Lemma 3.2. *Let the sequence of numbers* $\{w_n\}_{n=0}^N$, $\{v_n\}_{n=0}^N$ *satisfy* $|w_{n+1}|\le|w_n|+A\max\{|w_n|,|v_n|\}+B$, $|v_{n+1}|\le|v_n|+A\max\{|w_n|,|v_n|\}+B$, *for some given positive constants* A *and* B, *and denote* $U_n=|W_n|+|V_n|$, $0\le n\le N$, $U_n\le\overline{A}^nU_0+\overline{B}\dfrac{\overline{A}^n-1}{\overline{A}-1}$, $0\le n\le N$, *where* $\overline{A}=1+2\tilde{A}$ *and* $\overline{B}=2\tilde{B}$.

Theorem 3.1. *Let* $F^*(t,u,v)$ *and* $G^*(t,u,v)$, *belongs to* $C^{H-1}(K)$ *and let the partial derivatives of* F^*, G^* *be bounded over* K. *Then, for arbitrary fixed* $r: 0\le r\le 1$, *the approximate solutions converge uniformly in* t *to the exact solutions.*

4. Numerical Examples

Example 1. Consider the initial value problem

$$\begin{cases} y''(t) = -y(t) + t, \; t \geq 0 \\ y'(0) = [1.8 + 0.2r, 2.2 - 0.2r]. \end{cases}$$

Exact solution at $t = 1$ using (1) – differentiability is given by:

$$y1(t;r) = \left(\frac{4}{5} + \frac{1}{5}r\right)\sin(t) + \left(\frac{9}{10} + \frac{1}{10}r\right)\cos(t) + t$$

$$y2(t;r) = \left(\frac{6}{5} - \frac{1}{5}r\right)\sin(t) + \left(\frac{11}{10} - \frac{1}{10}r\right)\cos(t) + t.$$

Table 1 shows the approximate solution of first differential and differential at $t = 1$.

Table 1

r	$\underline{y}$	$\overline{y}$	$\underline{y}'$	$\overline{y}'$
0	2.159448	2.604097	0.674918	0.722745
0.2	2.203914	2.559633	0.679700	0.717962
0.4	2.248378	2.515167	0.684483	0.713180
0.6	2.292843	2.470702	0.689266	0.708397
0.8	2.337308	2.426238	0.694049	0.703614
1.0	2.381773	2.381773	0.698832	0.698832

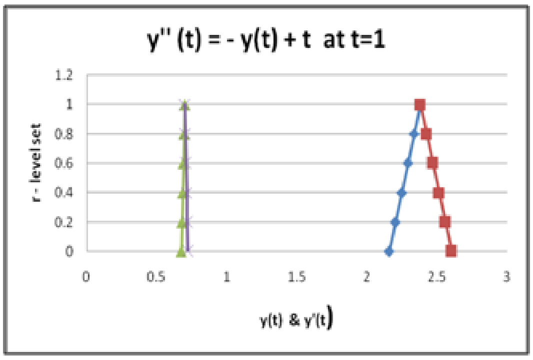

Figure 1

Example 2. Consider the following fuzzy linear initial value problem.

$$\begin{cases} y''(t) = -y(t), \; t \geq 0 \\ y(0) = [0.9 + 0.1r, 1.1 - 0.1r]. \end{cases}$$

The exact solution at $t = 1$ using (1) — differentiability is given by:

$$Y(t;r) = [(0.9 + 0.1r)\sin(t)(1.1 - 0.r)\sin(t)].$$

Table 2 shows the approximate solution at t=1 for first differential and differential **??? incomplete sentence.**

Table 2

r	$\underline{y}$	$\overline{y}$	$\underline{y}'$	$\overline{y}'$
0	0.757323	0.925617	0.486272	0.594333
0.2	0.774153	0.908788	0.497078	0.583527
0.4	0.790982	0.891958	0.507884	0.572721
0.6	0.807812	0.875129	0.518690	0.561915
0.8	0.824641	0.858300	0.529496	0.551109
1.0	0.841470	0.841470	0.540302	0.540302

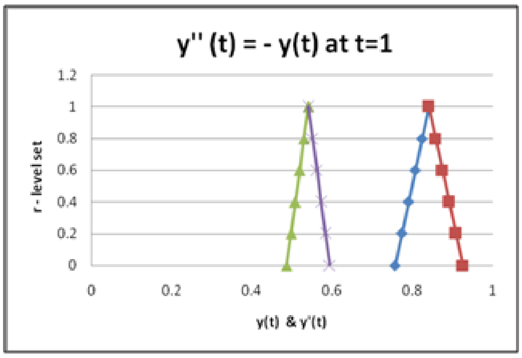

Figure 2

Remark 4.1. Usually, we are converting a second order linear fuzzy differentiable equation in system of simultaneous linear differential equation but here without converting we give the approximate solution converges with exact solution with $h = 0.1$ and $t = 1$.

The Fuzzy differential equations are applicable to a limited class which are presented by the method of solution using step by step methods to calculate a value of y over a limited range of x, for equal intervals of x. Taylor's series solution is agreed by this method up to the terms in h^r. Whether the differential equation is linear or non-linear, the operation is identical in this method.

5. Conclusion

The proposed RK-method formula have been successfully done to obtain approximate solution of second order Fuzzy differential equations. We can use these method in non linear fuzzy problem and fuzzy integration.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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