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Research Article

Cordial Labeling of Cartesian Product between two Balanced Bipartite Graphs

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Abstract. Cordial Labeling finds its application in Automated Routing algorithms, Communications relevant Adhoc Networks and many others. These networks and routing paths are best represented by a set of vertices and edges which forms a complex graph. In this paper an Algorithmic approach is devised in order to cordially label some of such complex graphs formed by the Cartesian product of two Balanced Bipartite Graphs. There is an absence of a universal algorithm that could label the entire family of $K_{n,n} \times K_{n,n}$ graphs, which is primarily because of the complexity of these graphs. We have attempted to redefine the $K_{n,n}$ graph so that on carrying out the Cartesian product we obtain a symmetrical graph. Subsequently we design an efficient and effective algorithm for cordially labeling $G = K_{n,n} \times K_{n,n}$. We also proof that the algorithm is running with polynomial time complexity. The results can be used to study and design algorithms for signed product cordial, total signed product cordial, prime cordial labeling etc. of the family of $K_{n,n} \times K_{n,n}$ graphs.

Keywords. Cordial labeling; Cartesian product; Balanced bipartite graphs; Algorithm; Time complexity

MSC. 05C76; 05C78

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1. Introduction

Cordial labeling is one of the most interesting graph labeling technique introduces by Cahit[3] as a weaker version of graceful labeling. Gallian [5] survey gives us a clear concept of different kind of labeling and their applications. According to Hegde [8] graph labeling problem describe in such a way where a set of values from which the labels are taken under a restricted environment and a procedure by which each edge label by a value under some efficient condition. Beineke and Hegde [2] explained graph labeling is an extreme limit between graphs structure and number theory.

The technique by which a graph is labeled can be applied on coding theory, missile guidance code etc. also the labeling techniques can be successfully applied on X-ray crystallography and communication network. Cahit [3, 4] has given the idea of both graceful [9] and harmonious [7] labellings. Cahit [4] proved the followings, tree is a cordial graph, K_n is cordial graph if and only if $n \leq 3$, $K_{m,n}$ is cordial graph $\forall\ m, n$, friendship graph is also a cordial graph with some restrictions, all fans are cordial etc. Babujee and Shobana [1] introduced the concepts of cordial languages and cordial numbers also some other labeling strategies are likewise presented with minor varieties in cordial theme like product cordial labeling, total product cordial labeling and prime cordial labeling. Vaidya and Shah [11–13] have contributed some new results on cordial labeling for the larger bistar graphs obtained by various graph operations.

Yegnanarayanan [14] explores some of the interesting applications of graph labeling. With advancement in technology the occurrence of more complex networking system is seen to have emerged. These systems form complex graph structure which are obtained by signed product or cartesian product of less complex graphs. In order to use the full potential of the system we need to develop an algorithm that not only labels such graphs but also has the lowest possible time complexity. It is a tedious task to deal with such complex structure, some of such complex graphs are studied and labeled by Ghosh and Pal [6, 10], the graph of interest in this paper i.e. $K_{n,n} \times K_{n,n}$ is one such complex graph. Due to its complex structure it is very difficult to visualize the graph and without proper visualization it is not possible to label these graphs. So, we develop a model that could simplify the graph for the ease of understanding and then using that information goes ahead to label the graph using a simple but effective algorithm. The graph $G = K_{n,n} \times K_{n,n}$ is a sub graph to Rook's graph so labeling this graph can aid in further development of models to label Rook's graph. Since Rook's graph has its application in solving complex assignment problems the labeling of $G = K_{n,n} \times K_{n,n}$ could help in writing a computer program with less time complexity for a practical assignment problem.

The paper is orchestrated in the following way. Section 2 contains some important and related definitions, Section 2.1 gives the physical representation and meaning of Cartesian Product between two $K_{n,n}$ graphs, Section 3 presents a way to simplify the graph obtained in previous section and also introduces an algorithms to label Cartesian product between two balanced bipartite graphs, proof of correctness of the algorithm is stated in this same section by means of Theorem 4.1, Section 5 presents an example to understand the methodology used and also to test the Algorithm, followed by conclusion.

2. Preliminaries

Definition 2.1 (Complete Bipartite Graphs)**.** A complete graph is simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A complete graph with n vertices is denoted by K_n. For this graph, for all vertices $x, y \in V$ there is an edge $(x, y) \in E$.

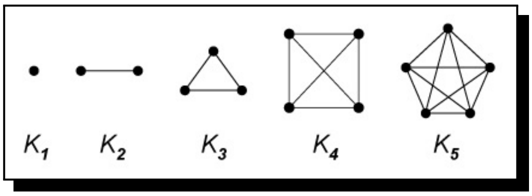

Figure 1. Complete graphs

A complete graph G is called a complete bipartite graph if its vertices can be partitioned into two subsets V_1 and V_2 such that no edges has both end points in the same subset, and each vertex of $V_1(V_2)$ is connected with all vertices of $V_2(V_1)$. Here $V_1 = \{x_1, x_2, \dots, x_m\}$ contains m vertices and $V_2 = \{x_{m+1}, x_{m+2}, \dots, x_{m+n}\}$ contains n vertices.

A complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.The total number of vertices in the graph is given by $(m+n)$ and the total number edges is given by mn. Therefore, the mathematical representation of the graph would be $G = ((m+n),(mn))$.

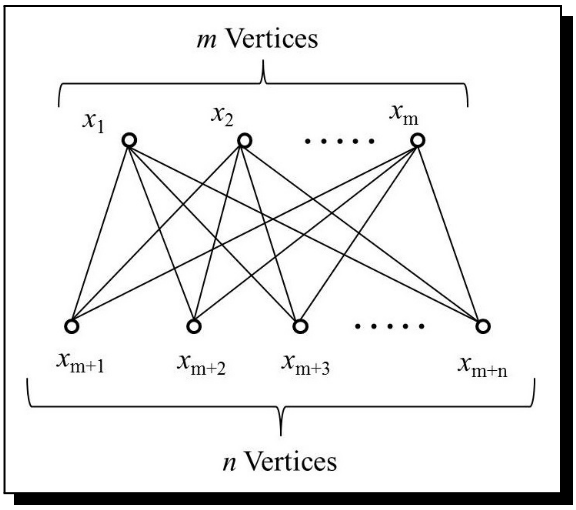

Figure 2. Generic form of Complete Bipartite graph $K_{m,n}$

Definition 2.2 (Balanced Bipartite Graph)**.** A complete bipartite graph G with vertex subset V_1 and V_2 is called balanced bipartite graph when $|V_1| = |V_2|$, that is if the two subsets have equal $cardinality$.

Definition 2.3 (Cartesian Products of Graphs)**.** Cartesian product of two graphs $G_1 = (V, E)$ and $G_2 = (V', E')$ is the Cartesian product between two set of vertices $V(G_1) \times V'(G_2)$ denoted by $G_1 \times G_2$, where (u, u') and (v, v') are the order pair of the Cartesian product will be adjacent in $G_1 \times G_2$ if and only if either

1. $u = v$ and u' is adjacent with v' in G_2, or
2. $u' = v'$ and u is adjacent with v in G_1.

The Cartesian product of two graphs are commutative.

Definition 2.4 (Cordial Labeling of Graphs)**.** Cordial labeling of a graph $G = (V,E)$ with vertex set V and edge set E where the vertex set is label by $f : V \to \{0,1\}$ and each edge xy is assign the value $|f(x)-f(y)|$. f is called cordial labeling if number of vertices labeled by 0 and number of vertices labeled by 1 is differ by at most 1, and the number of edges labeled by 0 and number of edges labeled by 1 is differ by at most 1.

So a graph $G = (V,E)$ is cordial if

$$|vf(0)-vf(1)| \le 1 \quad \text{and} \quad |ef(0)-ef(1)| \le 1.$$

2.1 Cartesian Product of two Balanced Bipartite $K_{n,n}$ graphs, i.e. $K_{n,n} \times K_{n,n}$

Consider two $K_{n,n}$ graphs named G_1 and G_2. In order to obtain the Cartesian product of these two graphs, we need to compute the vertex set of the resultant graph i.e. graph $G_1 \times G_2$. The vertex set is obtained by enumerating the ordered pair in the vertex set of $G_1 \times G_2$.
$V_1 = \{x_1, x_2, \dots, x_{2n}\}$ such that $V_1 \in G_1$ and $V_2 = \{x_1, x_2, \dots, x_{2n}\}$ such that $V_2 \in G_2$.

Therefore, the vertex set of $G = K_{n,n} \times K_{n,n}$ is defined by the following matrix:

$$V = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,2n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,2n} \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots & x_{3,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{2n,1} & x_{2n,2} & x_{2n,3} & \cdots & x_{2n,2n} \end{pmatrix}$$

After defining the vertex set of $G_1 \times G_2$ now we move on to defining the edge set of the same graph. Consider any two vertices of the graph G given by $x_{a,b}$ and $x_{c,d}$. An edge is defined between $x_{a,b}$ and $x_{c,d}$ if and only if the following two condition is fulfilled:

1. If $a = c$ and x_b and x_d are connected in graph G_1 or G_2
2. If $b = d$ and x_a and x_c are connected in graph G_1 or G_2

Therefore, for the Cartesian product of $K_{n,n} \times K_{n,n}$, The graph $K_{n,n}$ is cloned $2n$ times. Each sub graph $K_{n,n}$ has two set of vertices A, B where $|A| = n$ and $|B| = n$. Each set of vertices of $K_{n,n}$ for $2n$ clones represented by $\{(A_1,B_1),(A_2,B_2),(A_3,B_3),\dots,(A_{2n},B_{2n})\}$, where each $A_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,n}\}$ and $B_i = \{x_{i,(n+1)}, x_{i,(n+2)}, \dots, x_{i,2n}\}$.

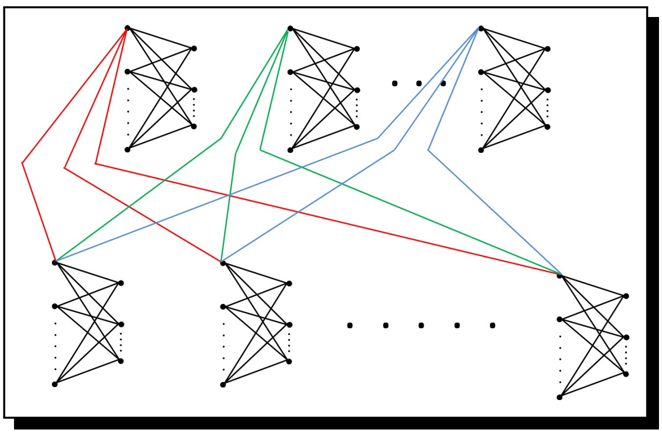

Figure 3. x_1 of each A_i is connected to x_1 of each B_i

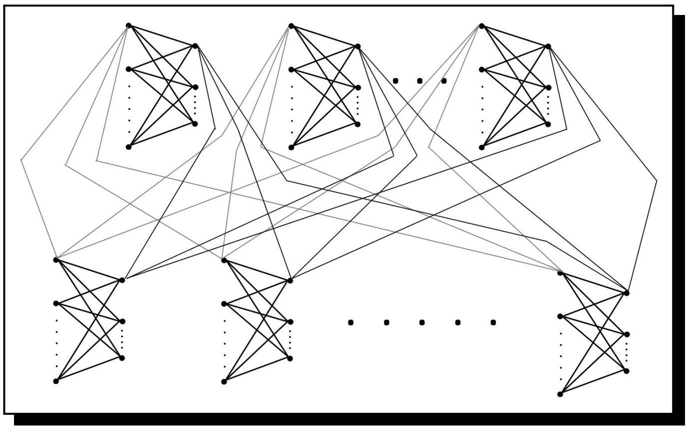

Figure 4. x_{n+1} of each A_i is connected to x_{n+1} of each B_i

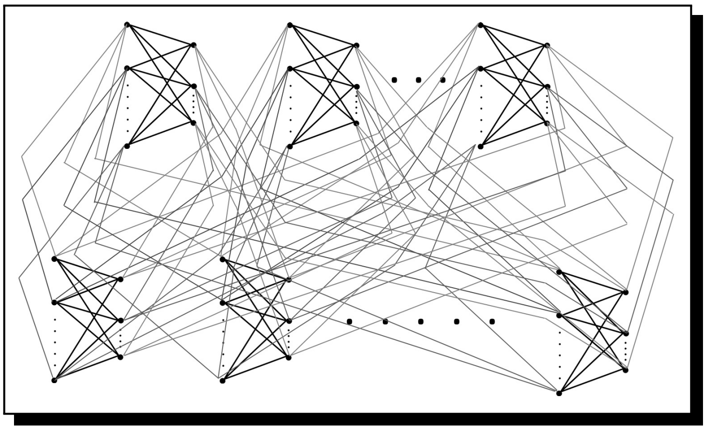

Figure 5. Cartesian product of $K_{n,n} \times K_{n,n}$

Lemma 2.1. *The degree of each vertex of graph G obtained from the cartesian product of two Balanced Bipartite Graph $K_{n,n}$ is $2n$.*

Proof. From the cartesian product of graph it is known that G is formed by $2n$ copies of $K_{n,n}$. Each vertex in these $K_{n,n}$ sub graph will have n edges originating from themselves responsible for the inter-connection of the $K_{n,n}$ graph. Along with this each vertex in G is connected with n other vertices belonging to n different clones of $K_{n,n}$ sub graphs. These edges are responsible for the intra-connection of the different clones of $K_{n,n}$ graphs. Therefore total number of edges originating from any given vertex is $2n$. So degree of each vertex of G is $2n$. □

Theorem 2.1. *For a graph G obtained from the cartesian product of two Balanced Bipartite Graphs (i.e. $G = K_{n,n} \times K_{n,n}$), the number of vertices of G is given by $4n^2$ and the number of edges is given by $4n^3$.*

Proof. Let G be a graph such that $G = K_{n,n} \times K_{n,n}$. By the definition of Cartesian Product of the graphs we know that the vertex set of G is a matrix obtained from the cartesian product of the vertex sets of parent $K_{n,n}$ graphs.

Now, by definition of Balanced Bipartite Graph, for every $K_{n,n}$ there exists two vertex sets V_1 and V_2 each containing n vertices such that $V_1 = \{x_1, x_2, \dots, x_n\}$ and $V_2 = \{x_{n+1}, x_{n+2}, \dots, x_{2n}\}$.

Therefore, total number of vertices for the $K_{n,n}$ graph is $2n$. Now since $G = K_{n,n} \times K_{n,n}$, the total number of vertices for G is $(2n \times 2n)$ or $4n^2$.

Hence it is proved that number of vertices in graph $G = K_{n,n} \times K_{n,n}$ is

$$|V_G| = 4n^2. \tag{2.1}$$

For the graph G total number of vertices is $4n^2$ and degree of each vertex is $2n$. Therefore, from the *Handshaking Theorem* we know,

$$\sum \deg G(V) = 2|E|, \tag{2.2}$$

therefore

$$4n^2 \times 2n = 2|E|, \tag{2.3}$$

$$|E| = 4n^3. \tag{2.4}$$

Hence it is proved that number of edges in graph $G = K_{n,n} \times K_{n,n}$ is

$$|E_G| = 4n^3. \tag{2.5}$$

□

From equations (2.1) and (2.5) it is proved that the total number of vertices of G is $4n^2$ and total number of edges of G is $4n^3$. Therefore it is observed that the graph G contains EVEN number of vertices and edges.

3. Methodology

3.1 Reconstruction and Simplification of the Graph

Labeling such complex graph is not possible unless we somehow reduce it into a symmetrical structure. To achieve the same, we need to make a minor alteration in the $K_{n,n}$ graph, before we obtain the Cartesian product of the same graph with itself. The nomenclature of the nodes of the parent $k_{n,n}$ graphs is changed in the following manner.

Let $G = K_{n,n}$ be complete bipartite graphs and its vertex set be grouped into two subsets V_1 and V_2 such that $V_1 = \{x_1, x_3, \dots x_{(2n-1)}\}$ and $V_2 = \{x_2, x_4, \dots x_{2n}\}$, shown in Figure 6, where

1. V_1 is a set of first $2n$ *odd* natural numbers and,
2. V_2 is a set of first $2n$ *even* natural numbers.

After doing this simple alteration we can see that each edge is mapped from a vertex set x_i to vertex set x_j in such a way that one end of the edge is an even numbered vertex and the other end is a odd numbered vertex.

Now we find the Cartesian product of the graphs G_1 and G_2 and as illustrated in Figure 7 we obtain a fairly organized and symmetrical graph with vertex set $x_{i,j}$.

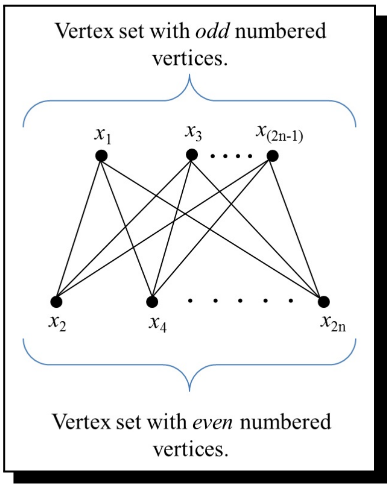

Figure 6. Redefined structure of $K_{n,n}$ graph

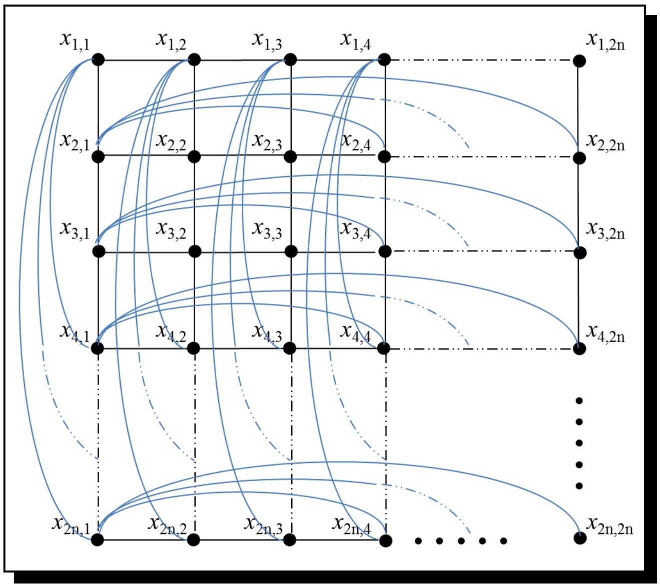

Figure 7. Symmetric representation of $K_{n,n} \times K_{n,n}$ graph

3.2 Labeling the Graph

As stated in Definition 2.4 in order to cordially label the graph G we need to assign $0s$ and $1s$ to its vertex set. Consider the graph G whose vertex set is given by $V = x_{i,j}$ where $i = j = \{1,2,3,\dots 2n\}$. The vertex set V is stored in a matrix with each elements denoted by $x_{i,j}$. For the array of elements $x_{i,j}$ in the matrix V, such that $i = 1$, and $j = \{1,2,3,\dots 2n\}$ we assign 0 to this complete array of $x_{1,j}$. Now, for $i = 2$, and $j = \{1,2,3,\dots 2n\}$ we assign 1 to this complete array of $x_{2,j}$ and so on. Therefore,

1. Element $x_{i,j}$ is assigned 0 if i is NOT divisible by 2 for all j and,
2. Element $x_{i,j}$ is assigned 1 if i is divisible by 2 for all j.

Algorithm 1 Algorithm for Cordially labeling the Cartesian product of two Balanced Bipartite Graph (BBG) $K_{n,n} \times K_{n,n}$ $(CLCPBBG)$

Input: A graph $G = (K_{n,n} \times K_{n,n})$. With vertex set $V = [x_{i,j}]$

Output: Cordially labeled graph $G = (K_{n,n} \times K_{n,n})$.

Step 1 Initialize $V = x[i][j]$
Step 2 for $i = 1$ to $2n$
Step 3 for $j = 1$ to $2n$
Step 4 if $(i \bmod 2) == 0$
$f(x_{ij}) = 1$
Step 5 else
$f(x_{ij}) = 0$
end j loop
end i loop

4. Analysis of the Algorithm

4.1 Proof of Correctness of the Algorithm

Theorem 4.1. *If a graph G, defined as $G = (K_{n,n} \times K_{n,n})$, follows the CLCPBBG Algorithm, then the graph G is Cordially Labeled.*

Proof. *Case* 1. *Checking for Vertex Cordial Labeling.*

By definition of G, the vertex set of G is given by:

$$V = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \dots & x_{1,2n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \dots & x_{2,2n} \\ x_{3,1} & x_{3,2} & x_{3,3} & \dots & x_{3,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{2n,1} & x_{2n,2} & x_{2n,3} & \dots & x_{2n,2n} \end{pmatrix}$$

or, $V = [x_{i,j}]$, for $i = j = \{1,2,3,\dots,n,(n+1),(n+2),\dots 2n\}$.

According to the Algorithm, for every $x_{i,j}$, a vertex is labeled 0 if i is EVEN and 1 if ODD. So, we see that the entire 1st row of the vertex matrix is labeled 0, 2nd row 1, 3rd row 0 and so on i.e. all the rows are alternatively labeled 0 and 1.

Since $i = \{1,2,3,\dots,n,(n+1),(n+2),\dots 2n\}$ we can conclude that there are even number of rows in the vertex matrix. And since the rows are alternatively labeled 0 and 1, there will be exactly half rows labeled 0 and half rows labeled 1. Therefore,

(a) n rows are labeled with 0 and,
(b) n rows are labeled with 1

Now we also know that for every i there exists a j which varies from 1 to $2n$ i.e. for every row of the vertex matrix there are $2n$ columns. So the total number of vertices labeled 0 would be $n \times 2n$ which is $2n^2$. Similarly vertices labeled 0 would be $2n^2$.

From the definition of cordial labeling, we know, the vertices are cordially labeled if

$$|vf(0) - vf(1)| \le 1.$$

For $G = (K_{n,n} \times K_{n,n})$,

$$|vf(0)| = 2n^2 \quad \text{also,} \quad |vf(1)| = 2n^2, \tag{4.1}$$

therefore

$$|vf(0) - vf(1)| = 2n^2 - 2n^2, \tag{4.2}$$

$$|vf(0) - vf(1)| = 0. \tag{4.3}$$

Hence it is proved that the vertices of G are Cordially Labeled.

Case 2. *Checking for Edge Cordial Labeling.*

Considering the graph G with vertex set $[x_{ij}]$, if the value of j is held to be constant and $i = \{1,2,3,\dots,n,(n+1),(n+2),\dots,2n\}$, we can observe that it essentially represents a $K_{n,n}$ graph such that, its vertex set is divided into two sub sets v'_1 and v'_2 where,

$v_1' = [x_{pj}]$ such that j = constant, $p \in i$ and $p = \{1,3,5,\ldots,(2n-1)\}$
$v_2' = [x_{qj}]$ such that j = constant, $q \in i$ and $q = \{2,4,6,\ldots,2n\}$.

Also every vertex in v_1' is connected to every vertex in v_2' with an edge. On applying the *CLCPBBG* algorithm we observe that $f(v_1') = 0$ and $f(v_2') = 1$. Therefore from the definition of Cordial Labeling, the labeling of the edge is given by the absolute difference between the labeling of the vertices it connects, i.e.

$$f(e(x_{pj}x_{qj})) = |f(e(x_{pj})) - f(e(x_{qj}))|, \tag{4.4}$$

$$f(e(x_{pj}x_{qj})) = |0-1|, \tag{4.5}$$

$$f(e(x_{pj}x_{qj})) = 1. \tag{4.6}$$

So all the edges within the $K_{n,n}$ sub graph is labeled as 1.

Now, for the graph G with vertex set $[x_{ij}]$ if i is held to be constant and $j = \{1,2,3,\ldots,n,(n+1),(n+2),\ldots 2n\}$ then an edge defined between vertex x_{ir} and vertex x_{is} where $r,s \in j$ shows the connection between the clones of the $K_{n,n}$ sub graphs.

Since for any two vertices x_{ir} and x_{is} the value of i is constant therefore both these vertices would have same vertex labeling, i.e.

(a) $f(x_{ir}) = f(x_{is}) = 0$, if i is *odd* and,
(b) $f(x_{ir}) = f(x_{is}) = 1$, if i is *even*

In either case the edge connecting x_{ir} and x_{is} would be labeled as the absolute difference of $f(x_{ir})$ and $f(x_{is})$ that is always 0.

From this above analysis we conclude that the edges within the clones of $K_{n,n}$ sub graphs will always be labeled as 1 and the edges connecting these clones with each will be labeled as 0.

Now by the definition of $G = (K_{n,n} \times K_{n,n})$, the graph G must contain $2n$ clones of the $K_{n,n}$ sub graphs and each $K_{n,n}$ sub graphs contains n^2 edges all of which are labeled 1. Therefore total number of edges labeled 1 is $(2n \times n^2)$ or $2n^3$ and rest all of the edges are labeled 0 as proved above.

So, number of edges labeled 0 could be obtained by subtracting number of edges labeled 1 from the total number of edges in graph G. Therefore number of edges labeled 0 is $(4n^3 - 2n^3)$ which is $2n^3$.

For $G = (K_{n,n} \times K_{n,n})$,

$$|ef(0)| = 2n^3 \quad \text{also,} \quad |ef(1)| = 2n^3, \tag{4.7}$$

therefore

$$|ef(0) - ef(1)| = 2n^3 - 2n^3 \tag{4.8}$$

$$|ef(0) - ef(1)| = 0 \tag{4.9}$$

Hence it is proved that the edges of G are Cordially Labeled.

Since in the graph G, defined as $G = (K_{n,n} \times K_{n,n})$, both vertices and edges are cordially labeled, therefore the graph G is Cordially Labeled. □

4.2 Time Complexity Analysis of the Algorithm

The algorithm consists of a nested *for* loop having two levels each iterating from 1 to $2n$. The decision criteria is a set of *if else* conditions which has linear time complexity. Considering the inner for loop iterating from $j = 1$ to $2n$, for each of these $2n$ iterations the decision statement would be executed for atleast once. Therefore the time complexity of this inner loop is $2n$, or $T \propto \bigcirc(n)$. Now considering the outer loop iterating from $i = 1$ to $2n$, for each value of i the decision statements will be evaluated for $2n$ times. So for $2n$ values of i the decision statements would be evaluated for $2n \times 2n$ times i.e. $4n^2$ times. Therefore, The time complexity of the entire nested loop is $4n^2$, or

$$T \propto \bigcirc(n^2).$$

Therefore the Algorithm 1 runs with *Quadratic* time complexity.

5. Illustration: Cordially Labeling $G = (K_{4,4} \times K_{4,4})$

***Step* 1: Constructing the $K_{4,4}$ Graph**

Consider a graph $G = K_{4,4}$, as shown in Figure 8 G is a complete bipartite graph and its vertex set V is grouped into two subsets V_1' and V_2' such that $V_1' = \{x_1, x_3, x_5, x_7\}$ and $V_2' = \{x_2, x_4, x_6, x_8\}$.

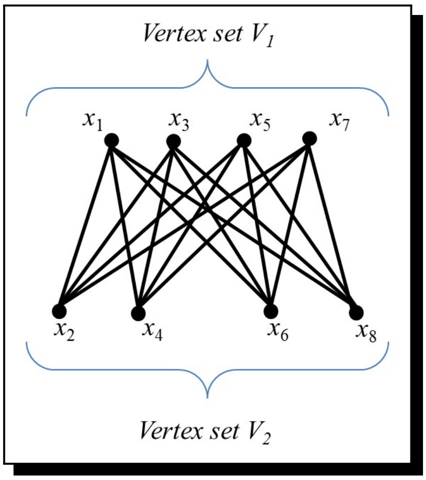

Figure 8. Complete Bipartite Graph $K_{4,4}$

***Step* 2: Constructing $G = K_{4,4} \times K_{4,4}$ Graph**

Consider two such $K_{4,4}$ graphs named G_1 and G_2 as described in Step 1. In order to obtain the Cartesian product of these two graphs, we need to compute the vertex set of the resultant graph i.e. graph $G_1 \times G_2$. The vertex set is obtained by enumerating the ordered pair in the vertex set of $G_1 \times G_2$.

$$V_1 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \text{ such that } V_1 \in G_1$$

and

$$V_2 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \text{ such that } V_2 \in G_2.$$

Therefore, the vertex set of $G = K_{4,4} \times K_{4,4}$ is characterized by the following matrix:

$$V = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} & x_{4,6} & x_{4,7} & x_{4,8} \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & x_{5,6} & x_{5,7} & x_{5,8} \\ x_{6,1} & x_{6,2} & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} & x_{6,7} & x_{6,8} \\ x_{7,1} & x_{7,2} & x_{7,3} & x_{7,4} & x_{7,5} & x_{7,6} & x_{7,7} & x_{7,8} \\ x_{8,1} & x_{8,2} & x_{8,3} & x_{8,4} & x_{8,5} & x_{8,6} & x_{8,7} & x_{8,8} \end{pmatrix}.$$

We have obtained the vertex set of G. Now following the steps as described in Section 2.1 we define the edges of the graph $G = K_{4,4} \times K_{4,4}$ hence obtaining a graph as shown in Figure 9.

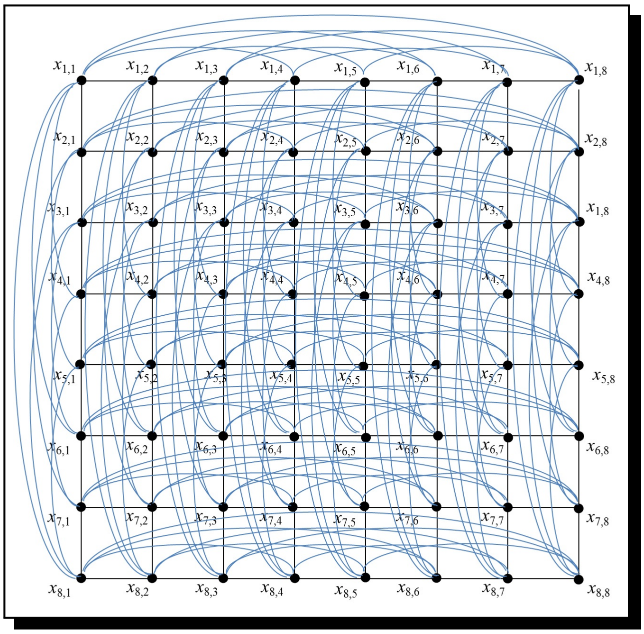

Figure 9. Graph $K_{4,4} \times K_{4,4}$

1. According to the lemma in Section 2.1, we can verify that the degree of each vertex in graph G is given by $2n$. Since for $G = K_{4,4} \times K_{4,4}$ the value of n is 4, therefore degree of each vertex is 8. This can be verified from Figure 9.
2. Theorem 2.1 in Section 2.1 states that the number of vertices of G is given by $4n^2$ and the number of edges is given by $4n^3$. Since $n = 4$, the number of vertices of G is $4 \times 4^2 = 64$ and the number of edges is given by $4 \times 4^3 = 256$. These can again be verified from Figure 9.

***Step* 3: Cordially Labeling the $K_{4,4} \times K_{4,4}$ Graph**

As stated in Algorithm 1 in Section 3.2, a vertex $x_{i,j}$ is labeled 0 if $i \bmod 2 \neq 0$ else its labeled 1 if $i \bmod 2 = 0$. So, Vertex $x_{1,j}, x_{3,j}, x_{5,j}, x_{7,j}$ are assigned 0 and
Vertex $x_{2,j}, x_{4,j}, x_{6,j}, x_{8,j}$ are assigned 1, for $j = \{1,2,3,4,5,6,7,8\}$.

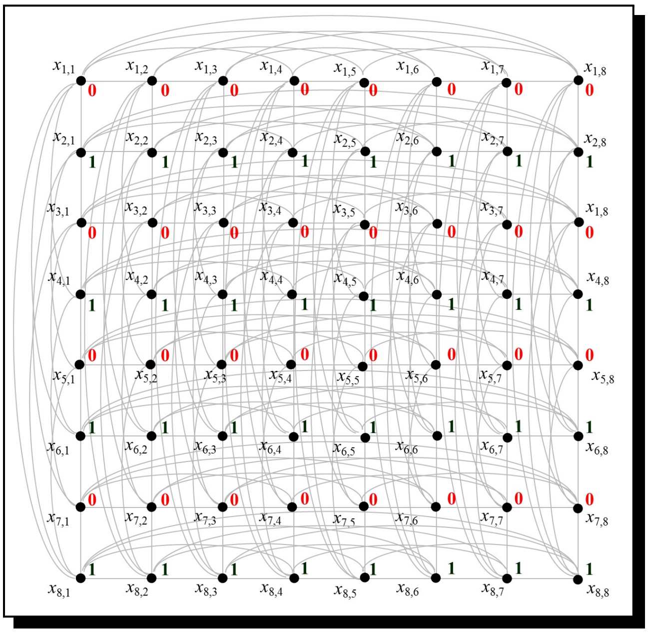

Figure 10. Cordially Labeled Graph $K_{4,4} \times K_{4,4}$

As proved in Theorem 4.1 the graph $G = K_{4,4} \times K_{4,4}$ is cordially labeled.

As described in Case 1 of Theorem 4.1, there are $2n^2 = 32$ vertices labeled 0 and another $2n^2 = 32$ vertices labeled 1. This can be observed in the Vertex Labeling Value matrix of each sub graph of $K_{4,4}$ in graph G.

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

therefore

$$|vf(0) - vf(1)| = |32 - 32| = 0 \tag{5.1}$$

In Case 2 of Theorem 4.1 the labeling of the edges is proved. There are 8 clones of $K_{4,4}$ in graph G and as proved in the same section all the inter-connections of these $K_{4,4}$ sub graphs are labeled 1 so, $2n^3$ i.e. 128 edges are labeled 1. So the Edge Labeling Value for each $K_{4,4}$ sub graph is given as:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix} \quad \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$
$$\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix} \quad \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$
$$\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix} \quad \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$

The intra-connection between these clones of $K_{4,4}$ graphs are labeled as 0. So, the remaining $2n^3$ i.e. 128 edges are labeled 0.

$$\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$
$$\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

therefore

$$|ef(0) - ef(1)| = |128 - 128| = 0. \tag{5.2}$$

Results

As described in Section 3.1, the complex $K_{4,4} \times K_{4,4}$ graph was reduced to a simplified form for the ease of study. The graph $G = K_{4,4} \times K_{4,4}$ had 64 vertices and 256 edges and degree of each vertex was 8 as found using Theorem 2.1 and lemma in Section 2.1. On imposing Algorithm 1 on graph G, the graph is *Cordially Labeled*, as evident from equations (5.1) and (5.2). Also, the time complexity is $\bigcirc(n^2)$.

6. Conclusions

In this paper we enroot an universal algorithm that cordially labels the complete family of graphs obtained by the cartesian product of two balanced bipartite graphs. As illustrated in

Section 5 the algorithm successfully labels and analyzes the time complexity of any graph obtained from cartesian product $K_{n,n} \times K_{n,n}$. The algorithm is found to run with a *Quadratic* time complexity in $\bigcirc(n^2)$. The family of graphs discussed in this paper are the sub graph of *Rook's* graph which is a graph obtained from the cartesian product of two complete graphs. So this algorithm can be further developed to study and label Rook's graph with less time complexity. Also, this algorithm can be used as a basis for signed product cordial, total signed product cordial, prime cordial labeling of $G = K_{n,n} \times K_{n,n}$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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