Journal of Informatics and Mathematical Sciences
Vol. 10, No. 4, pp. 573–581, 2018
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v10i4.1130

http://www.rgnpublications.com

Research Article

(ε,δ)-Characteristic Fuzzy Sets Approach to the Ideal Theory of *BCK/BCI*-Algebras

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Abstract. The notion of (ε,δ)-characteristic fuzzy sets is introduced. Given an ideal F of a BCK/BCI-algebra X, conditions for the (ε,δ)-characteristic fuzzy set in X to be an $(\in,\in\vee q)$-fuzzy ideal, an $(\in,q)$-fuzzy ideal, an $(\in,\in\wedge q)$-fuzzy ideal, a (q,q)-fuzzy ideal, a $(q,\in)$-fuzzy ideal, a $(q,\in\vee q)$-fuzzy ideal and a $(q,\in\wedge q)$-fuzzy ideal are provided. Using the notions of (α,β)-fuzzy ideal $\mu_F^{(\varepsilon,\delta)}$, conditions for the F to be an ideal of X are investigated where (α,β) is one of $(\in,\in\vee q)$, $(\in,\in\wedge q)$, $(\in,q)$, $(q,\in\vee q)$, $(q,\in\wedge q)$, $(q,\in)$ and (q,q).

Keywords. (ε,δ)-characteristic fuzzy set; (Fuzzy) ideal; (α,β)-fuzzy ideal

MSC. 06F35; 03G25; 06D72

Received: November 18, 2018 **Accepted:** December 3, 2018

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [10], played a vital role to generate some different types of fuzzy subgroups, called (α,β)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. The concept of (α,β)-fuzzy subalgebras in BCK/BCI-algebras is also important and useful generalization of the well-known concepts, called fuzzy subalgebras (see for e.g., [3], [4], [5] and [11]). Recently, Muhiuddin *et al*. studied the fuzzy set theoretical approach to the BCK/BCI-algebras on various aspects (see for e.g., [7], [8], [9]).

In this paper, we introduce the notion of (ε,δ)-characteristic fuzzy sets in BCK/BCI-algebras. Given an ideal F of a BCK/BCI-algebra X, we provide conditions for the (ε,δ)-characteristic fuzzy set in X to be an $(\in,\in\vee q)$-fuzzy ideal, an $(\in,q)$-fuzzy ideal, an $(\in,\in\wedge q)$-fuzzy ideal, a (q,q)-fuzzy ideal, a $(q,\in)$-fuzzy ideal, a $(q,\in\vee q)$-fuzzy ideal and a $(q,\in\wedge q)$-fuzzy ideal. Using the notions of (α,β)-fuzzy ideal $\mu_F^{(\varepsilon,\delta)}$, we investigate conditions for the F to be an ideal of X where (α,β) is one of $(\in,\in\vee q)$, $(\in,\in\wedge q)$, $(\in,q)$, $(q,\in\vee q)$, $(q,\in\wedge q)$, $(q,\in)$ and (q,q).

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X,*,0)$ of type $(2,0)$ satisfying the axioms:

(a1) $((x*y)*(x*z))*(z*y)=0$,

(a2) $(x*(x*y))*y=0$,

(a3) $x*x=0$,

(a4) $x*y=y*x=0 \Rightarrow x=y$,

for all $x,y,z\in X$.

We can define a partial ordering $\le$ by $x\le y$ if and only if $x*y=0$. If a BCI-algebra X satisfies the axiom

(a5) $0*x=0$ for all $x\in X$,

then we say that X is a *BCK-algebra*. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

(I1) $0\in A$,

(I2) $(\forall\, x\in X)\,(\forall\, y\in A)\,(x*y\in A \implies x\in A)$.

We refer the reader to the books [2] and [6] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y):=\begin{cases} t\in(0,1] & \text{if } y=x,\\ 0 & \text{if } y\neq x,\end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [10] introduced the symbol $x_t\alpha\mu$, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$. To say that $x_t\in\mu$ (resp. $x_t\,q\,\mu$), we mean $\mu(x)\ge t$ (resp. $\mu(x)+t>1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t\in\vee q\,\mu$ (resp. $x_t\in\wedge q\,\mu$), we mean $x_t\in\mu$ or $x_t\,q\,\mu$ (resp. $x_t\in\mu$ and $x_t\,q\,\mu$). To say that $x_t\,\overline{\alpha}\,\mu$, we mean $x_t\alpha\mu$ does not hold, where $\alpha\in\{\in,q,\in\vee q,\in\wedge q\}$.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy ideal* of X if it satisfies:

$$\mu(0)\ge\mu(x)\ge\min\{\mu(x*y),\mu(y)\} \tag{2.1}$$

for all $x,y\in X$.

Proposition 2.1 ([3]). *Let X be a BCK/BCI-algebra. A fuzzy set μ in X is a fuzzy ideal of X if and only if the following assertions are valid.*

$$x_t\in\mu \implies 0_t\in\mu, \tag{2.2}$$

$$(x*y)_t \in \mu,\ y_s \in \mu \implies x_{\min\{t,s\}} \in \mu \tag{2.3}$$

for all $x,y \in X$ and $t,s \in (0,1]$.

3. Ideals of *BCK/BCI*-Algebras Based on (α,β)-Type Fuzzy Sets

In what follows, let X denote a BCK/BCI-algebra and let $\varepsilon,\delta \in [0,1]$ such that $\varepsilon > \delta$ unless otherwise specified.

For any non-empty subset F of X, define a fuzzy set $\mu_F^{(\varepsilon,\delta)}$ in X as follows:

$$\mu_F^{(\varepsilon,\delta)}(x) := \begin{cases} \varepsilon & \text{if } x \in F, \\ \delta & \text{otherwise.} \end{cases}$$

We say that $\mu_F^{(\varepsilon,\delta)}$ is an (ε,δ)-*characteristic fuzzy set* in X over F (see [9]). In particular, $(1,0)$-characteristic fuzzy set $\mu_F^{(1,0)}$ in X over F is the characteristic function χ_F of F.

Theorem 3.1. *For any non-empty subset F of X, the following are equivalent:*

(1) *F is an ideal of X.*

(2) *The (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a fuzzy ideal of X.*

Proof. Assume that F is an ideal of X. Since $0 \in F$, clearly $\mu_F^{(\varepsilon,\delta)}(0) = \varepsilon \ge \mu_F^{(\varepsilon,\delta)}(x)$ for all $x \in X$. Let $x,y \in X$. If $y \in F$ and $x*y \in F$, then $x \in F$ and so

$$\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \min\left\{\mu_F^{(\varepsilon,\delta)}(y), \mu_F^{(\varepsilon,\delta)}(x*y)\right\}.$$

If $y \notin F$ or $x*y \notin F$, then $\mu_F^{(\varepsilon,\delta)}(y) = \delta$ or $\mu_F^{(\varepsilon,\delta)}(x*y) = \delta$. Hence

$$\mu_F^{(\varepsilon,\delta)}(x) \ge \delta = \min\left\{\mu_F^{(\varepsilon,\delta)}(y), \mu_F^{(\varepsilon,\delta)}(x*y)\right\}.$$

Therefore $\mu_F^{(\varepsilon,\delta)}$ is a fuzzy ideal of X for all $\varepsilon,\delta \in [0,1]$ with $\varepsilon > \delta$.

Conversely, suppose that (2) is valid. Obviously, $0 \in F$. Let $x,y \in X$ be such that $y \in F$ and $x*y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(y) = \varepsilon$ and $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon$. It follows that

$$\mu_F^{(\varepsilon,\delta)}(x) \ge \min\left\{\mu_F^{(\varepsilon,\delta)}(y), \mu_F^{(\varepsilon,\delta)}(x*y)\right\} = \varepsilon.$$

Thus $x \in F$, and therefore F is an ideal of X. □

Definition 3.2 ([3]). A fuzzy set μ in X is said to be an (α,β)-*fuzzy ideal* of X, where $\alpha,\beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq\ \in\wedge q$, if it satisfies the following condition:

$$(\forall\, x \in X)\ (\forall\, t \in (0,1])\left(x_t \alpha \mu \ \Rightarrow\ 0_t \beta \mu\right), \tag{3.1}$$

$$(\forall\, x,y \in X)\ (\forall\, t_1,t_2 \in (0,1])\left((x*y)_{t_1} \alpha \mu,\, y_{t_2} \alpha \mu \ \Rightarrow\ x_{\min\{t_1,t_2\}} \beta \mu\right). \tag{3.2}$$

Lemma 3.3 ([3]). *A fuzzy set μ in X is an $(\in,\in\vee q)$-fuzzy ideal of X if and only if it satisfies:*

(1) $(\forall\, x \in X)(\mu(0) \ge \min\{\mu(x), 0.5\})$,

(2) $(\forall\, x,y \in X)(\mu(x) \ge \min\{\mu(x*y), \mu(y), 0.5\})$.

Theorem 3.4. *If F is an ideal of X, then the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in,\in\vee q)$-fuzzy ideal of X.*

Proof. Assume that F is an ideal of X. Since $0 \in F$, we have

$$\mu_F^{(\varepsilon,\delta)}(0) = \varepsilon \ge \min\left\{\mu_F^{(\varepsilon,\delta)}(x), 0.5\right\}$$

for all $x \in X$. For any $x, y \in X$, if $x * y \in F$ and $y \in F$, then $x \in F$ and so

$$\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon \geq \min\left\{\mu_F^{(\varepsilon,\delta)}(x * y), \mu_F^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

If $x \notin F$ or $y \notin F$, then $\mu_F^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_F^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_F^{(\varepsilon,\delta)}(x * y) \geq \delta \geq \min\left\{\mu_F^{(\varepsilon,\delta)}(x), \mu_F^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

It follows from Lemma 3.3 that $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in \vee q)$-fuzzy ideal of X. □

We consider the converse of Theorem 3.4.

Theorem 3.5. *For any $\varepsilon, \delta \in [0,1]$ such that $\delta < \varepsilon \leq 0.5$, if the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in \vee q)$-fuzzy ideal of X then F is an ideal of X.*

Proof. If $0 \notin F$, then $\mu_F^{(\varepsilon,\delta)}(0) = \delta < \varepsilon = \mu_F^{(\varepsilon,\delta)}(x)$ for some $x \in F$. Hence $x_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$, and so $0_\varepsilon \in \vee q\, \mu_F^{(\varepsilon,\delta)}$ since $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in \vee q)$-fuzzy ideal of X. But $\mu_F^{(\varepsilon,\delta)}(0) = \delta \ngeq \varepsilon$ and $\mu_F^{(\varepsilon,\delta)}(0) + \varepsilon = \delta + \varepsilon \not> 1$. This is a contradiction, and so $0 \in F$. Let $x, y \in F$ be such that $x * y \in F$ and $y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$. Using Lemma 3.3, we have

$$\mu_F^{(\varepsilon,\delta)}(x) \geq \min\left\{\mu_F^{(\varepsilon,\delta)}(x * y), \mu_F^{(\varepsilon,\delta)}(y), 0.5\right\} = \min\{\varepsilon, 0.5\} = \varepsilon,$$

and so $x \in F$. Therefore F is an ideal of X. □

Corollary 3.6. *A non-empty subset F of X is an ideal of X if and only if the characteristic function χ_F of F is an $(\in, \in \vee q)$-fuzzy ideal of X.*

Proof. The necessity is by taking $\varepsilon = 1$ and $\delta = 0$ in Theorem 3.4.

Conversely, suppose that the characteristic function χ_F of F is an $(\in, \in \vee q)$-fuzzy ideal of X. Obviously, $0 \in F$ by Lemma 3.3(1). Let $x, y \in X$ be such that $x * y \in F$ and $y \in F$. Then $\chi_F(x * y) = 1 = \chi_F(y)$, which implies from Lemma 3.3(2) that

$$\chi_F(x) \geq \min\{\chi_F(x * y), \chi_F(y), 0.5\} = \min\{1, 0.5\} = 0.5.$$

Hence $x \in F$, and therefore F is an ideal of X. □

Theorem 3.7. *Assume that if any element t in $(0,1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta < t$ and $1 - t < \varepsilon$. If F is an ideal of X, then the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy ideal of X.*

Proof. Let $x \in X$ and $t \in (0,1]$ be such that $x_t \in \mu_F^{(\varepsilon,\delta)}$. Since $0 \in F$ and $1 - t < \varepsilon$, we have $\mu_F^{(\varepsilon,\delta)}(0) + t = \varepsilon + t > 1$. Hence $0_t\, q\, \mu_F^{(\varepsilon,\delta)}$. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $(x * y)_{t_1} \in \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2} \in \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x * y) \geq t_1 > \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) \geq t_2 > \delta$. It follows that $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $x * y \in F$ and $y \in F$. Since F is an ideal of X, we have $x \in F$. Hence $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon$, and thus $\mu_F^{(\varepsilon,\delta)}(x) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1$ which shows that $x_{\min\{t_1,t_2\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Therefore $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy ideal of X. □

We consider the converse of Theorem 3.7.

Theorem 3.8. *If $\varepsilon + \delta \leq 1$ and the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy ideal of X, then F is an ideal of X.*

Proof. Assume that $\varepsilon+\delta\le 1$ and the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in,q)$-fuzzy ideal of X. Suppose that $0\notin F$. Then $\mu_F^{(\varepsilon,\delta)}(0)=\delta<\varepsilon=\mu_F^{(\varepsilon,\delta)}(x)$ for some $x\in X$, and so $x_\varepsilon\in\mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is an $(\in,q)$-fuzzy ideal of X, it follows that $0_\varepsilon\, q\,\mu_F^{(\varepsilon,\delta)}$, that is, $\mu_F^{(\varepsilon,\delta)}(0)+\varepsilon>1$. This is a contradiction, and thus $0\in F$. Let $x,y\in X$ be such that $x*y\in F$ and $y\in F$. Then $\mu_F^{(\varepsilon,\delta)}(x*y)=\varepsilon=\mu_F^{(\varepsilon,\delta)}(y)$, and so $(x*y)_\varepsilon\in\mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon\in\mu_F^{(\varepsilon,\delta)}$. Hence $x_\varepsilon=x_{\min\{\varepsilon,\varepsilon\}}\, q\,\mu_F^{(\varepsilon,\delta)}$, which implies that $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon>1$. Therefore $\mu_F^{(\varepsilon,\delta)}(x)>1-\varepsilon\ge\delta$, and thus $\mu_F^{(\varepsilon,\delta)}(x)=\varepsilon$, that is, $x\in F$. Consequently, F is an ideal of X. □

If we take $\varepsilon=1$ and $\delta=0$ in Theorems 3.7 and 3.8, then we have the following corollary.

Corollary 3.9. *A non-empty subset F of X is an ideal of X if and only if the characteristic function χ_F of F is an $(\in,q)$-fuzzy ideal of X.*

Theorem 3.10. *Let $\varepsilon,\delta\in[0,1]$ such that $\varepsilon>\delta$. If F is an ideal of X, then the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy ideal of X whenever if any element t in $(0,1]$ satisfies $x_t\in\mu_F^{(\varepsilon,\delta)}$ for $x\in X$ then $\delta\le 1-t<\varepsilon$.*

Proof. Since $0\in F$, we have $\mu_F^{(\varepsilon,\delta)}(0)+t=\varepsilon+t>1$, that is, $0_t\, q\,\mu_F^{(\varepsilon,\delta)}$ for any $x\in X$ and $t\in(0,1]$ with $x_t\, q\,\mu_F^{(\varepsilon,\delta)}$. Let $x,y\in X$ and $t_1,t_2\in(0,1]$ be such that $(x*y)_{t_1}\, q\,\mu_F^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\,\mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x*y)+t_1>1$ and $\mu_F^{(\varepsilon,\delta)}(y)+t_2>1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x*y)>1-t_1\ge\delta$ and $\mu_F^{(\varepsilon,\delta)}(y)>1-t_2\ge\delta$. It follows that $\mu_F^{(\varepsilon,\delta)}(x*y)=\varepsilon=\mu_F^{(\varepsilon,\delta)}(y)$ and so that $x*y\in F$ and $y\in F$. Since F is an ideal of X, we have $x\in F$ and so $\mu_F^{(\varepsilon,\delta)}(x)=\varepsilon$. Thus

$$\mu_F^{(\varepsilon,\delta)}(x)+\min\{t_1,t_2\}=\varepsilon+\min\{t_1,t_2\}>1,$$

that is, $x_{\min\{t_1,t_2\}}\, q\,\mu_F^{(\varepsilon,\delta)}$. This shows that $\mu_F^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy ideal of X. □

Theorem 3.11. *Let $\varepsilon,\delta\in[0,1]$ such that $\varepsilon>\max\{\delta,0.5\}$ and $\varepsilon+\delta\le 1$. If the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy ideal of X, then F is an ideal of X.*

Proof. Assume that $0\notin F$. Then $\mu_F^{(\varepsilon,\delta)}(0)=\delta<\varepsilon=\mu_F^{(\varepsilon,\delta)}(x)$ for some $x\in X$, which implies that $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon=2\varepsilon>1$, that is, $x_\varepsilon\, q\,\mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy ideal of X, it follows that $0_\varepsilon\, q\,\mu_F^{(\varepsilon,\delta)}$ and so that $\delta+\varepsilon=\mu_F^{(\varepsilon,\delta)}(0)+\varepsilon>1$. This is a contradiction, and therefore $0\in F$. Let $x,y\in X$ be such that $x*y\in F$ and $y\in F$. Then $\mu_F^{(\varepsilon,\delta)}(x*y)=\varepsilon=\mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x*y)+\varepsilon=\varepsilon+\varepsilon>1 \quad\text{and}\quad \mu_F^{(\varepsilon,\delta)}(y)+\varepsilon=\varepsilon+\varepsilon>1,$$

that is, $(x*y)_\varepsilon\, q\,\mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\,\mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a (q,q)-fuzzy ideal of X, it follows that $x_\varepsilon=x_{\min\{\varepsilon,\varepsilon\}}\, q\,\mu_F^{(\varepsilon,\delta)}$. Hence $\mu_F^{(\varepsilon,\delta)}(x)>1-\varepsilon\ge\delta$, and therefore $\mu_F^{(\varepsilon,\delta)}(x)=\varepsilon$. This proves that $x\in F$, and F is an ideal of X. □

If we take $\varepsilon=1$ and $\delta=0$ in Theorems 3.10 and 3.11, then we have the following corollary.

Corollary 3.12. *A non-empty subset F of X is an ideal of X if and only if the characteristic function χ_F of F is a (q,q)-fuzzy ideal of X.*

Theorem 3.13. *Let $\varepsilon,\delta\in[0,1]$ such that $\varepsilon>\delta$. If F is an ideal of X, then the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in)$-fuzzy ideal of X whenever if any element t in $(0,1]$ satisfies $x_t\in\mu_F^{(\varepsilon,\delta)}$ for $x\in X$ then $\delta\le 1-t$ and $t<\varepsilon$.*

Proof. Obviously, $0_t \in \mu_F^{(\varepsilon,\delta)}$ for all $x \in X$ and $t \in (0,1]$ with $x_t\, q\, \mu_F^{(\varepsilon,\delta)}$. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $(x*y)_{t_1}\, q\, \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\, \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x*y)+t_1 > 1$ and $\mu_F^{(\varepsilon,\delta)}(y)+t_2 > 1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x*y) > 1-t_1 \geq \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) > 1-t_2 \geq \delta$. Hence $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $x*y \in F$ and $y \in F$. Since F is an ideal of X, we have $x \in F$ and thus

$$\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon \geq \min\{t_1, t_2\},$$

that is, $x_{\min\{t_1,t_2\}} \in \mu_F^{(\varepsilon,\delta)}$. This shows that $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in)$-fuzzy ideal of X. □

Theorem 3.14. *Let $\varepsilon,\delta \in [0,1]$ such that $\varepsilon > \max\{\delta, 0.5\}$. If the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in)$-fuzzy ideal of X, then F is an ideal of X.*

Proof. If $0 \notin F$, then $\mu_F^{(\varepsilon,\delta)}(0) = \delta < \varepsilon = \mu_F^{(\varepsilon,\delta)}(x)$ for some $x \in X$. Hence $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon = 2\varepsilon > 1$, and so $x_\varepsilon\, q\, \mu_F^{(\varepsilon,\delta)}$. It follows that $\mu_F^{(\varepsilon,\delta)}(0) \geq \varepsilon$ since $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in)$-fuzzy ideal of X. This is a contradiction, and thus $0 \in F$. Let $x, y \in X$ be such that $x*y \in F$ and $y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x*y)+\varepsilon = \varepsilon+\varepsilon > 1 \quad \text{and} \quad \mu_F^{(\varepsilon,\delta)}(y)+\varepsilon = \varepsilon+\varepsilon > 1,$$

that is, $(x*y)_\varepsilon\, q\, \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in)$-fuzzy ideal of X, it follows that $x_\varepsilon = x_{\min\{\varepsilon,\varepsilon\}} \in \mu_F^{(\varepsilon,\delta)}$ and so that $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon$, that is, $x \in F$. Therefore F is an ideal of X. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.13 and 3.14, then we have the following corollary.

Corollary 3.15. *A non-empty subset F of X is an ideal of X if and only if the characteristic function χ_F of F is a $(q,\in)$-fuzzy ideal of X.*

Theorem 3.16. *Let $\varepsilon,\delta \in [0,1]$ such that $\varepsilon > \delta$. If F is an ideal of X, then the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy ideal of X whenever if any element t in $(0,1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta < t$ and $1-t < \varepsilon$.*

Proof. Obviously $0_t \in \mu_F^{(\varepsilon,\delta)}$ since $0 \in F$. Now, $\mu_F^{(\varepsilon,\delta)}(0)+t = \varepsilon+t > 1$, and so $0_t\, q\, \mu_F^{(\varepsilon,\delta)}$. Thus $0_t \in\wedge q\, \mu_F^{(\varepsilon,\delta)}$. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $(x*y)_{t_1} \in \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2} \in \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x*y) \geq t_1 > \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) \geq t_2 > \delta$, which imply that $x*y \in F$ and $y \in F$ and $\varepsilon \geq \min\{t_1,t_2\}$. Since F is an ideal of X, we have $x \in F$. Hence $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon \geq \min\{t_1,t_2\}$, i.e., $x_{\min\{t_1,t_2\}} \in \mu_F^{(\varepsilon,\delta)}$. Now, $\mu_F^{(\varepsilon,\delta)}(x)+\min\{t_1,t_2\} = \varepsilon+\min\{t_1,t_2\} > 1$ and so $x_{\min\{t_1,t_2\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Therefore $x_{\min\{t_1,t_2\}} \in\wedge q\, \mu_F^{(\varepsilon,\delta)}$, and consequently $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy ideal of X. □

Theorem 3.17. *Let $\varepsilon,\delta \in [0,1]$ such that $\varepsilon > \delta$. If $\varepsilon+\delta \leq 1$ and the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy ideal of X, then F is an ideal of X.*

Proof. Assume that $\varepsilon+\delta \leq 1$ and the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy ideal of X. If $0 \notin F$, then $\mu_F^{(\varepsilon,\delta)}(0) = \delta < \varepsilon = \mu_F^{(\varepsilon,\delta)}(x)$ for some $x \in X$. Thus $x_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$, which implies that $0_\varepsilon \in\wedge q\, \mu_F^{(\varepsilon,\delta)}$ since $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy ideal of X. But $\mu_F^{(\varepsilon,\delta)}(0) < \varepsilon$ implies that $0_\varepsilon \overline{\in} \mu_F^{(\varepsilon,\delta)}$. Also, $\mu_F^{(\varepsilon,\delta)}(0)+\varepsilon = \delta+\varepsilon \leq 1$, i.e., $0_\varepsilon\, \overline{q}\, \mu_F^{(\varepsilon,\delta)}$. Hence $0_\varepsilon \overline{\in\wedge q}\, \mu_F^{(\varepsilon,\delta)}$, a contradiction. Therefore $0 \in F$. Let $x, y \in X$ be such that $x*y \in F$ and $y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $(x*y)_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$. Hence $x_\varepsilon = x_{\min\{\varepsilon,\varepsilon\}} \in\wedge q\, \mu_F^{(\varepsilon,\delta)}$, that is, $x_\varepsilon = x_{\min\{\varepsilon,\varepsilon\}} \in \mu_F^{(\varepsilon,\delta)}$ and $x_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Hence $\mu_F^{(\varepsilon,\delta)}(x) \geq \varepsilon$ and $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon > 1$. If $\mu_F^{(\varepsilon,\delta)}(x) \geq \varepsilon$, then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon$

and thus $x\in F$. If $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon>1$, then $\mu_F^{(\varepsilon,\delta)}(x)>1-\varepsilon\geq\delta$ and so $\mu_F^{(\varepsilon,\delta)}(x)=\varepsilon$, which shows that $x\in F$. Therefore F is an ideal of X. □

If we take $\varepsilon=1$ and $\delta=0$ in Theorems 3.16 and 3.17, then we have the following corollary.

Corollary 3.18. *A non-empty subset F of X is an ideal of X if and only if the characteristic function χ_F of F is an $(\in,\in\wedge q)$-fuzzy ideal of X.*

Theorem 3.19. *Let $\varepsilon,\delta\in[0,1]$ such that $\varepsilon>\delta$. If F is an ideal of X, then the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\wedge q)$-fuzzy ideal of X under the condition that if any element t in $(0,1]$ satisfies $x_t\in\mu_F^{(\varepsilon,\delta)}$ for $x\in X$ then $\delta\leq 1-t$ and $t<\varepsilon$.*

Proof. Let $x\in X$ and $t\in(0,1]$ be such that $x_t\,q\,\mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x)>1-t\geq\delta$, and so $\mu_F^{(\varepsilon,\delta)}(x)=\varepsilon>1-t$. Since $0\in F$, we have $\mu_F^{(\varepsilon,\delta)}(0)=\varepsilon>t$, i.e., $0_t\in\mu_F^{(\varepsilon,\delta)}$ and $\mu_F^{(\varepsilon,\delta)}(0)+t=\varepsilon+t>1-t+t=1$, i.e., $0_t\,q\,\mu_F^{(\varepsilon,\delta)}$. Thus $0_t\in\wedge q\,\mu_F^{(\varepsilon,\delta)}$. Let $x,y\in X$ and $t_1,t_2\in(0,1]$ be such that $(x*y)_{t_1}\,q\,\mu_F^{(\varepsilon,\delta)}$ and $y_{t_2}\,q\,\mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x*y)+t_1>1$ and $\mu_F^{(\varepsilon,\delta)}(y)+t_2>1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x*y)>1-t_1\geq\delta$ and $\mu_F^{(\varepsilon,\delta)}(y)>1-t_2\geq\delta$. Hence $\mu_F^{(\varepsilon,\delta)}(x*y)=\varepsilon=\mu_F^{(\varepsilon,\delta)}(y)$, and so $\varepsilon>\max\{1-t_1,1-t_2\}$. Thus $x*y\in F$ and $y\in F$. Since F is an ideal of X, we have $x\in F$ and thus

$$\mu_F^{(\varepsilon,\delta)}(x)=\varepsilon\geq\min\{t_1,t_2\},$$

that is, $x_{\min\{t_1,t_2\}}\in\mu_F^{(\varepsilon,\delta)}$. Now, $\mu_F^{(\varepsilon,\delta)}(x)+\min\{t_1,t_2\}=\varepsilon+\min\{t_1,t_2\}>1$, and so $x_{\min\{t_1,t_2\}}\,q\,\mu_F^{(\varepsilon,\delta)}$. Hence $x_{\min\{t_1,t_2\}}\in\wedge q\,\mu_F^{(\varepsilon,\delta)}$, and $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\wedge q)$-fuzzy ideal of X. □

Theorem 3.20. *Let $\varepsilon,\delta\in[0,1]$ such that $\varepsilon>\max\{\delta,0.5\}$. If the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\wedge q)$-fuzzy ideal of X, then F is an ideal of X.*

Proof. If $0\notin F$, then $\mu_F^{(\varepsilon,\delta)}(0)=\delta<\varepsilon=\mu_F^{(\varepsilon,\delta)}(x)$ for some $x\in X$. Hence $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon=2\varepsilon>1$, and thus $x_\varepsilon\,q\,\mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\wedge q)$-fuzzy ideal of X, it follows that $0_\varepsilon\in\wedge q\,\mu_F^{(\varepsilon,\delta)}$, i.e., $0_\varepsilon\in\mu_F^{(\varepsilon,\delta)}$ and $0_\varepsilon\,q\,\mu_F^{(\varepsilon,\delta)}$. This is a contradiction. Therefore $0\in F$. Assume that $x*y\in F$ and $y\in F$ for all $x,y\in X$. Then $\mu_F^{(\varepsilon,\delta)}(x*y)=\varepsilon=\mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x*y)+\varepsilon=\varepsilon+\varepsilon>1 \text{ and } \mu_F^{(\varepsilon,\delta)}(y)+\varepsilon=\varepsilon+\varepsilon>1,$$

that is, $(x*y)_\varepsilon\,q\,\mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon\,q\,\mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\wedge q)$-fuzzy ideal of X, it follows that $x_\varepsilon=x_{\min\{\varepsilon,\varepsilon\}}\in\wedge q\,\mu_F^{(\varepsilon,\delta)}$ and so that $\mu_F^{(\varepsilon,\delta)}(x)\geq\varepsilon$. Hence $x\in F$ and F is an ideal of X. □

If we take $\varepsilon=1$ and $\delta=0$ in Theorems 3.19 and 3.20, then we have the following corollary.

Corollary 3.21. *A non-empty subset F of X is an ideal of X if and only if the characteristic function χ_F of F is a $(q,\in\wedge q)$-fuzzy ideal of X.*

Theorem 3.22. *Assume that*

$$(\forall\ x\in X)(\forall\ t\in(0,1])\left(x_t\in\mu_F^{(\varepsilon,\delta)}\ \Rightarrow\ \delta\leq 1-t\right).$$

If F is an ideal of X, then the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\vee q)$-fuzzy ideal of X.

Proof. Let $x \in X$ and $t \in (0,1]$ be such that $x_t \, q \, \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x) > 1-t \geq \delta$, and so $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon > 1-t$. Since $0 \in F$, we have $\mu_F^{(\varepsilon,\delta)}(0)+t = \varepsilon+t > 1-t+t = 1$, that is, $0_t \, q \, \mu_F^{(\varepsilon,\delta)}$. Thus $0_t \in\vee q \, \mu_F^{(\varepsilon,\delta)}$. Let $x,y \in X$ and $t_1,t_2 \in (0,1]$ be such that $(x*y)_{t_1} \, q \, \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2} \, q \, \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x*y)+t_1 > 1$ and $\mu_F^{(\varepsilon,\delta)}(y)+t_2 > 1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x*y) > 1-t_1 \geq \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) > 1-t_2 \geq \delta$. Hence $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $\varepsilon > \max\{1-t_1, 1-t_2\}$. Thus $x*y \in F$ and $y \in F$. Since F is an ideal of X, we have $x \in F$ and thus $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon$ which implies that $\mu_F^{(\varepsilon,\delta)}(x)+\min\{t_1,t_2\} = \varepsilon+\min\{t_1,t_2\} > 1$, i.e., $x_{\min\{t_1,t_2\}} \, q \, \mu_F^{(\varepsilon,\delta)}$. It follows that $x_{\min\{t_1,t_2\}} \in\vee q \, \mu_F^{(\varepsilon,\delta)}$. Therefore $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\vee q)$-fuzzy ideal of X. □

Theorem 3.23. *Let $\varepsilon,\delta \in [0,1]$ such that $\varepsilon > \max\{\delta, 0.5\}$ and $\varepsilon+\delta \leq 1$. If the (ε,δ)-characteristic fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\vee q)$-fuzzy ideal of X, then F is an ideal of X.*

Proof. Assume that $0 \notin F$. Then $\mu_F^{(\varepsilon,\delta)}(0) = \delta < \varepsilon = \mu_F^{(\varepsilon,\delta)}(x)$ for some $x \in X$. Hence $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon = 2\varepsilon > 1$, and thus $x_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\vee q)$-fuzzy ideal of X, we get $0_\varepsilon \in\vee q \, \mu_F^{(\varepsilon,\delta)}$ which implies that $0_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$ or $0_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$. If $0_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$, then $\mu_F^{(\varepsilon,\delta)}(0) \geq \varepsilon$, a contradiction. If $0_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$, then $\delta+\varepsilon = \mu_F^{(\varepsilon,\delta)}(0)+\varepsilon > 1$ which is a contradiction. Therefore $0 \in F$. Suppose that $x*y \in F$ and $y \in F$ for all $x,y \in X$. Then $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x*y)+\varepsilon = \varepsilon+\varepsilon > 1 \quad \text{and} \quad \mu_F^{(\varepsilon,\delta)}(y)+\varepsilon = \varepsilon+\varepsilon > 1,$$

that is, $(x*y)_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q,\in\vee q)$-fuzzy ideal of X, it follows that $x_\varepsilon = x_{\min\{\varepsilon,\varepsilon\}} \in\vee q \, \mu_F^{(\varepsilon,\delta)}$, that is, $\mu_F^{(\varepsilon,\delta)}(x) \geq \varepsilon$ or $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon > 1$. If $\mu_F^{(\varepsilon,\delta)}(x) \geq \varepsilon$, then $x \in F$. If $\mu_F^{(\varepsilon,\delta)}(x)+\varepsilon > 1$, then $\mu_F^{(\varepsilon,\delta)}(x) > 1-\varepsilon \geq \delta$ and so $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon$. Thus $x \in F$, and therefore F is an ideal of X. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.22 and 3.23, then we have the following corollary.

Corollary 3.24. *A non-empty subset F of X is an ideal of X if and only if the characteristic function χ_F of F is a $(q,\in\vee q)$-fuzzy ideal of X.*

Conclusions

We have introduced the notion of (ε,δ)-characteristic fuzzy sets in BCK/BCI-algebras. Given an ideal F of a BCK/BCI-algebra X, we have provided conditions for the (ε,δ)-characteristic fuzzy set in X to be an $(\in,\in\vee q)$-fuzzy ideal, an $(\in,q)$-fuzzy ideal, an $(\in,\in\wedge q)$-fuzzy ideal, a (q,q)-fuzzy ideal, a $(q,\in)$-fuzzy ideal, a $(q,\in\vee q)$-fuzzy ideal and a $(q,\in\wedge q)$-fuzzy ideal. Using the notions of (α,β)-fuzzy ideal $\mu_F^{(\varepsilon,\delta)}$, we have investigated conditions for the F to be an ideal of X where (α,β) is one of $(\in,\in\vee q)$, $(\in,\in\wedge q)$, $(\in,q)$, $(q,\in\vee q)$, $(q,\in\wedge q)$, $(q,\in)$ and (q,q).

Acknowledgments

The authors are thankful to the anonymous referees for their valuable comments and several useful suggestions. The first author was partially supported by the research grant S-0064-1439, Deanship of Scientific Research, University of Tabuk, Tabuk-71491, Saudi Arabia.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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