Journal of Informatics and Mathematical Sciences
Vol. 12, No. 1, pp. 83–98, 2020
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v12i1.1094

http://www.rgnpublications.com

Research Article

Effects of Radiation and Triaxiality of Primaries on Existence and Stability of Collinear Equilibrium Points in Elliptical Restricted Three Body Problem

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Abstract. This paper deals with the motion of infinitesimal mass around primaries whose are radiating and triaxial. They are moving around each other in elliptic orbits about the common barycenter in the neighborhood of collinear equilibrium points. It is observed that location and stability of collinear points are effected by radiation and triaxiality of primaries. The results shows that collinear points L_1 and L_2 are having unstable behavior, while L_3 is showing stable behavior for some values of radiation and triaxiality.

Keywords. ER3BP; Lagrangian points; Triaxiality; Dynamical system; Collinear points

MSC. 37N05; 70F15

Received: September 13, 2018 **Accepted:** August 9, 2019

1. Introduction

The restricted three-body problem has a wider range of application as compared to the general three body problem in space dynamics, celestial mechanics and analytical dynamics. The *elliptical restricted three body problem* (ER3BP) model, the motion of infinitesimal mass which moves under the influence of two massive bodies revolving around their common centre of mass in an elliptical orbit, being influenced but not influencing the two primaries. The circular restricted three body problem has been generalized by the introduction of the elliptic orbit, thus improving its applicability and retaining some useful properties of the circular model suitable to the elliptical case. Various authors [1], [6], [7], [10], [11], [12], [13], [16], [17], [19], [20], [21], [23], [24] and [27] have studied the effects of radiation pressure on the motion of the infinitesimal body by taking one or both primaries as a source of radiation.

The bodies in celestial model of the problem were considered as spherical, but many celestial bodies are either oblate spheroids or triaxial or both, and not spheres. For instance, the mars, Jupiter, Saturn, Neutron stares, Regulus and white dwarfs are oblate spheroids, where as the Moon and Pluto and its moon Charon are triaxial. This oblateness and triaxiality of primaries causes perturbations in the system. That is why many researchers have included these charecterisations in their study of elliptical restricted three-body problem. The study of existence and stability of L_4 and L_5 have been conducted [14] under the assumption that both the primaries are radiating and triaxial. The characteristic exponents of triangular solutions in ER3BP has been analyzed [15]. The study of collinear point have been conducted [4], [9], [18], [22], [25].

The study of motion of infinitesimal around the collinear point is useful for spacecraft mission. These are the suitable to set permanent observatories of the Sun, the magnetosphere of the Earth links with the hidden part of the Moon and others [8]. In this paper, we have derived location of collinear points and their stability around binary system when the primaries are radiating as well as triaxial. The study of the stability of infinitesimal around the collinear points are important as these point can serve as a possible fuel depot for future space probe in the lunar mission.

2. Equation of Motion

The differential equations of motion of the infinitesimal mass in the *elliptical restricted three body problem* (ER3BP) under radiating and triaxial primaries in a barycentric, pulsating system are given below Narayan *et al*. [14]. The differential equation of motion of the third body in non-dimensional barycentre, pulsating and non-uniformly rotating coordinate system (x, y) is written in the form:

$$x'' - 2y' = \frac{1}{1+e\cos v}\left(\frac{\partial\Omega}{\partial x}\right); \quad y'' + 2x' = \frac{1}{1+e\cos v}\left(\frac{\partial\Omega}{\partial y}\right), \tag{1}$$

where $'$ denotes differentiation with respect to v, and

$$\Omega = \frac{x^2+y^2}{2} + \frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^3} + \frac{\mu(2\sigma_1'-\sigma_2')q_2}{2r_2^3} - \frac{3(1-\mu)(\sigma_1-\sigma_2)y^2q_1}{2r_1^5} - \frac{3\mu(\sigma_1'-\sigma_2')y^2q_2}{2r_2^5}\right] \tag{2}$$

where

$$n^2 = 1 + \frac{3}{2}(2\sigma_1-\sigma_2) + \frac{3}{2}(2\sigma_1'-\sigma_2'); \; r_1^2 = (x+\mu)^2+y^2; \; r_2^2 = (x+\mu-1)^2+y^2; \; \mu = \frac{m_2}{m_1+m_2}, \tag{3}$$

where m_1 and m_2 are masses of the primaries. q_1, q_2 are the radiation pressure parameters.

σ_1, σ_2, σ_1' and σ_2' are triaxial parameters, while e and v are the eccentricity of orbits and true anomaly one of the primaries, respectively.

Then equation (1) can be written as:

$$x'' - 2y' = \frac{1}{1+e\cos v}\left\{x\left[\left(1-k+\frac{3(1-\mu)(\sigma_1-\sigma_2)q_1}{n^2r_1^5}+\frac{3\mu(\sigma_1'-\sigma_2')q_2}{n^2r_2^5}\right)\right]\right.$$
$$-\frac{\mu(1-\mu)}{n^2}\left[\frac{q_1}{r_1^3}-\frac{q_2}{r_2^3}+\frac{3(2\sigma_1-\sigma_2)q_1}{2r_1^5}-\frac{3(2\sigma_1'-\sigma_2')q_2}{2r_2^5}-\frac{15(\sigma_1-\sigma_2)y^2q_1}{2r_1^7}\right.$$
$$\left.\left.+\frac{15(\sigma_1'-\sigma_2')y^2q_2}{2r_2^7}\right]\right\} \tag{4}$$

and

$$y'' + 2x' = \frac{1}{1+e\cos v}(1-k), \tag{5}$$

where

$$k = \frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1^3}+\frac{\mu q_2}{r_2^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5}+\frac{3\mu(2\sigma_1'-\sigma_2')q_2}{2r_2^5}\right.$$
$$\left.-\frac{15(1-\mu)(\sigma_1-\sigma_2)y^2q_1}{2r_1^7}-\frac{15\mu(\sigma_1'-\sigma_2')y^2q_2}{2r_2^7}\right]. \tag{6}$$

3. Location of Collinear Equilibrium Points

The collinear equilibrium points of the system are the saddle points. The points where the resources consumed minimally is referred as equilibrium points of the system. So they are represented as follows:

$$\left.\begin{aligned}\frac{\partial\Omega}{\partial x} &= 0,\\ \frac{\partial\Omega}{\partial y} &= 0,\end{aligned}\right\} \tag{7}$$

where Ω is given by the equation (2), but the collinear points lies on x-axis; hence are given by the conditions:

$$\frac{\partial\Omega}{\partial x} = 0, \quad \frac{\partial\Omega}{\partial y} = 0, \quad y = 0. \tag{8}$$

Therefore using the above condition we get:

$$f(x) = \left\{x - \frac{1}{n^2}\left[\frac{(1-\mu)(x+\mu)q_1}{r_1^3}+\frac{\mu(x-1+\mu)q_2}{r_2^3}+\frac{3(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5}\right.\right.$$
$$\left.\left.+\frac{3\mu(x-1+\mu)(2\sigma_1'-\sigma_2')q_2}{2r_2^5}\right]\right\} = 0. \tag{9}$$

There are three collinear equilibrium points. These are denoted by L_1, L_2 and L_3. The L_1 lies between bigger and smaller primary ($-\mu < x < 1-\mu$); L_2, lying to the right of smaller primary ($x > 1-\mu$) and L_3, lying to the left of the bigger primary ($x < -\mu$).

3.1 Location of L_1

In order to find the solution for L_1, substituting $x = x_2 - \rho = 1 - \mu - \rho$, such that $r_2 = \rho$ and $r_1 = 1 - \rho$ in equation (9), we have:

$$1-\mu-\rho-\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{(1-\rho)^2}-\frac{\mu q_2}{\rho^2}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2(1-\rho)^4}-\frac{3\mu(2\sigma_1'-\sigma_2')q_2}{2\rho^4}\right\}=0, \tag{10}$$

where $q_1 = 1-\varepsilon_1'$, $q_2 = 1-\varepsilon_2'$.

Now, rearranging the terms in equation (10), after simplification, we get:

$$\frac{n^2(1-\rho)^5-(1-\rho)^2q_1-\frac{3}{2}(2\sigma_1-\sigma_2)q_1}{(1-\rho)^4}=-\frac{\mu}{1-\mu}\left[\frac{q_2+\frac{\frac{3}{2}(2\sigma_1'-\sigma_2')q_2}{\rho}^2-n^2\rho^3}{\rho^2}\right] \tag{11}$$

on simplifying, we get:

$$\begin{aligned}
&\rho^3\left[1-\frac{\left\{-3-5(2\sigma_1-\sigma_2)-8(2\sigma_1'-\sigma_2')-\frac{5}{3}\varepsilon_1'\right\}\rho}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}+\frac{\left\{-\frac{10}{3}+5(2\sigma_1-\sigma_2)+3(2\sigma_1'-\sigma_2')-\frac{4}{3}\varepsilon_1'\right\}\rho^2}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}\right]\\
&\quad=\frac{\mu}{3(1-\mu)}\frac{\left\{1+15(2\sigma_1'-\sigma_2')+\varepsilon_2'\right\}}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}\\
&\qquad\times(1-\rho)^4\left[1-30(2\sigma_1'-\sigma_2')\rho+\frac{45(2\sigma_1'-\sigma_2')\rho^2}{2}-\left\{n^2+6(2\sigma_1'-\sigma_2')\right\}\rho^3\right].
\end{aligned} \tag{12}$$

Now, let

$$\left[\frac{\mu}{3(1-\mu)}\frac{\left\{1+15(2\sigma_1'-\sigma_2')+\varepsilon_2'\right\}}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}\right]^{1/3}=\lambda \tag{13}$$

then, we have

$$\begin{aligned}
&\rho^3\left[1-\frac{\left\{-3-5(2\sigma_1-\sigma_2)-8(2\sigma_1'-\sigma_2')-\frac{5}{3}\varepsilon_1'\right\}\rho}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}+\frac{\left\{-\frac{10}{3}+5(2\sigma_1-\sigma_2)+3(2\sigma_1'-\sigma_2')-\frac{4}{3}\varepsilon_1'\right\}\rho^2}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}\right]\\
&\quad=\lambda^3(1-\rho)^4\left[1-30(2\sigma_1'-\sigma_2')\rho+\frac{45(2\sigma_1'-\sigma_2')\rho^2}{2}-\left\{n^2+6(2\sigma_1'-\sigma_2')\right\}\rho^3\right].
\end{aligned} \tag{14}$$

Using the series expansion given as follows:

$$\rho=\lambda\left(1+c_1\lambda+c_2\lambda^2+\dots\right). \tag{15}$$

Using the series as in equation (15) into the equation (14), the value of ρ is given as follows:

$$\begin{aligned}
\rho=\lambda&\left[1-\frac{1}{3}\frac{\left\{1+15(2\sigma_1-\sigma_2)+46(2\sigma_1'-\sigma_2')+\frac{\varepsilon_1'}{3}\right\}}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}\lambda\right.\\
&\left.-\frac{1}{9}\frac{\left\{1+270(2\sigma_1-\sigma_2)+\frac{2031}{2}(2\sigma_1'-\sigma_2')-\frac{44\varepsilon_1'}{3}\right\}}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}^2}\lambda^2+\dots\right].
\end{aligned} \tag{16}$$

Hence, the solution for collinear point L_1 is given by

$$x = 1-\mu-\lambda\left[1-\frac{1}{3}\frac{\left\{1+15(2\sigma_1-\sigma_2)+46(2\sigma_1'-\sigma_2')+\frac{\varepsilon_1'}{3}\right\}}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}}\lambda\right.$$
$$\left.-\frac{1}{9}\frac{\left\{1+270(2\sigma_1-\sigma_2)+\frac{2031}{2}(2\sigma_1'-\sigma_2')-\frac{44\varepsilon_1'}{3}\right\}}{\left\{1+\frac{5}{2}(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{\varepsilon_1'}{3}\right\}^2}\lambda^2+\dots\right]. \tag{17}$$

3.2 Location of L_2

To find the location of L_2, putting $x = x_2+\rho$ such that $r_2=\rho$ and $r_1 = 1+\rho$ in equation (9), we get:

$$1-\mu+\rho-\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{(1+\mu)^2}+\frac{\mu q_2}{\rho^2}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2(1+\rho)^4}+\frac{3\mu(2\sigma_1'-\sigma_2')q_2}{2\rho^4}\right\}=0. \tag{18}$$

Rearranging the terms, we get:

$$\frac{n^2(1+\rho)^5-(1+\rho)^2q_1-\frac{3}{2}(2\sigma_1-\sigma_2)q_1}{(1+\rho)^4}=\frac{\mu}{1-\mu}\left[\frac{q_2+\frac{\frac{3}{2}(2\sigma_1'-\sigma_2')q_2}{\rho^2}-n^2\rho^3}{\rho^2}\right]. \tag{19}$$

Simplifying the above equation we get:

$$\rho^3\left[1+\frac{\left\{3+5(2\sigma_1-\sigma_2)+2(2\sigma_1'-\sigma_2')-\frac{5}{3}\varepsilon_1'\right\}\rho}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}}+\frac{\left\{\frac{10}{3}+5(2\sigma_1-\sigma_2)+7(2\sigma_1'-\sigma_2')+\frac{4}{3}\varepsilon_1'\right\}\rho^2}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}}\right]$$
$$=\frac{\mu}{3(1-\mu)}\frac{\left\{1+15(2\sigma_1'-\sigma_2')-\varepsilon_2'\right\}}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}}$$
$$\times(1+\rho)^4\left[1-30(2\sigma_1'-\sigma_2')\rho+\frac{45(2\sigma_1'-\sigma_2')\rho^2}{2}-\left\{n^2+6(2\sigma_1'-\sigma_2')\right\}\rho^3\right] \tag{20}$$

Let

$$\left[\frac{\mu}{3(1-\mu)}\frac{\left\{1+15(2\sigma_1'-\sigma_2')-\varepsilon_2'\right\}}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}}\right]^{1/3}=\lambda \tag{21}$$

Using series given in equation (15), the ρ can be given as follows:

$$\rho=\lambda\left[1-\frac{1}{3}\frac{\left\{1+7\,(2\sigma_1-\sigma_2)-22(2\sigma_1'-\sigma_2')+\frac{17\varepsilon_1'}{3}\right\}}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}}\lambda\right.$$
$$\left.-\frac{1}{9}\frac{\left\{1+7(2\sigma_1-\sigma_2)+41(2\sigma_1'-\sigma_2')-\frac{179\varepsilon_1'}{3}\right\}}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}^2}\lambda^2+\dots\right]. \tag{22}$$

Hence the solution for L_2 is given by:

$$x=1-\mu+\lambda\left[1-\frac{1}{3}\frac{\left\{1+7\,(2\sigma_1-\sigma_2)-22(2\sigma_1'-\sigma_2')+\frac{17\varepsilon_1'}{3}\right\}}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}}\lambda\right.$$

$$-\frac{1}{9}\frac{\left\{1+7(2\sigma_1-\sigma_2)+41(2\sigma_1'-\sigma_2')-\frac{179\varepsilon_1'}{3}\right\}}{\left\{1+3(2\sigma_1-\sigma_2)+\frac{5}{2}(2\sigma_1'-\sigma_2')+\varepsilon_1'\right\}^2}\lambda^2+\ldots\Bigg]. \tag{23}$$

3.3 Location of L_3

In order to find the solution for L_3, substituting $x=x_1-\rho$ such that $r_1=\rho$ and $r_2=1+\rho$ in equation (9), we get

$$\frac{\mu}{1-\mu}=\frac{\left[n^2\rho^3-q_1-\frac{3(2\sigma_1-\sigma_2)q_1}{2\rho^2}\right](1+\rho)^2}{\rho^2\left[q_2+\frac{3(2\sigma_1'-\sigma_2')q_2}{2(1+\rho)^2}-n^2(1+\rho)^3\right]} \tag{24}$$

Let, $\rho=1+\alpha$ and using the elementary algorithm for division upto $[\alpha^4]$, so the equation (24) can be written as follows:

$$\begin{aligned}\frac{\mu}{1-\mu}=&\Bigg[\left\{-\frac{4\varepsilon_1'}{7}+\frac{3}{14}(2\sigma_1-\sigma_2)-\frac{6}{7}(2\sigma_1'-\sigma_2')\right\}\\&+\left(-\frac{12\alpha}{7}\right)\left\{1-\frac{26\varepsilon_1'}{21}+\frac{145}{56}(2\sigma_1-\sigma_2)+\frac{1}{7}(2\sigma_1'-\sigma_2')\right\}\\&+\left(-\frac{12\alpha}{7}\right)^2\left\{1-\frac{139\varepsilon_1'}{84}+\frac{\varepsilon_2'}{6}+\frac{1801}{336}(2\sigma_1-\sigma_2)+\frac{3379}{672}(2\sigma_1'-\sigma_2')\right\}\\&+\left(-\frac{12\alpha}{7}\right)^3\left\{\frac{1567}{1728}-\frac{7316876\,\varepsilon_1'}{4148928}+\frac{23\varepsilon_2'}{48}+\frac{31547082}{4148928}(2\sigma_1-\sigma_2)\right.\\&\left.+\frac{4512629079}{406594944}(2\sigma_1'-\sigma_2')\right\}+\Omega\left(\alpha^4\right)+\ldots\Bigg].\end{aligned} \tag{25}$$

Now, by using Lagrange inversion formula and successive approximation [3], and retaining only linear terms in ε_1', ε_2', σ_1, σ_2, σ_1' and σ_2' we have:

$$\begin{aligned}\rho=&\Bigg[1-\left\{-\frac{\varepsilon_1'}{3}+\frac{1}{8}(2\sigma_1-\sigma_2)-\frac{1}{2}(2\sigma_1'-\sigma_2')\right\}\\&-\frac{7}{12}\left\{1-\frac{1471\varepsilon_1'}{1008}-\frac{1577752976}{1000000000}(2\sigma_1-\sigma_2)-\frac{2815}{672}(2\sigma_1'-\sigma_2')\right\}\left(\frac{\mu}{1-\mu}\right)\\&-\frac{7}{12}\left\{\frac{1313\varepsilon_1'}{1008}-\frac{1313}{2688}(2\sigma_1-\sigma_2)+\frac{2465}{672}(2\sigma_1'-\sigma_2')\right\}\left(\frac{\mu}{1-\mu}\right)^2\\&-\frac{7}{12}\left\{\frac{2910383598\varepsilon_1'}{1000000000}-\frac{8801}{8064}(2\sigma_1-\sigma_2)+\frac{8801}{2016}(2\sigma_1'-\sigma_2')\right\}\left(\frac{\mu}{1-\mu}\right)^3+\Omega\left(\frac{\mu}{1-\mu}\right)^4+\ldots\Bigg]\end{aligned} \tag{26}$$

Hence solution for collinear point L_3 is as follows:

$$\begin{aligned}x=-\mu-&\Bigg[1-\left\{-\frac{\varepsilon_1'}{3}+\frac{1}{8}(2\sigma_1-\sigma_2)-\frac{1}{2}(2\sigma_1'-\sigma_2')\right\}\\&-\frac{7}{12}\left\{1-\frac{1471\varepsilon_1'}{1008}-\frac{1577752976}{1000000000}(2\sigma_1-\sigma_2)-\frac{2815}{672}(2\sigma_1'-\sigma_2')\right\}\left(\frac{\mu}{1-\mu}\right)\\&-\frac{7}{12}\left\{\frac{1313\varepsilon_1'}{1008}-\frac{1313}{2688}(2\sigma_1-\sigma_2)+\frac{2465}{672}(2\sigma_1'-\sigma_2')\right\}\left(\frac{\mu}{1-\mu}\right)^2\end{aligned}$$

$$-\frac{7}{12}\left\{\frac{2910383598\varepsilon_1'}{1000000000}-\frac{8801}{8064}(2\sigma_1-\sigma_2)+\frac{8801}{2016}(2\sigma_1'-\sigma_2')\right\}\left(\frac{\mu}{1-\mu}\right)^3+\Omega\left(\frac{\mu}{1-\mu}\right)^4\Bigg]. \quad (27)$$

4. Linear Stability of Collinear Points

The stability of motion of the infinitesimal mass near the collinear equilibrium points is analyzed using the following lemma [26], defines the stability of motion of the collinear points. some modification have been incorporating in lemma for adapting to the present problem.

Lemma. *It states that, at collinear points:*

$$k=\frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1^3}+\frac{\mu q_2}{r_2^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5}+\frac{3\mu(2\sigma_1'-\sigma_2')q_2}{2r_2^5}\right.$$
$$\left.-\frac{15(1-\mu)(\sigma_1-\sigma_2)y^2q_1}{2r_1^7}-\frac{15\mu(\sigma_1'-\sigma_2')y^2q_2}{2r_2^7}\right]>1. \quad (28)$$

Proof. For an equilibrium point we have the condition [14]:

$$x-\frac{1}{n^2}\left[\frac{(1-\mu)(x+\mu)q_1}{r_1^3}+\frac{\mu q_2}{r_2^3}+\frac{3(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5}\right.$$
$$\left.+\frac{3\mu(x-1+\mu)(2\sigma_1'-\sigma_2')q_2}{2r_2^5}-\frac{15(1-\mu)(x+\mu)(\sigma_1-\sigma_2)y^2q_1}{2r_1^7}-\frac{15\mu(x-1+\mu)(\sigma_1'-\sigma_2')y^2q_2}{2r_2^5}\right]=0. \quad (29)$$

The collinear points lie on x-axis, hence $y=0$. Therefore, equation (29) become

$$x-\frac{1}{n^2}\left[\frac{(1-\mu)(x+\mu)q_1}{r_1^3}+\frac{\mu(x-1+\mu)q_2}{r_2^3}+\frac{3(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5}+\frac{3\mu(x-1+\mu)(2\sigma_1'-\sigma_2')q_2}{2r_2^5}\right]=0. \quad (30)$$

Now, rearranging the terms, the above equation (30), which can be written as follows:

$$\frac{1}{n^2}\left\{\frac{(1-\mu)(x+\mu)(r_1-r_1^{-2}q_1)}{r_1}+\frac{\mu(x+\mu-1)(r_2-r_2^{-2}q_2)}{r_2}+\frac{3(1-\mu)(x+\mu)(2\sigma_1-\sigma_2)(r_1-r_1^{-4}q_1)}{2r_1}\right.$$
$$\left.+\frac{3\mu(x+\mu-1)(2\sigma_1'-\sigma_2')(r_2-r_2^{-2}q_2)}{2r_2}\right\}=0. \quad (31)$$

Now, to prove equation (28), we analyze each collinear equilibrium point separately.

4.1 Stability at Collinear Point L_1

At collinear point L_1, $r_1+r_2=1$, where $r_1=x+\mu$ and $r_2=1-x-\mu$. using these values in equation (31) and simplifying using equation (6), we get:

$$\frac{1}{n^2}\left[\left\{1-k+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}+\frac{3\mu(2\sigma_1'-\sigma_2')}{2}\right\}r_1-\mu\left\{1-\frac{\mu q_2}{r_2^3}+\frac{3(2\sigma_1'-\sigma_2')}{2}-\frac{3(2\sigma_1'-\sigma_2')q_2}{2r_2^5}\right\}\right]=0.$$

Since $r_2<1$ and $\frac{1}{n^2}\neq 0$. Hence we have:

$$k=1+\left[\frac{\mu}{r_1}\left\{\frac{q_2}{r_2^3}-\frac{3(2\sigma_1'-\sigma_2')}{2}+\frac{3(2\sigma_1'-\sigma_2')q_2}{2r_2^5}-1\right\}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}+\frac{3\mu(2\sigma_1'-\sigma_2')}{2}\right]. \quad (32)$$

If ε'_1, ε'_2, σ_1, σ_2, σ'_1 and σ'_2 are negligible, then $k > 1$ for L_1,causing failure of stability condition.

4.2 Stability at Collinear Point L_2

For collinear point L_2, $r_1 - r_2 = 1$, where $r_1 = x + \mu$ and $r_2 = x + \mu - 1$. substituting these values in equation (31) and simplifying using equation (6), we get:

$$\frac{1}{n^2}\left[\left\{1-k+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}+\frac{3\mu(2\sigma'_1-\sigma'_2)}{2}\right\}r_1-\mu\left\{1-\frac{q_2}{r_2^3}+\frac{3(2\sigma'_1-\sigma'_2)}{2}-\frac{3(2\sigma'_1-\sigma'_2)q_2}{2r_2^5}\right\}\right]=0.$$

Since $r_2 < 1$ and $\frac{1}{n^2}$?0. Hence we have:

$$k=1+\left[\frac{\mu}{r_1}\left\{\frac{q_2}{r_2^3}-\frac{3(2\sigma'_1-\sigma'_2)}{2}+\frac{3(2\sigma'_1-\sigma'_2)q_2}{2r_2^5}-1\right\}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}+\frac{3\mu(2\sigma'_1-\sigma'_2)}{2}\right]. \tag{33}$$

If ε'_1, ε'_2, σ_1, σ_2, σ'_1 and σ'_2 are negligible, then $k > 1$ for L_2.

4.3 Stability at Collinear Point L_3

Similarly for collinear point L_3, $r_2 - r_1 = 1$, where $r_1 = -x - \mu$ and $r_2 = -x - \mu - 1$. Inserting these values in equation (31) and simplifying using equation (6), we have:

$$\frac{1}{n^2}\left[\left\{k-1-\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}-3\mu(2\sigma'_1-\sigma'_2)q_2\right\}r_1-\mu\left\{1-\frac{q_2}{r_2^3}+\frac{3(2\sigma'_1-\sigma'_2)}{2}-\frac{3(2\sigma'_1-\sigma'_2)q_2}{2r_2^5}\right\}\right]=0.$$

Since $r_2 < 1$ and $\frac{1}{n^2} \neq 0$. Hence we have:

$$k=1+\left[\frac{\mu}{r_1}\left\{1-\frac{q_2}{r_2^3}+\frac{3(2\sigma'_1-\sigma'_2)}{2}-\frac{3(2\sigma'_1-\sigma'_2)q_2}{2r_2^5}-1\right\}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2}+\frac{3\mu(2\sigma'_1-\sigma'_2)q_2}{2}\right]. \tag{34}$$

If ε'_1, ε'_2, σ_1, σ_2, σ'_1 and σ'_2 are negligibly small, then $k > 1$ for L_3.

Hence for all collinear points, we have $k > 1$. This completes the proof of lemma. □

Now in order to analyse the stability motion of primaries near collinear points, investigating the roots of the characteristic equations. For this assuming, that particle gets a small displacement from the equilibrium position. Then finding the variational equations of motion by inserting the coordinates of displaced point in the equation of motion (1) and expanding by Taylor's series about the collinear points and taking only the linear terms, we have the following equation [14] as:

$$\xi''-2\eta'=\varnothing\left[\xi\Omega^0_{xx}+\eta\Omega^0_{xy}\right],\quad \eta''+2\xi'=\varnothing\left[\xi\Omega^0_{yx}+\eta\Omega^0_{yy}\right], \tag{35}$$

where $\varnothing = \left[\frac{1}{1+e\cos v}\right]$ and (x_0, y_0) are the coordinates of the collinear points, respectively. The subscript of Ω denotes second order partial derivative of Ω with respect to x, y. Because all collinear points lies on x-axis, hence $y = 0$. Hence the values of Ω_{xx}, Ω_{yy} and Ω_{xy} are written as:

$$\Omega_{xx}=1-\frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1^3}-\frac{3(1-\mu)q_1}{r_1^3}+\frac{\mu q_2}{r_2^3}-\frac{3\mu q_2}{r_2^3}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5}\right.$$
$$\left.-\frac{15(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5}+\frac{3\mu(2\sigma'_1-\sigma'_2)q_2}{2r_2^5}-\frac{15\mu(2\sigma'_1-\sigma'_2)q_2}{2r_2^5}\right], \tag{36}$$

$$\Omega_{yy} = 1 - \frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu q_2}{r_2^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5} + \frac{3\mu(2\sigma_1'-\sigma_2')q_2}{2r_2^5} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{r_1^5} + \frac{3\mu(2\sigma_1'-\sigma_2')q_2}{r_2^5}\right] \tag{37}$$

and

$$\Omega_{xy} = 0. \tag{38}$$

Now, in order to investigate the stability of the motion of collinear points, new variables given by the equation below are introduced as:

$$x_1 = \xi,\ x_2 = \eta,\ x_3 = \frac{d\xi}{dv},\ x_4 = \frac{d\eta}{dv}.$$

Substituting these values in equation (35), the system of equations takes the form:

$$\frac{dx_i}{dv} = P_{i1}x_1 + P_{i2}x_2 + P_{i3}x_3 + P_{i4}x_4; \quad i = 1,2,3,4, \tag{39}$$

where $P_{11} = P_{12} = P_{13} = P_{14} = P_{22} = P_{23} = P_{33} = P_{44} = 0$
$P_{13} = 1$, $P_{24} = 1$, $P_{34} = 2$, $P_{43} = -2$.

Then, we get:

$$P_{31} = \frac{1}{(1+e\cos v)}\Omega_{xx}^0 = \phi\Omega_{xx}^0,$$

$$P_{42} = \frac{1}{(1+e\cos v)}\Omega_{yy}^0 = \phi\Omega_{yy}^0.$$

The coefficient in the system of equation (39), are the periodic functions of $'v'$ with period 2π. Taking the average over the system, we get:

$$\frac{dx_i^{(0)}}{dv} = P_{i1}^{(0)}x_1^{(0)} + P_{i2}^{(0)}x_2^{(0)} + P_{i3}^{(0)}x_3^{(0)} + P_{i4}^{(0)}x_4^{(0)}, \tag{40}$$

where $P_{is}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} P_{is}(v)dv$, $i,s = 1,2,3,4$.

Hence, we get

$$P_{31}^0 = \frac{1}{\sqrt{1-e^2}}\Omega_{xx}^0,$$

$$P_{42}^0 = \frac{1}{\sqrt{1-e^2}}\Omega_{yy}^0,$$

where subscript '0', wherever appears indicates the value of the corresponding collinear points L_1, L_2, L_3.

The characteristic equation for the system is:

$$\lambda^4 + Q\lambda^2 + R = 0, \tag{41}$$

where

$$Q = -(4 - P_{31}^{(0)} + P_{42}^{(0)}); \quad R = P_{31}^{(0)}P_{42}^{(0)}. \tag{42}$$

The motion of the infinitesimal particle will be stable near the collinear points, when given a small displacement and small velocity, the particle oscillates for a considerable time about the points. The system will be stable if the roots of characteristic equation are purely imaginary [9].

Therefore the condition for stable roots can be given as:

$$Q<0;\ R>0$$

Taking the second inequality, the condition of stability can be written as:

$$\begin{aligned}&\frac{-1}{2}-\frac{3(2\sigma_1-\sigma_2)}{4}-\frac{3(2\sigma_1'-\sigma_2')}{4}-\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5}-\frac{3\mu(2\sigma_1'-\sigma_2')}{2r_2^5}\\&\quad<k<1+\frac{3(2\sigma_1-\sigma_2)}{2}+\frac{3(2\sigma_1'-\sigma_2')}{2}-\frac{3(1-\mu)(\sigma_1-\sigma_2)}{r_1^5}-\frac{3\mu(\sigma_1'-\sigma_2')}{r_2^5}.\end{aligned}$$

If ε_1', ε_2', σ_1, σ_2, σ_1' and σ_2' are negligible, then

$$\frac{-1}{2}<k<1. \tag{43}$$

But from the above discussion, it can be empirically stated that for all the collinear equilibrium points, we have $k>1$. hence the collinear points, L_1, L_2, L_3 are unstable using the condition given in equation (43).

Now, the roots of the characteristic equation (41) are represented by as follows:

$$\lambda_{1,2}^2=\frac{\left(\begin{aligned}&\left[k-2(1-e^2)+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu(2\sigma_1{}'-\sigma_2{}')}{r_2^5}\right]\\&\pm\left[9k^2-8k+\frac{18(1-\mu)k(2\sigma_1-\sigma_2)}{r_1^5}+\frac{18\mu k(2\sigma_1'-\sigma_2')}{r_2^5}-\frac{24(1-\mu)(2\sigma_1-\sigma_2)}{r_1^5}-\frac{24\mu(2\sigma_1{}'-\sigma_2{}')}{r_2^5}\right.\\&\left.+8e^2\left\{1+\frac{k}{2}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{3\mu(2\sigma_1{}'-\sigma_2{}')}{2r_2^5}\right\}\right]^{1/2}\end{aligned}\right)}{2\sqrt{1-e^2}} \tag{44}$$

If $\lambda_i^2=\sigma_i$, $i=1,2$. Then, the equation (44) can be represented as:

$$\sigma_1=\frac{\left(\begin{aligned}&\left[k-2(1-e^2)+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu(2\sigma_1'-\sigma_2')}{r_2^5}\right]\\&+\left[9k^2-8k+\frac{18(1-\mu)k(2\sigma_1-\sigma_2)}{r_1^5}+\frac{18\mu k(2\sigma_1'-\sigma_2')}{r_2^5}-\frac{24(1-\mu)(2\sigma_1-\sigma_2)}{r_1^5}-\frac{24\mu(2\sigma_1'-\sigma_2')}{r_2^5}\right.\\&\left.+8e^2\left\{1+\frac{k}{2}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{3\mu(2\sigma_1'-\sigma_2')}{2\ r_2^5}\right\}\right]^{1/2}\end{aligned}\right)}{2\sqrt{1-e^2}} \tag{45}$$

and

$$\sigma_2=\frac{\left(\begin{aligned}&\left[k-2(1-e^2)+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{r_1^5}+\frac{3\mu(2\sigma_1'-\sigma_2')}{r_2^5}\right]\\&-\left[9k^2-8k+\frac{18(1-\mu)k(2\sigma_1-\sigma_2)}{r_1^5}+\frac{18\mu k(2\sigma_1'-\sigma_2')}{r_2^5}-\frac{24(1-\mu)(2\sigma_1-\sigma_2)}{r_1^5}-\frac{24\mu(2\sigma_1'-\sigma_2')}{r_2^5}\right.\\&\left.+8e^2\left\{1+\frac{k}{2}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{3\mu(2\sigma_1'-\sigma_2')}{2r_2^5}\right\}\right]^{1/2}\end{aligned}\right)}{2\sqrt{1-e^2}}. \tag{46}$$

Since, for collinear points $k>1$. Hence for $\sigma_1,\sigma_2<1$, $e<1$, we get:

$$\left[9k^2-8k+\frac{18(1-\mu)k(2\sigma_1-\sigma_2)}{r_1^5}+\frac{18\mu k(2\sigma_1'-\sigma_2')}{r_2^5}-\frac{24(1-\mu)(2\sigma_1-\sigma_2)}{r_1^5}-\frac{24\mu(2\sigma_1'-\sigma_2')}{r_2^5}\right.$$

$$+8e^2\left\{1+\frac{k}{2}+\frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5}+\frac{3\mu(2\sigma_1'-\sigma_2')}{2r_2^5}\right\}\Bigg] > 1. \tag{47}$$

As $\lambda^2 = s$, from equation (45), $\sigma_1 > 0$ and it gives two real roots of opposite signs, similarly from equation (46), $\sigma_2 < 0$ and it provides two imaginary roots. Therefore, the solution for equation (44) can be given as:

$$\lambda_i = C_{i1}\epsilon^{p_1\nu} + C_{i2}\epsilon^{p_2\nu} + C_i\cos(p_3\nu - C_{i4}), \quad i = 1,2, \tag{48}$$

where p_1, p_2, p_3 are the roots of equation (41). The first and second term of equation (48) show the exponential growth in the value of the roots of λ_i and dominates the third term. Hence from equation (45) and (46), it is clear that the motion is unstable near collinear points.

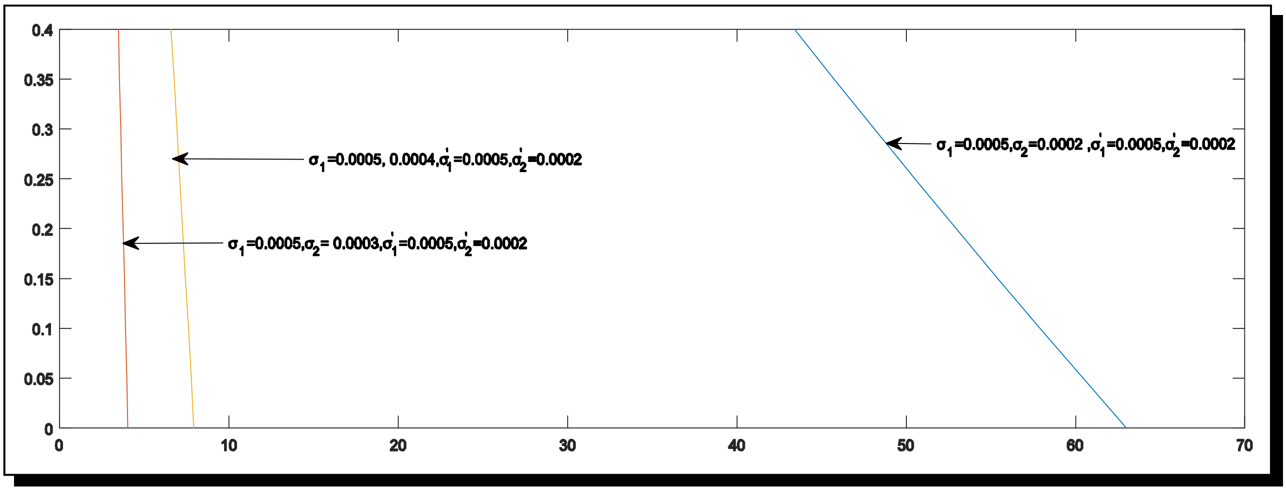

Figure 1. Correlation of characteristic root λ_1 and ε_1' for L_1 ($\sigma_1 = 0.0005$, $\sigma_2 = 0.0002$ to 0.0004, $\sigma_1' = 0.0005$, $\sigma_2' = 0.0002$, $\varepsilon_2' = 0.002$)

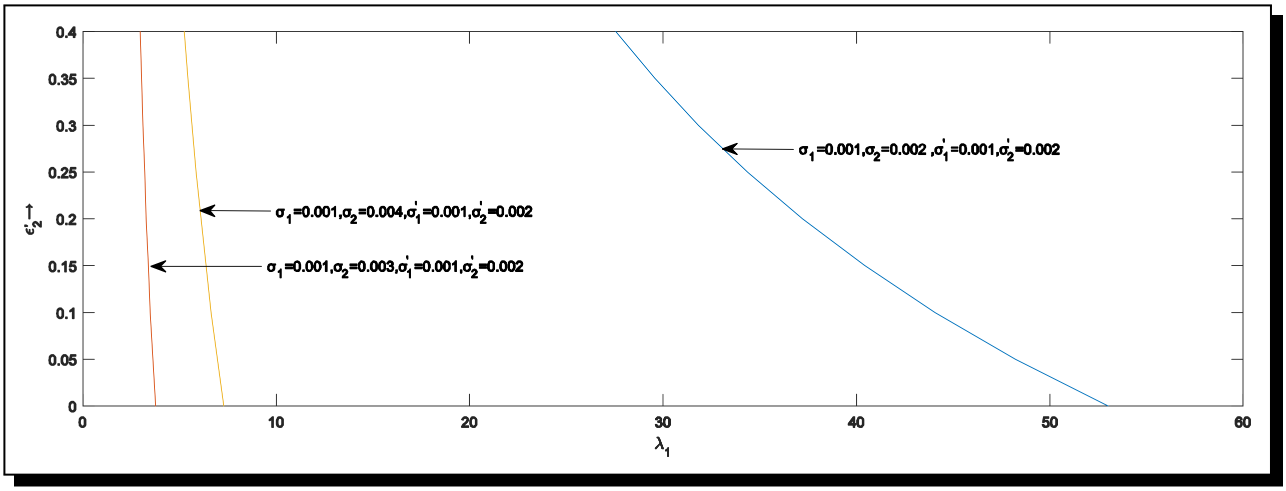

Figure 2. Correlation of characteristic root λ_1 and ε_2' for L_1 ($\sigma_1 = 0.001$, $\sigma_2 = 0.002$ to 0.004, $\sigma_1' = 0.001$, $\sigma_2' = 0.002$, $\varepsilon_1' = 0.002$)

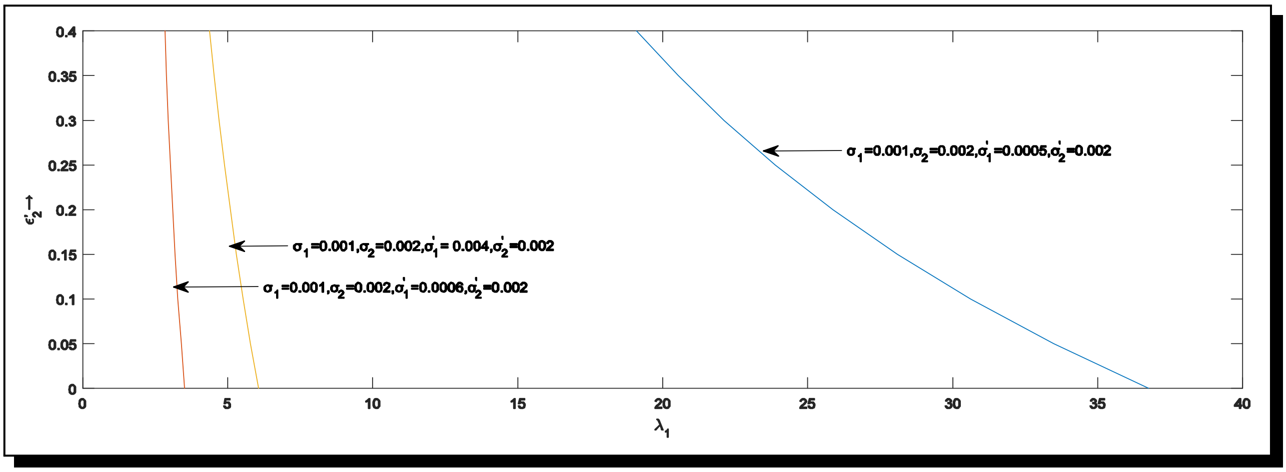

Figure 3. Correlation of characteristic root $\lambda_1$and ε_2' for L_1 ($\sigma_1 = 0.001$, $\sigma_2 = 0.002$, $\sigma_1' = 0.0005$ to 0.004, $\sigma_2' = 0.002$, $\varepsilon_2' = 0.002$)

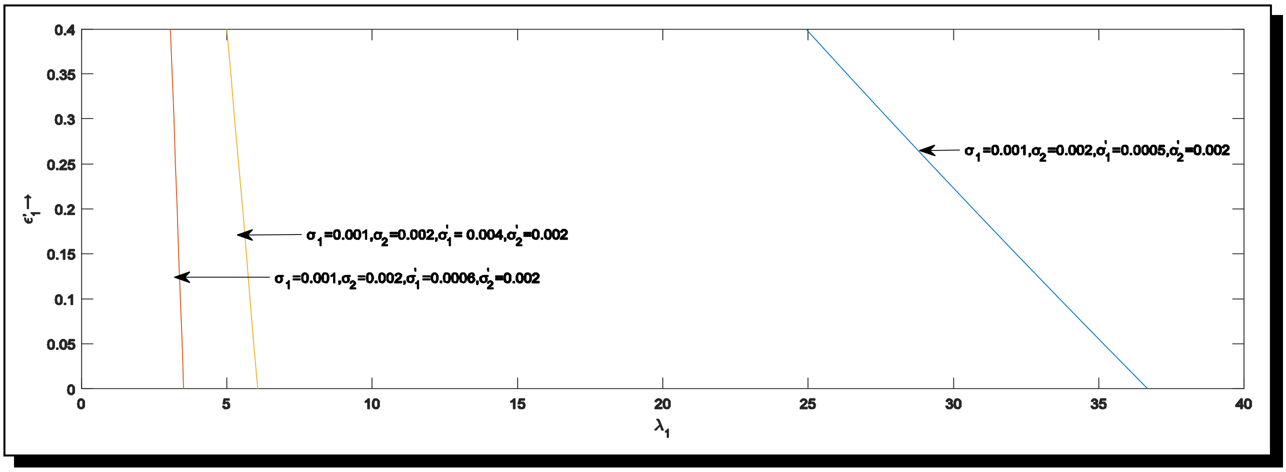

Figure 4. Correlation of characteristic root $\lambda_1$and ε_1' for L_1 ($\sigma_1 = 0.001$, $\sigma_2 = 0.002$, $\sigma_1' = 0.0005$ to 0.004, $\sigma_2' = 0.002$, $\varepsilon_1' = 0.002$)

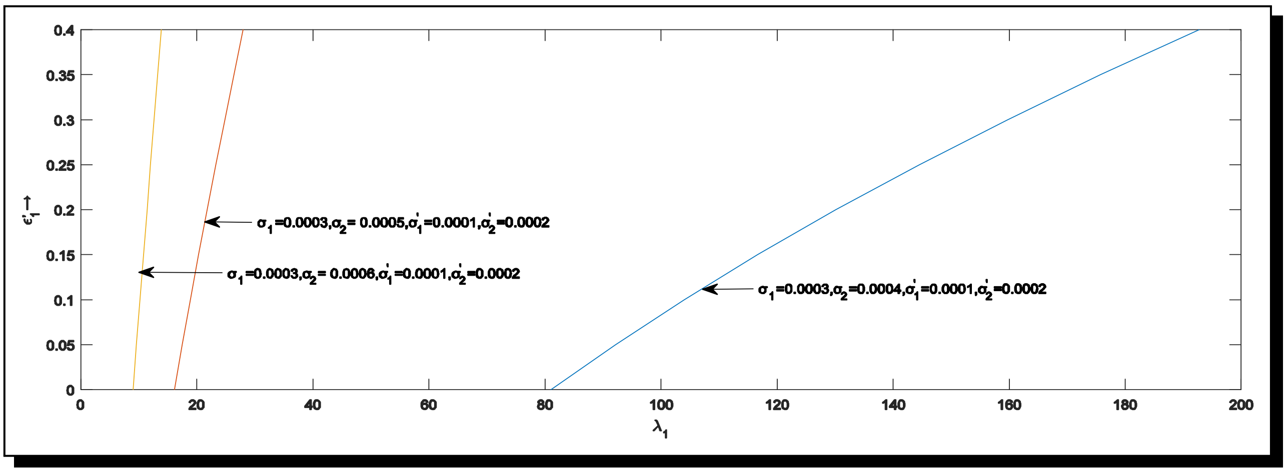

Figure 5. Correlation of characteristic root $\lambda_1$and ε_1' for L_2 ($\sigma_1 = 0.0003$, $\sigma_2 = 0.0004$ to 0.0006, $\sigma_1' = 0.0001$, $\sigma_2' = 0.0002$, $\varepsilon_2' = 0.002$)

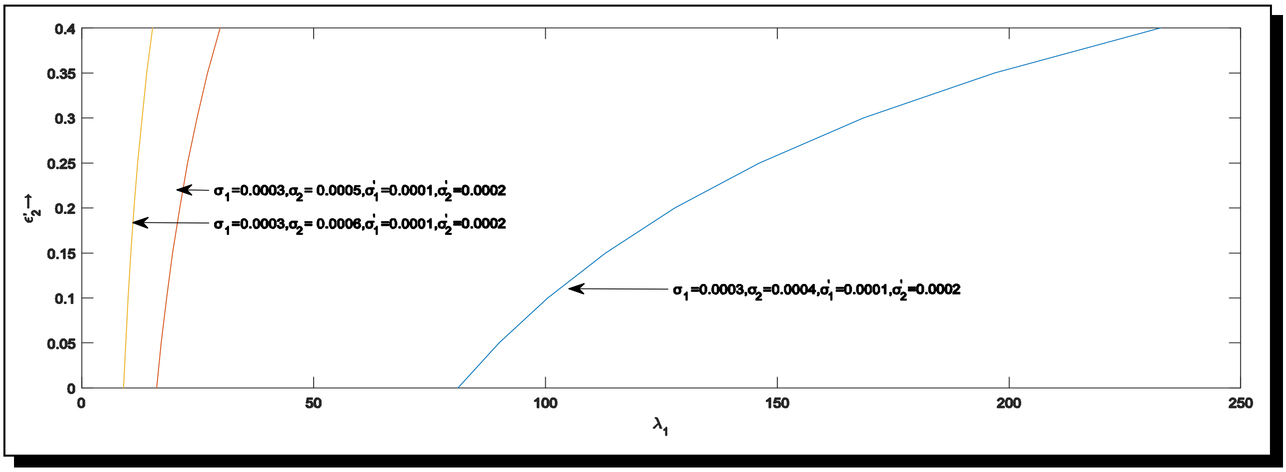

Figure 6. Correlation of characteristic root $\lambda_1$and ε_2' for L_2 ($\sigma_1 = 0.0003$, $\sigma_2 = 0.0004$ to 0.0006, $\sigma_1' = 0.0001$, $\sigma_2' = 0.0002$, $\varepsilon_1' = 0.002$)

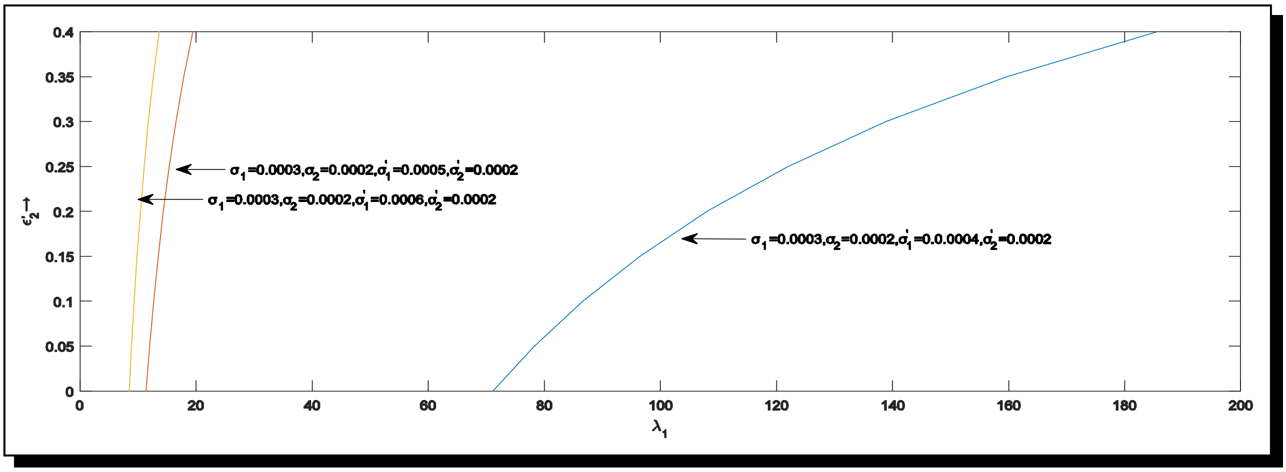

Figure 7. Correlation of characteristic root λ_1 and ε_2' for L_2 ($\sigma_1 = 0.0003$, $\sigma_2 = 0.0002$, $\sigma_1' = 0.0004$ to 0.0006, $\sigma_2' = 0.0002$, $\varepsilon_1' = 0.002$)

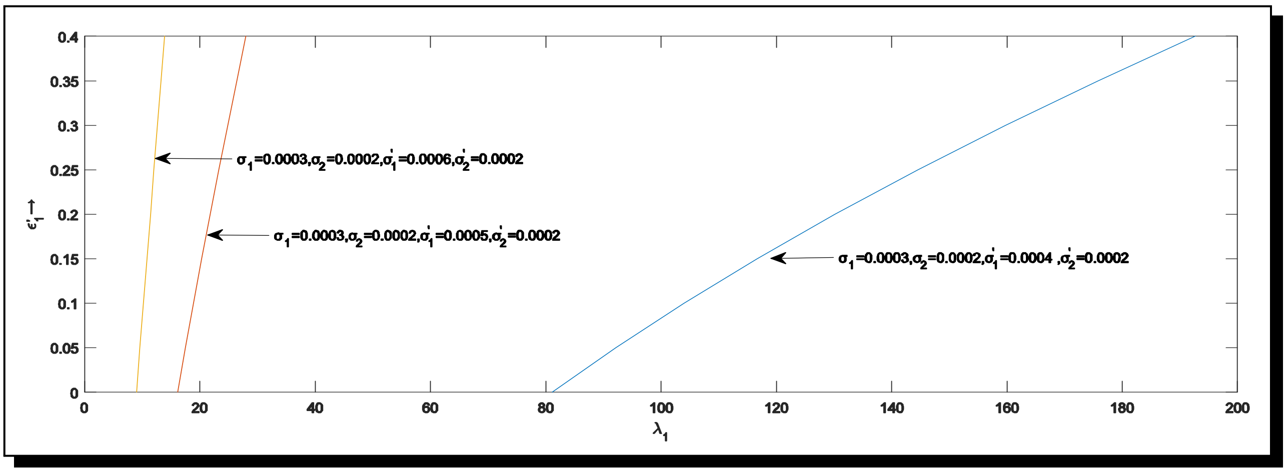

Figure 8. Correlation of characteristic root λ_1 and ε_1' for L_2 ($\sigma_1 = 0.0003$, $\sigma_2 = 0.0002$, $\sigma_1' = 0.0004$ to 0.0006, $\sigma_2' = 0.0002$, $\varepsilon_2' = 0.002$)

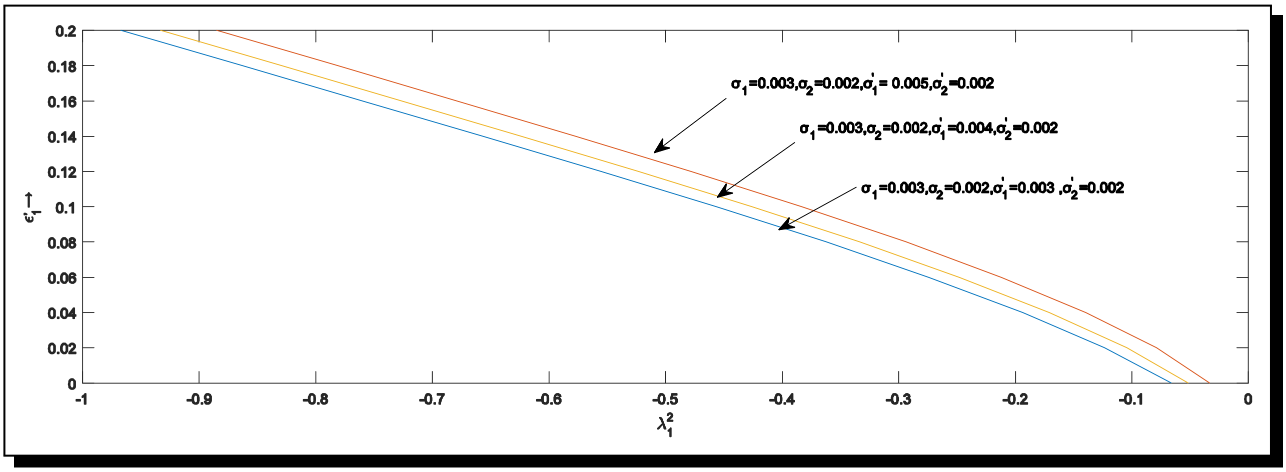

Figure 9. Correlation of characteristic root λ_1^2 and ε_1' for L_3 ($\sigma_1 = 0.003$, $\sigma_2 = 0.002$, $\sigma_1' = 0.003$ to 0.005, $\sigma_2' = 0.002$, $\varepsilon_2' = 0.002$)

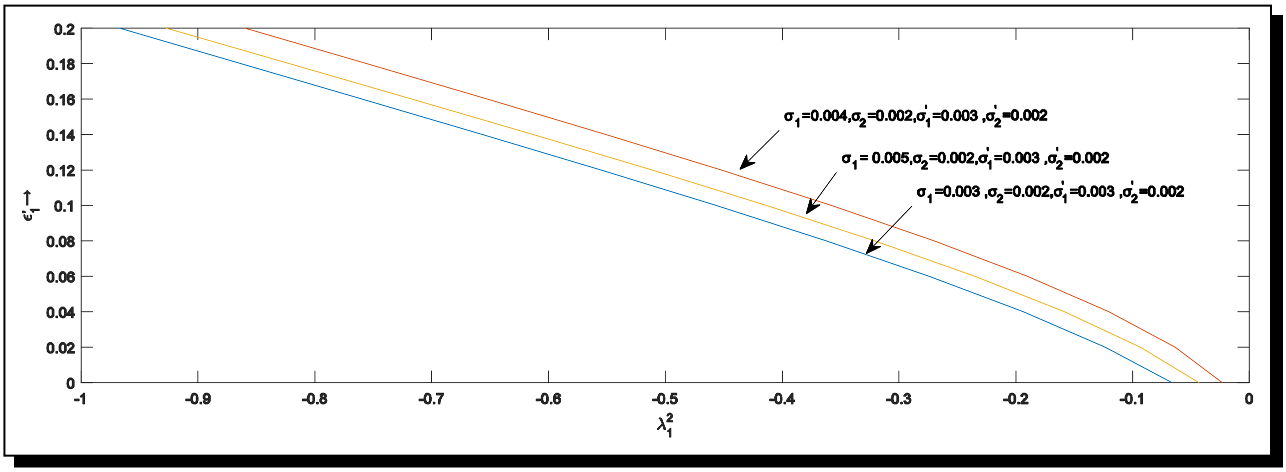

Figure 10. Correlation of characteristic root λ_1^2 and ε_1' for L_3 ($\sigma_1 = 0.003$ to 0.005, $\sigma_2 = 0.002$, $\sigma_1' = 0.003$, $\sigma_2' = 0.002$, $\varepsilon_2' = 0.002$)

5. Conclusion

The formula derived in this paper can be applied to the binary system as two primaries and a space craft as third body.

(i) The motion around the collinear point L_1 is unstable for different values of ε_1', ε_2', σ_1, σ_2, σ_1' and σ_2' as $k > 1$ and $\lambda_1^2 > 0$, $\lambda_2^2 < 0$. This can be analysed from Figure 1, Figure 2, Figure 3 and Figure 4.

(ii) The collinear point L_2 also shows the instability of motion in its vicinity as $k > 1$ and $\lambda_1^2 > 0$, $\lambda_2^2 < 0$. This is evident from Figure 5, Figure 6, Figure 7 and Figure 8 for different values of ε_1', ε_2', σ_1, σ_2, σ_1' and σ_2'.

(iii) The motion of infinitesimal around L_3 is stable for some values of ε_1', ε_2', σ_1, σ_2, σ_1' and σ_2' because $\lambda_{1,2}^2 < 0$ as well as $k < 1$. It is also observed that increment in values of ε_1', ε_2',

σ_1, σ_2, σ_1' and σ_2' tends the system to be unstable. This can be seen from Figure 9 and Figure 10 as roots become imaginary.

For different values of ε_1', ε_2', σ_1, σ_2, σ_1' and σ_2' our result is in conformity of the results of Usha and Narayan [25] and Narayan and Singh [18]. The existence and stability of collinear equilibrium points of the elliptic restricted three body problem with different conditions has been analysed. The figures are drawn using MATLAB R2016a.

Hence we arrived at the conclusion that motion around collinear point L_1 and L_2 are unstable, while motion around L_3 is conditionally stable for some values of ε_1', ε_2', σ_1, σ_2, σ_1'.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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