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Research Article

An Effective Numerical Method for Singularly Perturbed Nonlocal Boundary Value Problem on Bakhvalov Mesh

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Abstract. The present study focuses on obtaining an absolutely accurate computational solution of a linear singularly perturbed problem with integral boundary condition on Bakhvalov mesh. A finite difference scheme was constructed and the approximation of the presented problem was obtained. Based on the ε-perturbation parameter, it was established that the first-order uniform convergence was within the discrete maximum norm. A numerical experiment was performed in order to demonstrate the effectiveness and accuracy of the presented method. The results were confirmed through the relevant table and figures.

Keywords. Singular perturbation; Finite difference scheme; Bakhvalov mesh; Uniformly convergence; Integral boundary condition

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1. Introduction

Nonlocal problems were first studied by Bitsadze and Samarskii [2]. The singularly perturbed problem with integral boundary conditions that contained nonlocal condition was indicated by

$$\varepsilon u''(x)+a(x)u'(x)=g(x), \qquad 0<x<1, \tag{1}$$

$$u'(0)=\frac{A}{\varepsilon}, \tag{2}$$

$$\int_0^1 b(x)u(x)dx = B, \tag{3}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, A and B are given constants; $a^* > a(x) \geq \alpha > 0$; $g(x)$ and $b(x)$ are assumed to be sufficiently smooth functions in $[0,1]$. Furthermore, the solution $u(x)$ has a boundary layer near $x = 0$.

It is acknowledged that multiplying a very small parameter ε by the highest-order derivative in a differential equation results with a singularly perturbed differential equation. Classical numerical methods usually fail to solve these equations due to the parameter ε. Therefore, it is necessary to apply appropriate numerical methods such as the finite difference method, finite element method, etc. The finite difference method is known to be appropriate to confirm ε-uniform convergence. Therefore, the present study focuses on using the finite difference method. Singularly perturbed equation with integral boundary condition is commonly used in applied mathematics and other scientific applications. These applications include human pupil light reflex, first-exit problems in neurobiology, models of physiological processes and diseases, optimal control theory, optical bistable devices, signal transmission, Reynolds number flow in fluid dynamics, heat transfer problems, hydrodynamics, transonic gas dynamics, chemical-reactor theory, control theory, oceanography, fluid mechanics, quantum mechanics, hydro mechanical problems, meteorology, electrical networks and other physical models [3, 6, 19–21, 23–26]. The references of the present study provides further information on the areas of application. Reference [22] provided the existence-uniqueness study for the solution of a singularly perturbed problem with multipoint boundary conditions. Recently, various researchers focused on using different numerical methods for these problems [1, 2, 4, 5, 7–18]. The present study provides approximate solutions to these problems through the employment of a non-uniform mesh. The study is composed of the following main sections: Section 2 provided the asymptotic behavior of the exact solution through the proof of Lemma 2.1 in order to be used in latter sections. In Section 3, the difference scheme was constructed via the integral identities method with quadrature rules. Section 4 provided the error analysis. The error was determined as first-order on Bakhvalov mesh. Conclusively, an experiment was presented in order to support the theory.

In the following sections, C and C_0 were used to denote the positive constants that were independent of perturbation parameter ε and the mesh parameter.

2. Asymptotic Behaviour Estimates of the Exact Solution

In this section, behaviour of the exact solution and its derivative was provided in order to analyze the numerical solution in the latter sections. Accordingly, the priori estimates for the exact solution and its derivative for the problem (1)-(3) was obtained via Lemma 2.1:

Lemma 2.1. *For the condition* $a(x)$, $g(x) \in C^2[0,1]$, $b(x) \in C^1[0,1]$ *and*

$$\|b\|_1 = \int_0^1 b(x)dx \neq 0, \tag{4}$$

the exact solution and its derivative for the problem (1)-(3) *yielded the following properties:*

$$|u(x)| \le C_0, \tag{5}$$

and

$$|u'(x)| \le C\left\{1+\frac{1}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}}\right\}. \tag{6}$$

Proof. Let us apply $u'(x)v(x)$ in the equation (1) and the solution of the problem (1)-(3) became

$$\varepsilon v' + a(x)v(x) = g(x), \tag{7}$$

$$v(0) = \frac{A}{\varepsilon}. \tag{8}$$

The solution of the problem (7)-(8) became:

$$v(x) = \frac{A}{\varepsilon}e^{\frac{-1}{\varepsilon}\int_0^x a(\xi)d\xi} + \int_0^x g(\tau)e^{\frac{-1}{\varepsilon}\int_\tau^x a(\eta)d\eta}d\tau. \tag{9}$$

Integrating equation (9) over $(0,x)$, we obtained,

$$u(x) = u(0) + \frac{A}{\varepsilon}\int_0^x e^{\frac{-1}{\varepsilon}\int_0^x a(\xi)d\xi}ds + \frac{1}{\varepsilon}\int_0^x\left(\int_0^x g(\tau)e^{\frac{-1}{\varepsilon}\int_\tau^x a(\eta)d\eta}d\tau\right)ds. \tag{10}$$

Once the equation (10) is written in equation (3), we obtained,

$$u(0) = \frac{1}{\int_0^1 b(x)dx}\Bigg(B - \frac{A}{\varepsilon}\int_0^1 b(x)\int_0^x e^{\frac{-1}{\varepsilon}\int_0^x a(\xi)d\xi}dsdx - \frac{1}{\varepsilon}\int_0^1 b(x)\int_0^x\int_0^x g(\tau)e^{\frac{-1}{\varepsilon}\int_\tau^x a(\eta)d\eta}d\tau dsdx\Bigg). \tag{11}$$

The evaluation of this equality resulted in,

$$|u(0)| \le \frac{|B|}{\|b\|_1} - \frac{|C|}{\|b\|_1} \le C. \tag{12}$$

Finally, from (10) and (12), we obtained,

$$|u(x)| \le C_0,$$

and it proved (5). Thereafter, it is essential to examine the inequality (6).

$$|u'(x)| \le |u'(0)|\left|e^{\frac{-1}{\varepsilon}\int_0^x a(\xi)d\xi} + \int_0^x g(\tau)e^{\frac{-1}{\varepsilon}\int_\tau^x a(\eta)d\eta}d\tau\right|. \tag{13}$$

Due to several calculations in equation (13), we obtained,

$$\begin{aligned}\left|u'(x)\right| &\le C + \frac{C}{\varepsilon}\left(e^{-\frac{\alpha x}{\varepsilon}}\right)\\ &\le C\left\{1+\frac{1}{\varepsilon}\left(e^{-\frac{\alpha x}{\varepsilon}}\right)\right\}.\end{aligned} \tag{14}$$

Eventually, inequality (6) was acquired, and subsequently, the proof of Lemma 2.1 was completed. □

3. Establishment of the Difference Scheme

This section focused on the discretization the problem (1)-(3) through the employment of a finite difference method on Bakhvalov type mesh. The Bakhvalov mesh utilized in the present study was introduced as follows.

3.1 Bakhvalov Mesh

The approximation to the solution $u(x)$ of the problem (1)-(3) was computed on a Bakhvalov mesh. For a positive integer N, that could be divided by two, the interval $[0,1]$ was divided into two subintervals $[0,\sigma]$ and $[\sigma,1]$. In practice, $\sigma \ll 1$ was usually employed as a transition point and it is represented as follows:

$$\sigma = \min\left\{\frac{1}{2}, -\alpha^{-1}\varepsilon\ln\varepsilon\right\}.$$

A set of the mesh $\bar{\omega}_N = \{x_i\}_{i=0}^N$ were introduced;

$$x_i = \begin{cases} -(\alpha)^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\frac{2i}{N}\right], & x_i\in[0,\sigma],\ i=0,\dots,\frac{N}{2},\ \sigma<\frac{1}{2};\\ -(\alpha)^{-1}\varepsilon\ln\left[1-(1-e^{\frac{-\alpha}{2\varepsilon}})\frac{2i}{N}\right], & x_i\in[0,\sigma],\ i=0,\dots,\frac{N}{2},\ \sigma=\frac{1}{2};\\ \sigma+\left(i-\frac{N}{2}\right)h^{(1)}, & x_i\in[\sigma,1],\ i=\frac{N}{2}+1,\dots,N,\ h^{(1)}=\frac{2(1-\sigma)}{N}. \end{cases}$$

3.2 Construction of the Difference Scheme on Shishkin Mesh

A non-uniform mesh was introduced on the interval $[0,1]$

$$\omega_N = \{0 < x_1 < x_2 < \ldots < x_{N-1} < 1\}$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = 1\}.$$

Prior to the description of the numerical method employed in the present study, it is essential to introduce several notations regarding the mesh functions. The following finite difference was defined for any mesh function $u_i = u(x_i)$ given on $\bar{\omega}_N$:

$$u_{\bar{x},i} = \frac{u_i - u_{i-1}}{h_i},\quad u_{x,i} = \frac{u_{i+1}-u_i}{h_{i+1}},\quad u_{\overset{0}{x},i} = \frac{u_{x,i}+u_{\bar{x},i}}{2},$$

$$u_{\hat{x},i} = \frac{u_{i+1}-u_i}{\hbar_i},\quad u_{\bar{x}\hat{x},i} = \frac{u_{x,i}-u_{\bar{x},i}}{\hbar_i},\quad \hbar_i = \frac{h_i+h_{i+1}}{2}$$

$$\|u\|_\infty \equiv \|u\|_{\infty,\bar{\omega}_N} := \max_{0<i<N}|u_i|.$$

For each $i \ge 1$, step-size was set as $h_i = x_i - x_{i-1}$, $i = 1,2,\dots,N$.

Herein, the difference scheme for the equation (1) could be constructed. First, the equation (1) was integrated over (x_{i-1}, x_{i+1}) as,

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} g(x)\varphi_i(x)dx,\ i=\overline{1,N-1}, \tag{15}$$

here $\{\varphi_i(x)\}_{i=1}^{N-1}$ represented the basis functions that $\{\varphi_i(x)\}_{i=1}^{N-1}$ took the following form

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \dfrac{e^{\frac{a_i(x-x_i)}{\varepsilon}}-1}{e^{\frac{a_ih_i}{\varepsilon}}-1}, & x_{i-1}<x<x_i,\\ \varphi_i^{(2)}(x) = -\dfrac{e^{\frac{a_i(x-x_{i+1})}{\varepsilon}}-1}{e^{\frac{-a_ih_{i+1}}{\varepsilon}}-1}, & x_i<x<x_{i+1},\\ 0, & x\notin(x_{i-1},x_{i+1}), \end{cases}$$

$\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$, were determined as the solutions of the problems, respectively:

$$\varepsilon\varphi'' - a_i\varphi' = 0, \quad x_{i-1} < x < x_i \qquad (16)$$
$$\varphi(x_{i-1}) = 0, \quad \varphi(x_i) = 1$$

$$\varepsilon\varphi'' - a_i\varphi' = 0, \quad x_i < x < x_{i+1} \qquad (17)$$
$$\varphi(x_i) = 1, \quad \varphi(x_{i+1}) = 0.$$

Due to several arrangements in the equation (15), we obtained,

$$-\varepsilon\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi'_i(x)dx + a_i\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i(x)dx = g_i + R_{a,i} + R_{g,i} \qquad (18)$$

where

$$R_{a,i} + R_{g,i} = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x_i) - a(x)]u'(x)\varphi_i(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} [g(x) - g(x_i)]\varphi_i(x)dx. \qquad (19)$$

The interpolating quadrature rules (2.1) and (2.2) from [2] were used with weight functions $\varphi_i(x)$ on subintervals (x_{i-1}, x_{i+1}) and the following precise relation was obtained

$$\varepsilon\theta_i u_{\bar{x}\hat{x},i} + \eta_i u_{\hat{x},i} = g_i + R_i, \quad i = \overline{1, N-1}, \qquad (20)$$

where

$$\theta_i = \frac{\frac{a_i h_i}{\varepsilon}}{e^{\frac{-a_i h_i}{\varepsilon}} - 1} \qquad (21)$$

and

$$\eta_i = \frac{\hbar_i^{-1} h_{i+1}}{1 - e^{\frac{-a_i h_{i+1}}{\varepsilon}}} - \frac{\hbar_i^{-1} h_i}{e^{\frac{a_i h_i}{\varepsilon}} - 1}. \qquad (22)$$

Herein, the difference scheme for the boundary condition (2) was achieved as follows: Initially, equation (2) was integrated over (x_0, x_1)

$$\int_{x_0}^{x_1} Lu(x)\varphi_0(x)dx = \int_{x_0}^{x_1} g(x)\varphi_0(x)dx, \qquad (23)$$

where, $\varphi_0(x)$ was the basis function in the following form:

$$\varphi_0(x) = \begin{cases} \frac{1 - e^{\frac{a_0(x_1 - x)}{\varepsilon}}}{1 - e^{\frac{-a_0 h_1}{\varepsilon}}}, & x_0 < x < x_1, \\ 0, & x \notin (x_0, x_1), \end{cases}$$

and $\varphi_0(x)$ was determined as the solution to the problem:

$$\varepsilon\varphi_0'' - a_0\varphi_0' = 0, \quad x_0 < x < x_1, \qquad (24)$$
$$\varphi(x_0) = 1, \quad \varphi(x_1) = 0.$$

As a result of using interpolation quadrature rules in the equation (23), we acquired,

$$\varepsilon\theta_1 u_{x,0} - g_0\theta_0 = A + r_0. \qquad (25)$$

Subsequently, the difference scheme for the boundary condition (3) was obtained as:

$$\int_0^1 b(x)u(x)dx = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} b(x)u(x)dx. \qquad (26)$$

Several calculations were completed for equation (26) and we obtained,

$$\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} b(x)u(x)dx = \sum_{i=1}^{N} g_i u_i h_i + r_1, \tag{27}$$

where

$$|r_1| = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} (x-x_{i-1})\frac{d}{dx}(b(x)u(x))dx. \tag{28}$$

Thus, the difference scheme for the boundary condition (3) was introduced as:

$$\sum_{i=1}^{N} b_i u_i h_i = B - r_1. \tag{29}$$

In case R_i, r_0 and r_1 were neglected in equations (20), (25) and (29), respectively, the following difference scheme for the approximation of the problem (1)-(3) could be proposed:

$$\varepsilon\theta_i y_{\bar{x}\hat{x},i} + \eta_i y_{\hat{x},i} = g_i, \quad i = \overline{1,N-1}, \tag{30}$$

$$\varepsilon\theta_1 y_{x,0} - g_0\theta_0 = A, \tag{31}$$

$$\sum_{i=1}^{N} b_i y_i h_i = B. \tag{32}$$

4. Estimates of the Uniform Error

The convergence of the method for the problem (1)-(3) on Bakhvalov mesh was examined in this section. The error function $z_i = y_i - u_i$, $i = 0,1,\ldots,N$, where z_i was the solution to the discrete problem, was provided:

$$\varepsilon\theta_i z_{\bar{x}\hat{x},i} + \eta_i z_{\hat{x},i} = R_i, \quad i = \overline{1,N-1}, \tag{33}$$

$$\varepsilon\theta_1 z_{x,0} = r_0, \tag{34}$$

$$\sum_{i=1}^{N} b_i z_i h_i = r_1. \tag{35}$$

Lemma 4.1. *Let z_i be the solution of the discrete problem* (33)-(35)*, then the estimate would hold:*

$$\|z\|_{\infty,\bar{\omega}_N} \le C\{\|R\|_{\infty,\omega_N} + |r_0| + |r_1|\}. \tag{36}$$

Proof. According to the maximum principle for (33)-(35) the following evaluation was obtained:

$$|z_i(x)| \le \beta^{-1}|r_0| + \gamma^{-1}|r_1| + \alpha^{-1}\|R\|_{\infty,\omega_N}. \tag{37}$$

Following equation (37)

$$\|z\|_{\infty,\bar{\omega}_N} \le C\{|r_0| + |r_1| + \|R\|_{\infty,\omega_N}\}. \tag{38}$$

Lemma 4.1 was proven. □

Lemma 4.2. *Based on the assumptions in Section 1 and Lemma 2.1, the solution of the problem* (1)-(3) *fulfilled the following estimates for the remainder terms R_i, r_0 and r_1,*

$$\|R\|_{\infty,\omega_N} \le CN^{-1}, \tag{39}$$

$$|r_0| \le CN^{-1}, \tag{40}$$

$$|r_1| \le CN^{-1}. \tag{41}$$

Proof. The remainder terms R_i, r_0 and r_1 were evaluated for the intervals $[0,\sigma]$ and $[\sigma,1]$, respectively. Given $|R_i| = R_{a,i} + R_{g,i}$ as sum of the error functions,

$$|R_i| \le \left| \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [a(x_i) - a(x)]u'(x)\varphi_i(x)dx + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [g(x) - g(x_i)]\varphi_i(x)dx \right|, \tag{42}$$

where

$$|a(x_i) - a(x)| \le Ch_i, \tag{43}$$

$$|g(x) - g(x_i)| \le Ch_i, \tag{44}$$

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)dx \le Ch_i. \tag{45}$$

Once the equations (43)-(45) were written in (42), we obtained,

$$|R_i| \le Ch_i. \tag{46}$$

In the first case, $x_i \in [0,\sigma]$:

(1) For $\sigma < \frac{1}{2}$,

$$x_{i-1} = \alpha^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{2(i-1)}{N}\right], \tag{47}$$

$$h_i = \alpha^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{2i}{N}\right] - \alpha^{-1}\varepsilon \ln\left[1 - (1-\varepsilon)\frac{2(i-1)}{N}\right]. \tag{48}$$

Applying the mean value theorem in (48), we obtained that

$$h_i = \alpha^{-1}\varepsilon \frac{2(1-\varepsilon)N^{-1}}{1 - 2i_1(1-\varepsilon)N^{-1}} \le CN^{-1}. \tag{49}$$

Consequently, it was possible to write from (46) and (49):

$$|R_i| \le CN^{-1}, \quad i = \overline{0, \frac{N}{2}}.$$

(2) For $\sigma = \frac{1}{2}$,

$$x_{i-1} = \alpha^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{\alpha}{2\varepsilon}})\frac{2(i-1)}{N}\right], \tag{50}$$

$$h_i = \alpha^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{\alpha}{2\varepsilon}})\frac{2i}{N}\right] - \alpha^{-1}\varepsilon \ln\left[1 - (1 - e^{\frac{=\alpha}{2\varepsilon}})\frac{2(i-1)}{N}\right]. \tag{51}$$

Applying the mean value theorem in (51), we obtain that

$$h_i = \alpha^{-1}\varepsilon \frac{2(1 - e^{\frac{\alpha}{2\varepsilon}})N^{-1}}{1 - 2i_1(1 - e^{\frac{\alpha}{2\varepsilon}})N^{-1}} \le CN^{-1}. \tag{52}$$

Thus, from (46) and (52), we can write

$$|R_i| \le CN^{-1}, \quad i = \overline{0, \frac{N}{2}}.$$

In the second case, $x_i \in [\sigma, 1]$:

$$x_i = \sigma + \left(i - \frac{N}{2}\right)h, \quad i = \overline{\frac{N}{2} + 1, N}, \tag{53}$$

where

$$h = \frac{2(1-\sigma)}{N}.$$

Following equations (46) and (53), we acquired

$$|R_i| \le Ch \le CN^{-1}.$$

Based on all situations described above, we obtained

$$|R_i| \le CN^{-1}.$$

Subsequently, the remainder term r_0 was estimated

$$|r_0| \le \left| \int_{x_0}^{x_1} [a(x) - a(0)]u'(x)\varphi_0(x)dx + \int_{x_0}^{x_1} [g(x) - g(0)]\varphi_0(x)dx \right|. \tag{54}$$

Herein, applying the mean value theorem for (54), we deduced that

$$|r_0| \le Ch_1 = C(x_1 - x_0). \tag{55}$$

Similar to the approach described above in (55) mesh points for $[0, \sigma]$ and $[\sigma, 1]$ were

$$|r_0| \le Ch_1 \le CN^{-1}.$$

Subsequently, the error function r_1 was evaluated as

$$|r_1| \le \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left| (x - x_{i-1}) \frac{d}{dx}[b(x)u(x)] \right| dx, \tag{56}$$

where

$$(x - x_{i-1}) \le h_i, \quad b^*(x) = \max|b(x)| \le C, \quad u(x) \le C_0. \tag{57}$$

Based on (57) and (58), it was possible to observe that

$$|r_1| \le Ch_i \le CN^{-1}. \tag{58}$$

Hence, the proof of Lemma 4.2 was achieved. □

It was possible to state the convergence result of the present study through Theorem 4.1, as follows:

Theorem 4.1. *Given* $u(x)$ *as the solution of the problem* (1)-(3) *and* y_i *as the solution of the difference scheme* (30)-(32) *the following uniform error estimate fulfills*

$$\|y - u\|_{\infty, \bar{\omega}_N} \le CN^{-1}.$$

Proof. The proof of this theorem is clear and easy from Lemma 4.1 and Lemma 4.2. □

5. The Algorithm and The Numerical Results

Here an effective algorithm for the solution of the difference scheme (1)-(3) was introduced and numerical results were presented in table and graphs.

- The algorithm for the solution of the difference scheme (30)-(32) was provided as follows:

$$\left(\frac{\varepsilon\theta_i}{\hbar_i h_i}\right) y_{i-1}^{(n)} - \left(\frac{\varepsilon\theta_i}{h_i h_{i+1}} + \frac{\varepsilon\theta_i}{h_i h_i} + \frac{\eta_i}{h_i}\right) y_i^{(n)} + \left(\frac{\varepsilon\theta_i}{\hbar_i h_{i+1}} + \frac{\eta_i}{h_i}\right) y_{i+1}^{(n)} = -g_i, \quad i = \overline{1, N-1},$$

$$y_0^{(n)} = y_1^{(n)} + \frac{g_0\theta_0 + A}{\varepsilon\theta_1 h_1^{-1}}, \quad y_N^{(n)} = \left[B - \sum_{i=1}^{N=1} h_i b_i y_i^{(n-1)}\right] h_N^{-1} b_N^{-1},$$

$$A_i = \frac{\varepsilon\theta_i}{\hbar_i h_i}, \quad B_i = \frac{\varepsilon\theta_i}{h_i h_{i+1}} + \frac{\varepsilon\theta_i}{h_i h_i} + \frac{\eta_i}{h_i}, \quad C_i = \frac{\varepsilon\theta_i}{\hbar_i h_{i+1}} + \frac{\eta_i}{h_i},$$

$$\alpha_1 = 1, \; \beta_1 = \frac{g_0\theta_0 + A}{\varepsilon\theta_1 h_1^{-1}},$$

$$\alpha_{i+1} = \frac{B_i}{C_i - A_i\alpha_i}, \; \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - A_i\alpha_i}, \quad i = \overline{1, N-1},$$

$$y_i^{(n)} = \alpha_{i+1} y_{i+1}^{(n)} + \beta_{i+1}, \quad i = \overline{N-1, 1},$$

- The following problem was examined in order to determine how the method worked:

$$-\varepsilon u''(x) + u'(x) = 1, \quad 0 < x < 1,$$

$$u'(0) = \frac{1}{\varepsilon}, \quad \int_0^1 u(x)dx = \frac{1}{2}.$$

The exact solution of this problem was provided via

$$u(x) = x + \varepsilon(\varepsilon - 1)(1 - e^{\frac{-1}{\varepsilon}}) + (\varepsilon - 1)(1 - e^{\frac{-x}{\varepsilon}}).$$

The corresponding ε-uniform convergence rates were computed using the formula

$$P^N = \frac{\ln(e^N/e^{2N})}{\ln 2}.$$

The error estimates were denoted by

$$e^N = \max_{\varepsilon} e_\varepsilon^N, \quad e_\varepsilon^N = \|y - u\|_{\infty, \bar{\omega}_N}.$$

Table 1. The computed maximum pointwise errors e^N and rates of convergence p^N

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	0.0177133 1.00	0.0088031 1.00	0.0043856 1.00	0.0021889 0.99	0.0010958 0.99	0.0005487 0.99	0.0002748
2^{-2}	0.0593480 1.02	0.0291769 1.01	0.0144542 1.00	0.0071922 1.00	0.0035876 0.99	0.0017973 0.99	0.0008990
2^{-3}	0.1515934 1.06	0.0725140 1.03	0.0354105 1.01	0.0174912 1.00	0.0086917 1.00	0.0043356 1.00	0.0021556
2^{-4}	0.3680207 1.14	0.1659372 1.08	0.0784342 1.04	0.0380923 1.02	0.0187665 1.01	0.0093175 1.00	0.0046408
3^{-1}	0.0373369 1.01	0.0184625 1.00	0.0091735 1.00	0.0045713 1.00	0.0022830 1.01	0.0011326 1.00	0.0005641
3^{-2}	0.1758537 1.07	0.0835009 1.03	0.0406156 1.02	0.0200216 1.01	0.0099388 1.00	0.0049515 1.00	0.0024707
10^{-1}	0.2008266 1.08	0.0946524 1.04	0.0458562 1.02	0.0225589 1.01	0.0111871 1.00	0.0055690 1.00	0.0027710

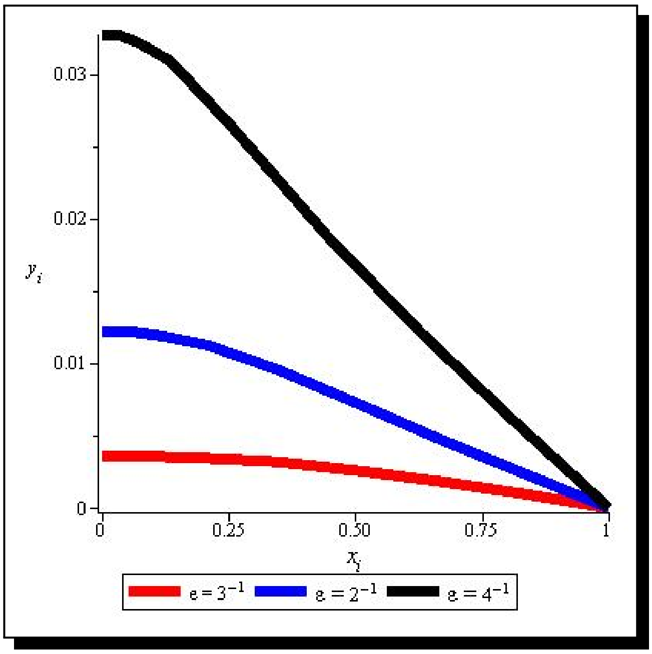

Figure 1. Error distribution for $N = 16$, $\varepsilon = 2^{-1}$, $\varepsilon = 3^{-1}$, $\varepsilon = 4^{-1}$

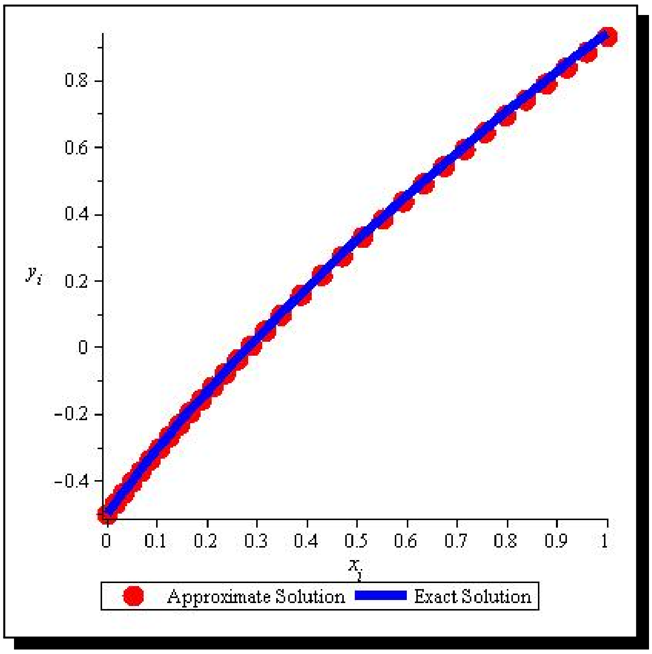

Figure 2. Comparison of approximate solution and exact solution for $N = 32$, $\varepsilon = 2^{-1}$

We applied the present method above the example. According to Table 1 and graphics, when N takes increasing values, it is seen that the convergence rate of the smooth convergence speed p^N is first-order. In Figure 2, the exact solution and approximate solution curves are almost identical. In Figure 1, the errors are maximum in the boundary layer region because of the irregularity caused by the sudden and rapid change of solution around $x = 0$ for different values ε. That is, numerical results show that the proposed scheme is working very effective.

6. Conclusion

In the present paper, singularly perturbed problem containing integral boundary conditions was examined through the finite difference method. It was determined that the method presented uniform convergence with respect to the perturbation parameter ε in the discrete maximum norm. The convergence was found to be first-order. In addition, the study of theoretical and numerical experiment provided results that were consistent with each other. In conclusion, it is possible to assert that this method could be used for more complicated nonlinear singularly perturbed problems. Furthermore, the convergence rate could be upgraded from first-order to higher orders with respect to the findings of this study.

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