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Research Article

Vorticity and Stress Tensor

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Abstract. The assumptions of the Navier-Stokes equations are reconsidered. Vorticity is one of the three basic isotropic motions of a fluid element about a point. Taking into consideration the isotropic and symmetric nature of the associated tensor, it was assumed that vorticity could not contribute to the viscous stresses and hence was omitted from subsequent derivations of the Navier-Stokes equations. Deviating from this classical approach, the effect of vorticity is included in the derivation of the stress tensor.The equation hence derived contains additional terms involving vortex viscosity. Analogous to Maxwell stress tensor in electromagnetic fields, another stress tensor is defined in a vorticity field. By treating vortices as physical structures, it is possible to define pressure and the shearing stress that deform the volume element.

Keywords. Navier-Stokes equation; Stress tensor; Vorticity; Coefficients of viscosity; Maxwell stress tensor

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1. Introduction

The Navier-Stokes equations have been a theoretical model that describes the motion of viscous fluids. In spite of their applications in almost all fields of fluid dynamics, it has not yet been proven that their solutions always exist in three dimensional flows, or even if they do whether they are unique and smooth. Recent studies have suggested that these equations are inadequate in accurately describing compressible flows as seen in the cases of rarefied gases, shock waves and gaseous flows through micro fluidic channels (Brenner [3]).

The reason for this failure or inadequacy can be traced back to the basic assumptions made by Stokes [19] in deriving the equations. The constitutive equations for the viscous stress tensor were derived based on the following assumptions.

(1) Stresses do not depend directly on the flow velocity, but only on the first order spatial derivatives of the velocity about a point. For Newtonian fluids, the stress σ_{ij} is linearly proportional to the velocity gradient $\nabla\vec{u}$.

(2) Vorticity and dilatation do not contribute to stress. Thus only the rate of strain is considered as a contributing factor to stress.

(3) The bulk viscosity $\kappa = 3\lambda + 2\mu$ is assumed to be negligible and the two coefficients depend on the relation $\lambda = -\frac{2}{3}\mu$, where λ is the second coefficient of viscosity and μ is the dynamic coefficient of viscosity. This is known as the Stokes' hypothesis. The above relation is equivalent to the assumption that isotropic dilatations of an elementary volume of fluid do not produce viscous stresses (Buresti [5]).

The lack of unique solutions to the Navier-Stokes equations, the evolution and stretching of vortex knots and also its inadequacy to explain turbulent flow have motivated researchers to reconsider the basic assumptions of the constitutive equations. Recently, Rajagopal [15] has considered an alternative approach by expressing the velocity gradient in terms of the stress, deviating from the classical Stokesian approach where the stress is represented in terms of the velocity gradient. Berdahl and Strang [2] reconsidered the second assumption made by Stokes and made an attempt to include the effect of rotation on the stresses. Truesdell [21, 22], Karim and Rosenhead [12, 16], Chapman and Cowling [6], Thompson [20], Emanuel and Argrow [9], Gad-el-Hak [11], Buresti [5] and many others have discussed, debated and experimented on the assumed relation between the coefficients of viscosity. Brenner ([4, 3]) made a revision of Newton's law of viscosity appearing in the role of the deviatoric stress tensor in the Navier-Stokes equations for compressible fluids and proposed a bi-velocity theory.

2. Effect of Vorticity in Stress Tensor

Let $\vec{u}$ be the velocity of the flow field. $\vec{\omega} = \nabla\times\vec{u}$ is the vorticity field that measures the local rotation at any point of the fluid. The velocity $\vec{u}$ of a fluid element about a point can be represented as the sum of translation, deformation and rotation [7]. $D_{ij} = (u_{i,j} - u_{j,i})$ is the rate of strain tensor (deformation tensor) and $W_{ij} = (u_{i,j} - u_{j,i}) = -\varepsilon_{ijk}\omega_k$ is the tensor associated with vorticity. $\frac{1}{2}D_{ij}$ is the symmetric part and $\frac{1}{2}W_{ij}$ is the skew symmetric part of $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, and ε_{ijk} is the Levi Civita symbol.

The momentum balance equation of a fluid is given as

$$\rho\frac{Du_i}{Dt} = \rho G_i + \sigma_{ij,j}, \tag{2.1}$$

where ρ is the fluid density, u_i is the ith component of the velocity, G_i is the ith component of the body force and σ_{ij} is the stress whose ith component of stress acts on an element of surface with unit normal in the jth direction.

It is assumed that the stress in the fluid is the sum of a pressure term and a diffusing viscous term (proportional to the first order gradients of velocity). Therefore

$$\sigma_{ij} = -p\delta_{ij} + A_{ijkl}u_{k,l}\,, \tag{2.2}$$

where p is the hydrostatic pressure, δ_{ij} is the Kronecker delta, A_{ijkl} is a fourth order tensor and $u_{k,l}$ is the fluid velocity gradient tensor [1]. The fourth order isotropic tensor A_{ijkl} can be expressed as $A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + v(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$ [17], where λ and μ are the coefficients of viscosity as stated before and v is the vortex viscosity. Substituting the above relation in (2.2), we get

$$\sigma_{ij} = -p\delta_{ij} + \lambda div\vec{u}\delta_{ij} + \mu D_{ij} + vW_{ij}\,. \tag{2.3}$$

Taking into consideration the symmetry and the isotropic nature of the tensor, Stokes assumed that vorticity cannot contribute to viscous stresses and hence set $v = 0$. Taking into account the incompressibility condition, the stress tensor hence derived is symmetric and can be diagonalised leaving only normal stresses. These normal stresses in the diagonalised tensor are termed as principal stresses and are always real and mutually orthogonal.

Deviating from Stokes line of reasoning Berdahl and Strang [2] included the effect of vorticity in the constitutive equations and this resulted in a vorticity influenced asymmetric stress tensor.

Impose the incompressibility condition and let $v \neq 0$ in (2.3). Thus by allowing the vortex viscosity to survive in the derivation of the constitutive relation, the resultant stress tensor will be

$$\sigma_{ij} = -p\delta_{ij} + \mu D_{ij} + vW_{ij}\,. \tag{2.4}$$

Substituting the above relation for the stress σ_{ij} in (2.1), the resulting equation will be of the form

$$\rho\frac{Du_i}{Dt} = \rho G_i - p_{,i} + (\mu + v)\nabla^2 u_i\,. \tag{2.5}$$

Note that the above equation includes an additional term due to vorticity.

The stress tensor σ_{ij} obtained above is an asymmetric tensor which contradicts the notion of the symmetry of a stress tensor and it is possible to construct an equivalent symmetric tensor $\widetilde{\sigma}_{ij}$ with the same physical significance such that $\widetilde{\sigma}_{ij,j} = \sigma_{ij,j}$ as suggested by Lautrup [13].

For example, consider an incompressible rotational flow described by $u = (2x^2 - xy + z^2, x^2 - 4xy + y^2, 2xy - yz + y^2)$. Let $C_{ij} = \mu D_{ij} + vW_{ij}$.

Evaluating C_{ij}, we get

$$C_{ij} = \begin{bmatrix} \mu(8x-2y) & [\mu(x-4y) - v(3x-4y)] & [\mu(2y+2z) - v(2y-2z)] \\ [\mu(x-4y) + v(3x-4y)] & \mu(-8x+4y) & [\mu(2x+2y-z) - v(2x+2y-z)] \\ [\mu(2y+2z) + v(2y-2z)] & [\mu(2x+2y-z) + v(2x+2y-z)] & -2\mu y \end{bmatrix}.$$

Here C_{ij} is asymmetric and hence $\sigma_{ij} = -p\delta_{ij} + C_{ij}$ is asymmetric.

Following the construction given by Lautrup, the corresponding symmetric tensor for theexample given above can be found to be

$$\widetilde{\sigma}_{ij} = -p\delta_{ij} + \widetilde{C}_{ij},$$

where

$$\widetilde{C}_{ij} = \begin{bmatrix} \mu(8x-2y) & [\mu(x-4y)+\upsilon(3x+4y)] & [\mu(2y+2z)+\upsilon(2z)] \\ [\mu(x-4y)+\upsilon(3x+4y)] & \mu(-8x+4y) & [\mu(2x+2y-z)+\upsilon(2y+z)] \\ [\mu(2y+2z)+\upsilon(2z)] & [\mu(2x+2y-z)+\upsilon(2y+z)] & -2\mu y \end{bmatrix}.$$

Note that the above symmetric tensor $\widetilde{C}_{ij}$ is not just the symmetric part of C_{ij} but contains additional terms involving the vortex viscosity υ.

We can also verify that

$$\widetilde{\sigma}_{ij,j} = \sigma_{ij,j} = -p_{,i} + (\mu+\upsilon)\nabla^2 u_i.$$

Thus, the two tensors $\widetilde{\sigma}_{ij,j}$ and $\sigma_{ij,j}$ are physically indistinguishable.

Thus, the inclusion of the effect of vorticity in the derivation of the stress tensor resulted in an asymmetric tensor. The example above illustrates that even if the vorticity influenced resultant tensor σ_{ij} is asymmetric; we can find an equivalent symmetric tensor $\widetilde{\sigma}_{ij}$ without compromising on the physical aspects of the problem.

Both the coefficients μ and υ depend on the first order velocity gradients, hence they should be of the same dimension $ML^{-1}T^{-1}$.

As mentioned above even though vorticity is associated with one of the three basic isotropic motions of a fluid about a point, it is left out of consideration in the derivation of the Navier-Stokes equations. Batchelor [1] noted that "it is taken for granted, in most expositions of fluid dynamics, that a deviatoric stress cannot be generated by pure rotation, irrespective of the structure of the fluid, simply on the grounds that there is no deformation of the fluid; however rigorous justification for this belief is elusive". Inclusion of the contribution to the stress from vorticity results in the introduction of one more coefficient of viscosity and it does not make any change in the nature of the equation except in the value of the dynamic coefficient of viscosity. Lautrup has also discussed about the symmetry of the stress tensor and he uses the ambiguity in the choice of the stress tensor to prove that even if the stress is asymmetric, it is possible to choose a symmetric stress tensor. The ambiguity arises from the fact that the relevant quantity for the dynamics of fluid flow is the contribution from $\sigma_{ij,j}$ to the effective density of force rather than the stress tensor σ_{ij} itself ([13]).

3. Maxwell Stress Tensor in Fluid Dynamics

The analogy between electromagnetic and fluid dynamic equations was first noted by Maxwell himself (Maxwell [14]). He suggested that the vector potential **A** of the magnetic induction **B** represents some kind of a fluid velocity field in ether. This was interpreted in the light of

Fizeaus experiment by Cook, Fearn and Millonni [8].

Another analogy between Maxwell stress tensor in electromagnetic continuum and a stress in incompressible inviscid fluids whose origin is vorticity has been explored [23].

Corresponding to magnetic energy, the energy associated with vorticity field is defined by $\frac{1}{2}\omega^2$. Thus the rate of change of energy is

$$\frac{\partial}{\partial t}\left(\frac{\omega^2}{2}\right) = -\vec{u}\cdot\left(\vec{f}\times\vec{\omega}\right) + \nabla\cdot\left[(\vec{u}\times\vec{\omega})\times\vec{\omega}\right],$$

where $\vec{f} = \nabla\times\vec{\omega}$ is the flexion field. If vorticity is confined to a sub-domain of the fluid, the divergence term vanishes on integrating over the entire volume.

Thus, we get

$$\frac{dM}{dt} = -\int_V \vec{u}\cdot\vec{F}dV.$$

Here $M = \int_v \frac{\omega^2}{2}dV$, is the total enstrophy and $\vec{F} = \vec{f}\times\vec{\omega}$ is the force analogous to Lorentz force.

The ith component of this force $\vec{F}$ is

$$F_i = \frac{\partial}{\partial x_j}\Pi_{ij},$$

where $\Pi_{ij} = \omega_i\omega_j - \frac{\omega^2}{2}\delta_{ij}$ and δ_{ij} is the Kronecker delta.

Analogous to the Maxwell Stress tensor associated to Lorentz force (Ferraro and Plumpton [10]), we get Π_{ij} as the vorticity stress tensor. This tensor is related to the enstrophy in the same way as magnetic stress tensor is associated to magnetic energy.

Π_{ij} being a symmetric matrix can be diagonalized. Choosing the principal axis OX_1 in the direction of vorticity, we get

$$\Pi_{ij} = \begin{bmatrix} \frac{\omega^2}{2} & 0 & 0 \\ 0 & \frac{-\omega^2}{2} & 0 \\ 0 & 0 & \frac{-\omega^2}{2} \end{bmatrix}.$$

Thus the principal stress tensor constitutes a tension $\frac{\omega^2}{2}$ along the line vortex and an equal pressure normal to it.

We can write this tensor as

$$\Pi_{ij} = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-\omega^2}{2} & 0 & 0 \\ 0 & \frac{-\omega^2}{2} & 0 \\ 0 & 0 & \frac{-\omega^2}{2} \end{bmatrix},$$

where the first matrix gives the effect of the force $\vec{F}$ and the second matrix represents the static pressure [23].

The above expressions for the vorticity stress tensor can be compared to the magnetic stress tensor as discussed by Stierstadt and Liu [18]. So the figures given by them apply to vortex stress tensor also. It is difficult to take into account the stretching of vortex lines in three dimensions. Most of the studies make use of *local induction approximation* (LIA) or perturbation methods. But the stress tensor associated with vorticity can be made use of in such studies.

4. Conclusion

We have analyzed the effect of vorticity in two different aspects. The omission of vorticity in the equation of motion of viscous fluids depends on the fact that, vorticity being related to rigid rotation cannot not contribute to viscous forces, while dilatation is not ignored on the same grounds. Many applications of the Navier-Stokes equations take into consideration the creation of vorticity in the flow domain. For example, the boundary layer is studied numerically considering the creation of vorticity on the boundary and its diffusion into the outer field. Therefore the reason why vorticity is still neglected in the derivation of the constitutive equations needs rigorous explanation. Another argument assumes the stress tensor to be symmetric on the basis of which it is proved that vorticity cannot contribute to the stress. This is a mere tautology. Stokes [14] in his original paper also has not given any justification for omitting vorticity. Retaining the dilatation in the derivation of constitutive equations gives rise to the second coefficient of viscosity and thereby the bulk viscosity. We have also derived a vorticity stress tensor based on the analogy between Maxwell's equations and ideal incompressible fluid flow equations. The stress tensor associated to vorticity can be applied in the studies leading to the explanation of stretching of vortex lines.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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