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Mixed Type Second-Order Duality for Variational Problems

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Abstract. A mixed type second-order dual to a variational problem is formulated as a unification of Wolfe and Mond-Weir type dual problems already treated in the literature and various duality results are validated under generalized second order invexity. Problems with natural boundary values are formulated and it also is pointed out that our duality results can be regarded as dynamic generalizations of those of (static) nonlinear programming.

1. Introduction

Duality in continuous programming problem has been investigated by many authors. Hanson [4] pointed out that some of the duality results in nonlinear programming have the analogues in calculus of variations. Exploring this relationship of mathematical programming and classical calculus of variation, Mond and Hanson [8] formulated a constrained variational problem in abstract space and using Valentine [9] optimality conditions for the same, constructed its Wolfe type dual variational problem for proving duality results under usual convexity conditions. Later Bector, Chandra and Husain [1] studied Mond-Weir type duality for the problem of Mond and Hanson [8] for relaxing its convexity requirement for duality to hold.

In view of Mond's [7] remarks that the second-order dual for a nonlinear programming problem gives a tighter bound and has computational advantage over a first order dual, it is natural to find its analogue in continuous programming. Motivated with this observation, Chen [3] formulated Wolfe type second order dual problem to the classical variational problem and studied usual duality results under invexity-like conditions on the function that appear in the formulation of the problem along with some strange and hard relations. Recently Husain *et*

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al. [5] presented Mond-Weir type second-order dual to the variational problems considered in [3] and establish various duality theorems under second-order generalized invexity conditions. In [5], the relationship between second-order duality results in calculus of variation and their counterparts in nonlinear programming is also pointed out.

The concept of mixed type duality seems to be interesting and useful both from theoretical and algorithmic point of view. In this research, in sprit of Xu [10], a mixed second-order dual to the variational problem [3] to combine Wolfe type dual and Mond-Weir type dual problems is presented. A pair of mixed type dual variational problem with natural boundary values is formulated and the validation of its duality results in indicated. The formulation of natural boundary value problems is essential for seeing our results as having anologues in nonlinear programming and hence it is pointed out that our duality results can be viewed as dynamic generalizations of nonlinear programming already existing in the literature.

2. **Definitions and Related Pre-requistes**

Let $I = [a, b]$ be a real interval, $f : I \times R^n \times R^n \to R$ and $g : I \times R^n \times R^n \to R^m$ be twice continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$ where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, denoted by f_x and $f_{\dot{x}}$, the partial derivative of f with respect to $x(t)$ and $\dot{x}(t)$, respectively, that is,

$$f_x = \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \frac{\partial f}{\partial x^2} \\ \vdots \\ \frac{\partial f}{\partial x^n} \end{pmatrix}, \quad f_{\dot{x}} = \begin{pmatrix} \frac{\partial f}{\partial \dot{x}^1} \\ \frac{\partial f}{\partial \dot{x}^2} \\ \vdots \\ \frac{\partial f}{\partial \dot{x}^n} \end{pmatrix};$$

Denote by f_{xx} the Hessian matrix of f with respect to x, that is,

$$f_{xx} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} & \frac{\partial^2 f}{\partial x^1 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^1 \partial x^n} \\ \frac{\partial^2 f}{\partial x^2 \partial x^1} & \frac{\partial^2 f}{\partial x^2 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^2 \partial x^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x^n \partial x^1} & \frac{\partial^2 f}{\partial x^n \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^n} \end{pmatrix}_{n \times n}$$

It is obvious that f_{xx} is a symmetric $n \times n$ matrix. Denote by g_x the $m \times n$ Jacobian matrix with respect to x, that is,

$$g_x = \begin{pmatrix} \frac{\partial g_1}{\partial x^1} & \frac{\partial g_1}{\partial x^2} & \cdots & \frac{\partial g_1}{\partial x^n} \\ \frac{\partial g_2}{\partial x^1} & \frac{\partial g_2}{\partial x^2} & \cdots & \frac{\partial g_2}{\partial x^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x^1} & \frac{\partial g_m}{\partial x^2} & \cdots & \frac{\partial g_m}{\partial x^n} \end{pmatrix}_{m\times n}.$$

Similarly $f_{\dot{x}}$, $f_{\dot{x}x}$, $f_{x\dot{x}}$ and $g_{\dot{x}}$ can be defined.

Denote by X the space of piecewise smooth functions $x : I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \quad \Longleftrightarrow \quad x(t) = \alpha + \int_a^t u(s)ds,$$

where α is given boundary value; thus $\frac{d}{dt} = D$ except at discontinuities.

We incorporate the following definitions which are needed for duality results to hold.

Definition 2.1 (*Second-order Invexity*). If there exists a vector function $\eta(t,x,\bar{x}) \in R^n$ where $\eta : I \times R^n \times R^n \to R^n$ with $\eta = 0$ at $t = a$ and $t = b$, such that for the functional $\int_I \phi(t,x,\dot{x})\,dt$ where $\phi : I \times R^n \times R^n \to R$ satisfies

$$\int_I \phi(t,x,\dot{x})\,dt - \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}p^T(t)Gp(t)\right\} dt$$
$$\geq \int_I \{\eta^T \phi_x(t,\bar{x},\dot{\bar{x}}) + (D\eta)^T \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \eta^T Gp(t)\}\,dt$$

where $G = \phi_{xx} - D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}}$ and $p \in C(I,R^n)$, the space of continuous n-dimensional continuous vector functions.

Definition 2.2 (*Second-order Pseudoinvexity*). If the functional $\int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\int_I \{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T Gp(t)\}\,dt \geq 0$$
$$\Rightarrow \quad \int_I \phi(t,x,\dot{x})\,dt \geq \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}p(t)^T Gp(t)\right\} dt,$$

then $\int_I \phi(t,x,\dot{x})\,dt$ is said to be second-order pseudoinvex with respect to η.

Definition 2.3 (*Strictly Second-order Pseudoinvexity*). If the functional $\int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\int_I \{\eta^T\phi_x + (D\eta)^T\phi_{\dot{x}} + \eta^T Gp(t)\}\,dt \geq 0$$

$$\Rightarrow \quad \int_I \phi(t,x,\dot{x})\,dt > \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}p(t)^T Gp(t)\right\} dt,$$

then $\int_I \phi(t,x,\dot{x})\,dt$ is said to be Strictly second-order pseudoinvex with respect to η.

Definition 2.4 (*Second order Quasi-invex*). If the functional $\int_I \phi(t,x,\dot{x})\,dt$ satisfies

$$\int_I \phi(t,x,\dot{x})\,dt \leq \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}p(t)^T(t)Gp(t)\right\} dt$$

$$\Rightarrow \quad \int_I \{\eta^T\phi_x + (D\eta)^T\phi_{\dot{x}} + \eta^T Gp(t)\}\,dt \leq 0,$$

then $\int_I \phi(t,x,\dot{x})\,dt$ is said to be second-order quasi-invex with respect to η.

If ϕ is independent of t, then the above definitions reduce to those given in [11].

Consider the following constrained variational problem:

(VP) Minimize $\int_I f(t,x,\dot{x})\,dt$

subject to $x(a) = 0 = x(b)$, (1)

$g(t,x,\dot{x}) \leq 0, \quad t \in I$ (2)

where $f: I \times R^n \times R^n \to R$ and $g: I \times R^n \times R^n \to R^m$ are continuously differentiable.

The Fritz-John optimality conditions for the problem (VP) derived in [2] are given in the proposition below.

Proposition 2.1 (*Fritz-John Conditions*). *If (VP) attains a local (or) global minimum at $x = \bar{x} \in X$, then there exist Lagrange multipliers $\tau \in R$ and piecewise smooth $y: I \to R^m$ such that*

$$\tau f_x(t,\bar{x},\dot{\bar{x}}) + y(t)^T g_x(t,\bar{x},\dot{\bar{x}}) - D[f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + y(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})] = 0, \quad t \in I,$$

$$y(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0, \quad t \in I$$

$$(\tau, y(t)) \geq 0, \quad t \in I$$

$$(\tau, y(t)) \neq 0, \quad t \in I$$

The Fritz John necessary conditions for (VP), become the Karush-Kuhn-Tucker conditions if $\tau = 1$. If $\tau = 1$, the solution $\bar{x}$ is said to be normal.

Chen [3] presented the following Wolfe type dual to (VP) in the spirit of Mangasarian [6] and proved various duality results under somewhat strange invexity-like condition.

(**WVD**) Maximize $\int_I \left\{ f(t,u,\dot{u}) + \alpha(t)^T g(t,u,\dot{u}) - \frac{1}{2} p(t)^T H(t,u,\dot{u},\alpha(t)) p(t) \right\} dt$

subject to

$$u(a) = 0 = u(b),$$

$$f_u(t,u,\dot{u}) + \alpha(t)^T g_u(t,u,\dot{u}) - D[f_{\dot{u}}(t,u,\dot{u}) + \alpha(t)^T g_{\dot{u}}(t,u,\dot{u})] + H(t)p(t) = 0, \quad t \in I,$$

$$\alpha(t) \in R^m_+, \ p(t) \in R^n$$

where

$$\begin{aligned} H = {} & f_{uu}(t,u(t),\dot{u}(t)) + (y(t)^T g_u(t,u(t),\dot{u}(t)))_u \\ & - 2D(f_{u\dot{u}}(t,u(t),\dot{u}(t)) + (y(t)^T g_u(t,u(t),\dot{u}(t)))_{\dot{u}}) \\ & + D^2(f_{\dot{u}\dot{u}}(t,u(t),\dot{u}(t)) + (y(t)^T g_{\dot{u}}(t,u(t),\dot{u}(t)))_{\dot{u}}). \end{aligned}$$

It is remarked here that f and g are independent of t then (WVD) becomes second-order dual problem studied by Mond [7]. Recently Husain *et al*. [5] presented the following Mond-Weir dual with the view to weaken the second order invexity requirements and proved duality theorems connecting the problems (CP) and (CD) under generalized second order invexity hypothesis.

(**CD**) Maximize $\int_I \left\{ f(t,u,\dot{u}) - \frac{1}{2} p(t)^T F(t) p(t) \right\} dt$

subject to

$$u(a) = 0 = u(b),$$

$$f_u + y(t)^T g_u - D(f_{\dot{u}} + y(t)^T g_{\dot{u}}) + (F(t) + G(t))p(t) = 0, \ \ t \in I,$$

$$\int_I \left\{ y(t)^T g(t,u,\dot{u}) - \frac{1}{2} p(t)^T G(t) p(t) \right\} dt \geq 0,$$

$$y(t) \geq 0.$$

where $F(t) = f_{uu} - Df_{u\dot{u}} + D^2 f_{\dot{u}\dot{u}}$ and $G(t) = (y(t)^T g_u)_u + D(y(t)^T g_u)_{\dot{u}} + D^2(y(t)^T g_{\dot{u}})_{\dot{u}}$ where D is defined as earlier.

3. **Mixed Type Second Order Duality**

In this section we construct a mixed type second-order dual model for the variational problem (VP):

$$(\textbf{MixVD}) \quad \text{Maximize} \int_I \left\{ f(t,u,\dot{u}) + \sum_{i\in I} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T H^0(t,u,\dot{u},y) p(t) \right\} dt$$

subject to $u(a) = 0 = u(b)$, (3)

$$f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) - D[f_{\dot{u}}(t,u,\dot{u}) + y(t)^T g_{\dot{u}}(t,u,\dot{u})] + H(t)p(t) = 0, \quad t \in I, \tag{4}$$

$$\int_I \left\{ \sum_{i\in I_\alpha} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T G(\alpha,t) p(t) \right\} dt \geq 0, \quad \alpha = 1,2,\ldots,r \tag{5}$$

$$y(t) \geq 0, \quad t \in I, \ p(t) \in R^n \tag{6}$$

where

(i) $H^0(t) = f_{uu} + \sum_{i\in I_0} (y^i(t) g^i_u(t,u,\dot{u}))_u - D\left(f_{u\dot{u}} + \sum_{i\in I_0} (y^i(t) g^i_u(t,u,\dot{u}))_{\dot{u}} \right) + D^2 \left(f_{\dot{u}\dot{u}} + \sum_{i\in I_0} (y^i(t) g^i_{\dot{u}}(t,u,\dot{u}))_{\dot{u}} \right)$

(ii) $G(\alpha,t) = \sum_{i\in I_\alpha} (y^i(t) g^i_u(t,u,\dot{u}))_u - D \sum_{i\in I_\alpha} (y^i(t) g^i_u(t,u,\dot{u}))_{\dot{u}} + D^2 \sum_{i\in I_\alpha} (y^i(t) g^i_{\dot{u}}(t,u,\dot{u}))_{\dot{u}}$

and

(iii) $I_\alpha \subset M = \{1,2,\ldots,m\}$, $\alpha = 0,1,2,\ldots,r$ with $M = \bigcup_{\alpha=0} I_\alpha$ and $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$.

We present the following duality theorems for the pair of dual problems (VP) and (MixVD).

Theorem 3.1 (*Weak Duality*)**.** *Let $x(t) \in X$ be a feasible solution of* (VP) *and $(u(t), y(t), p(t))$ be a feasible solution of* (MixVD). *If for all feasible $(x(t), u(t), y(t), p(t))$,*

$$\int_I \left\{ f(t,\cdot,\cdot) + \sum_{i\in I_0} (y^i(t) g^i(t,\cdot,\cdot)) \right\} dt$$

be second-order pseudo-invex and $\sum_{i\in I_0} \int_I (y^i(t) g^i(t,\cdot,\cdot))$ be second-order quasi-invex with respect to the same $\eta : I \times R^n \times R^n \to R^n$ satisfying $\eta = 0$ at $t = a$ and $t = b$, then

$$\int_I f(t,x,\dot{x}) dt \geq \int_I \left\{ f(t,u,\dot{u}) + \sum_{i\in I_0} (y^i(t) g^i(t,u,\dot{u})) - \frac{1}{2} p(t)^T H^0(t,u,\dot{u},y) p(t) \right\} dt.$$

Proof. The relation $g(t,x,\dot{x}) \leq 0$, $t \in I$ and $y(t) \geq 0$ yield

$$\int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,u,\dot{u}) \right\} dt \leq 0, \quad \alpha = 1,2,\ldots,r\,.$$

This together with (5) implies

$$\int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,x,\dot{x}) \right\} dt$$
$$\leq \int_I \left\{ \sum_{i \in I_\alpha} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T G(\alpha,t) p(t) \right\} dt, \quad \alpha = 1,2,\ldots,r$$

This, because of second-order quasi-invexity of

$$\int_I \left\{ \sum_{i \in I_0} y^i(t) g^i(t,\cdot,\cdot) \right\} dt, \quad \alpha = 1,2,\ldots,r,$$

gives

$$0 \geq \int_I \left\{ \eta^T \left(\sum_{i \in I_0} y^i(t) g^i_u \right) + (D\eta)^T \left(\sum_{i \in I_0} y^i(t) g^i_u \right) + \eta^T G(\alpha,t) p(t) \right\} dt$$
$$= \int_I \eta^T \left\{ \sum_{i \in I_0} y^i(t) g^i_u - D \sum_{i \in I_0} y^i(t) g^i_{\dot{u}} + G(\alpha,t) p(t) \right\} dt$$
$$+ \eta \sum_{i \in I_0} y^i(t) g^i_{\dot{u}}(t,u,\dot{u}) \Big|_{t=a}^{t=b}. \qquad \text{(by integration by part)}$$

Using $\eta(t,u,\dot{u})\Big|_{t=a}^{t=b} = 0$, we have

$$\int_I \eta^T \left\{ \sum_{i \in I_\alpha} y^i(t) g^i_u(t,u,\dot{u}) - D \sum_{i \in I_\alpha} y^i(t) g^i_{\dot{u}}(t,u,\dot{u}) + G(\alpha,t) p(t) \right\} dt \leq 0,$$
$$\alpha = 1,2,\ldots,r$$

Hence

$$\int_I \eta^T \left\{ \sum_{M-I_0} y^i(t) g^i_u(t,u,\dot{u}) - D \sum_{M-I_0} y^i(t) g^i_{\dot{u}}(t,u,\dot{u}) + G(\alpha,t) p(t) \right\} dt \leq 0$$

By (4), this yield

$$\int_I \eta^T \left\{ \left(f_u(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g^i_u(t,u,\dot{u}) \right) \right.$$
$$\left. - D \left(f_{\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g^i_u(t,u,\dot{u}) \right) + H^0(t) p(t) \right\} dt \geq 0.$$

Integrating by parts, this gives,

$$0 \leq \int_I \Big\{ \eta^T \Big(f_u(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g_u^i(t,u,\dot{u}) \Big) + (D\eta)^T \Big(f_{\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) \Big) + \eta^T H^0(t) p(t) \Big\} dt - \eta^T \Big(f_{\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) \Big) \Big|_{t=a}^{t=b}$$

Using $\eta = 0$ at $t = a$ and $t = b$ in the above inequality, we obtain

$$\int_I \Big\{ \eta^T \Big(f_u(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g_u^i(t,u,\dot{u}) \Big) + (D\eta)^T \Big(f_{\dot{u}}(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g_{\dot{u}}^i(t,u,\dot{u}) \Big) + \eta^T H^0(t) p(t) \Big\} dt \geq 0$$

This, in view of second-order invexity of $\int_I \Big\{ f(t,\cdot,\cdot) + \sum_{i \in I_0} y^i(t) g^i(t,\cdot,\cdot) \Big\} dt$ with respect to η gives

$$\int_I \Big\{ f(t,x,\dot{x}) + \sum_{i \in I_0} y^i(t) g^i(t,x,\dot{x}) \Big\} dt \geq \int_I \Big\{ f(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T H^0 p(t) \Big\} dt$$

Since $y(t) \geq 0,\ t \in I$ and $g(t,x,\dot{x}) \leq 0,\ t \in I$ yielding $\sum_{i \in I_0} y^i(t) g^i(t,x,\dot{x}) \leq 0$, $t \in I$, (7) gives

$$\int_I f(t,x,\dot{x}) dt \geq \int_I \Big\{ f(t,u,\dot{u}) + \sum_{i \in I_0} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2} p(t)^T H^0 p(t) \Big\} dt. \qquad \square$$

Theorem 3.2 (*Strong duality*). *If $\bar{x}(t) \in X$ is an optimal solution of* (VP) *and meets the normality condition, then there exists a piecewise smooth functions $\bar{y} : I \to R^m$ such that $(\bar{x}(t), \bar{y}(t), \bar{p}(t) = 0)$, $t \in I$ is a feasible for* (Mix VD) *and the two objective values are equal. Furthermore, if the hypothesis of* Theorem 1 *holds, then $(\bar{x}(t), \bar{y}(t), \bar{p}(t))$ is optimal for* (Mix VD).

Proof. From Proposition 1 of [2], there exist piecewise smooth functions $\bar{y} : I \to R^m$ satisfying the following conditions:

$$(f_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}})) - D(f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})) + H(t)\bar{p}(t) = 0, \quad t \in I \ \text{ with } \ \bar{p}(t) = 0$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0, \quad \bar{y}(t) \geq 0, \ t \in I.$$

The last relation implies

$$\sum_{i\in I_0} \bar{y}^i(t) g^i(t,\bar{x},\dot{\bar{x}}) = 0 = \sum_{i\in I_\alpha} \bar{y}^i(t) g^i(t,\bar{x},\dot{\bar{x}}), \quad \alpha = 1,2,\ldots,r$$

$$\int_I \left\{ \sum_{i\in I_\alpha} \bar{y}^i(t) g^i(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2} p(t)^T G(\alpha,t) p(t) \right\} = 0, \quad \alpha = 1,2,\ldots,r \quad \text{with } p(t) = 0$$

From the above relation it implies that $(\bar{x}(t), \bar{y}(t), p(t) = 0)$ is feasible for (Mix VD).

In view of $\sum_{i\in I_0} \bar{y}^i(t) g^i(t,\bar{x},\dot{\bar{x}}) = 0$, $t \in I$ and $p(t) = 0$, $t \in I$, we have

$$\int_I f(t,\bar{x},\dot{\bar{x}})\,dt = \int_I \left\{ f(t,\bar{x},\dot{\bar{x}}) + \sum_{i\in I_0} y^i(t) g^i(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2} p(t)^T H^0(t) p(t) \right\} dt$$

Furthermore, for every feasible solution $(u(t), y(t), p(t))$, from the condition we have,

$$\begin{aligned} &\int_I \left\{ f(t,\bar{x},\dot{\bar{x}}) + \sum_{i\in I_0} y^i(t) g^i(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2} p(t)^T H^0(t) p(t) \right\} dt \\ &= \int_I f(t,\bar{x},\dot{\bar{x}})\,dt \\ &\geq \int_I \left\{ f(t,u(t),\dot{u}(t)) + \sum_{i\in I_0} y^i(t) g^i(t,u(t),\dot{u}(t)) - \frac{1}{2} p(t)^T H^0(t) p(t) \right\} dt \end{aligned}$$

So, $(\bar{x}(t), \bar{y}(t), \bar{p}(t))$ is also an optimal solution of (MixVD). □

The theorem given below is the Mangasarian type converse duality theorem:

Theorem 3.3 (*Strict Converse duality*)**.** *Let* $\bar{x}$ *be an optimal solution of* (VP) *and normal. If* $(\widehat{u}, \widehat{y}, \widehat{p})$ *is an optimal solution to* (Mix VD) *and if* $\int_I \left\{ f(t,\cdot,\cdot) + \sum_{i\in I_0} y^i(t) g^i(t,\cdot,\cdot) \right\} dt$ *is second order strict pseudoinvex and* $\sum_{i\in I_\alpha} \int_I \widehat{y}^i(t) g^i(t,\cdot,\cdot)\,dt$*,* $(\alpha = 1,2,\ldots,r)$ *is a second-order quasi-invex with respect to* $\eta = \eta(t,\bar{x},\widehat{u})$*, then* $\bar{x} = \widehat{u}$*, i.e.* $\widehat{u}$ *is an optimal solution of* (VP).

Proof. We assume that $\bar{x}(t) \neq \widehat{u}(t)$, $t \in I$ and show that the contradiction occurs. Since $\bar{x}$ is an optimal solution of (VP) and normal, it follows from Theorem 2 that there exists piecewise smooth $\bar{y} : R \to R^m$ with $\bar{y}(t) = (\bar{y}^1(t), \bar{y}^2(t), \ldots, \bar{y}^m(t))^T$ such that $(\bar{x}(t), \bar{y}(t), \bar{p}(t))$ is optimal for (MixVD) and

$$\int_I \{f(t,\bar{x},\dot{\bar{x}})\}\,dt = \int_I \left\{ f(t,\bar{x},\dot{\bar{x}}) + \sum_{i\in I_0} \bar{y}^i(t) g^i(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2} \bar{p}(t)^T H^0(t) \bar{p}(t) \right\} dt$$

$$= \int_I \left\{ f(t, \widehat{u}, \dot{\widehat{u}}) + \sum_{i \in I_0} \widehat{y}^i(t) g^i(t, \widehat{u}, \dot{\widehat{u}}) - \frac{1}{2} \bar{p}(t)^T H^0(t) \bar{p}(t) \right\} dt \tag{7}$$

Since $\bar{x}(t)$ is feasible for (VD) and $\widehat{u}(t)$, $\widehat{y}(t)$, $\widehat{p}(t)$ is feasible for (MixVD), we have

$$\sum_{i \in I_\alpha} \int_I \widehat{y}^i(t) g^i(t, \bar{x}, \dot{\bar{x}}) \, dt \leq 0, \quad \alpha = 1, 2, \ldots, r$$

This, together with the feasibility $\widehat{u}(t), \widehat{y}(t), \widehat{p}(t)$ for the dual problem (MixVD)

$$\sum_{i \in I_\alpha} \int_I \widehat{y}^i(t) g^i(t, \bar{x}, \dot{\bar{x}}) \, dt$$

$$\leq \int_I \left\{ \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i(t, \widehat{u}, \dot{\widehat{u}}) - \frac{1}{2} \bar{p}(t)^T H^0(t) \bar{p}(t) \right\} dt, \quad (\alpha = 1, 2, \ldots, r)$$

This, in view of second-order quasi-invexity of

$$\sum_{i \in I_\alpha} \int_I \widehat{y}^i(t) g^i(t, \cdot, \cdot) \qquad (\alpha = 1, 2, \ldots, r)$$

gives

$$\int_I \left\{ \eta^T \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\widehat{u}}(t, \widehat{u}, \dot{\widehat{u}}) + (D\eta)^T \left(\sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\dot{\widehat{u}}}(t, \widehat{u}, \dot{\widehat{u}}) \right) + \eta^T G(\alpha, t) \widehat{p}(t) \right\} dt \leq 0.$$

This, by integration by parts, gives

$$0 \geq \int_I \left\{ \eta^T \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\widehat{u}}(t, \widehat{u}, \dot{\widehat{u}}) + (D\eta)^T \left(\sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\dot{\widehat{u}}}(t, \widehat{u}, \dot{\widehat{u}}) \right) + \eta^T G(\alpha, t) \widehat{p}(t) \right\} dt$$

$$+ \eta^T \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\dot{\widehat{u}}}(t, \widehat{u}, \dot{\widehat{u}}) \Big|_{t=a}^{t=b}$$

Using $\eta \Big|_{t=a}^{t=b} = 0$, this gives

$$\int_I \eta^T \left\{ \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\widehat{u}}(t, \widehat{u}, \dot{\widehat{u}}) + D \left(\sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\dot{\widehat{u}}}(t, \widehat{u}, \dot{\widehat{u}}) \right) + G(\alpha, t) \widehat{p}(t) \right\} dt \leq 0.$$

From (4) we have,

$$\int_I \eta^T \left\{ f_{\widehat{u}} + \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\widehat{u}}(t, \widehat{u}, \dot{\widehat{u}}) + D \left(f_{\dot{\widehat{u}}} + \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\dot{\widehat{u}}}(t, \widehat{u}, \dot{\widehat{u}}) \right) + H^0(t) \widehat{p}(t) \right\} dt \geq 0.$$

This inequality, by integration by parts, gives,

$$\int_I \left\{ \eta^T \left(f_{\widehat{u}} + \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\widehat{u}}(t, \widehat{u}, \dot{\widehat{u}}) \right) \right.$$

$$\left. + (D\eta)^T \left(f_{\dot{\widehat{u}}} + \sum_{i \in I_\alpha} \widehat{y}^i(t) g^i_{\dot{\widehat{u}}}(t, \widehat{u}, \dot{\widehat{u}}) \right) + \eta^T H^0(t) \widehat{p}(t) \right\} dt \geq 0.$$

which in view of second-order strict pseudoinvexity of

$$\int_I \left\{ f(t,\cdot,\cdot) + \sum_{i\in I_0} \widehat{y}^i g^i(t,\cdot,\cdot) \right\} dt$$

gives

$$\int_I \left\{ f(t,\bar{x},\dot{\bar{x}}) + \sum_{i\in I_0} \widehat{y}^i g^i(t,\bar{x},\dot{\bar{x}}) \right\} dt$$
$$> \int_I \left\{ f(t,\widehat{u},\dot{\widehat{u}}) + \sum_{i\in I_0} \widehat{y}^i g^i(t,\widehat{u},\dot{\widehat{u}}) - \frac{1}{2}\widehat{p}(t)^T H^0(t)\widehat{p}(t) \right\} dt$$

Using $\sum_{i\in I_0} \widehat{y}^i g^i(t,\bar{x},\dot{\bar{x}}) \le 0$, $t \in I$ this yields

$$\int_I \{f(t,\bar{x},\dot{\bar{x}})\}\, dt$$
$$> \int_I \left\{ f(t,\widehat{u},\dot{\widehat{u}}) + \sum_{i\in I_0} \widehat{y}^i g^i(t,\widehat{u},\dot{\widehat{u}}) - \frac{1}{2}\widehat{p}(t)^T H^0(t)\widehat{p}(t) \right\} dt$$

This contradicts the relation (7). Hence $\bar{x}(t) = \widehat{u}(t)$, $t \in I$ i.e. $\widehat{u}(t)$ is optimal solution of (VP).

4. **Special Cases**

If I_α is empty for each $\alpha \in 1,2,\ldots,r$, then $H^0(t) = H(t)$ (MixVD) reduces to the following Wolfe type second-order dual variational problem treated by Chen [3].

If I_0 is empty, then (MixVD) reduces to the following Mond-weir type second-order dual variational problem recently treated by Husain *et al.* [5].

5. **Natural Boundary Values**

In this section, we present dual variational problem with natural boundary values rather than fixed end points.

$(\mathbf{VP_0})$ Minimize $\int_I f(t,x(t),\dot{x}(t))\,dt$

subject to $g(t,x,\dot{x}) \le 0, \quad t \in I$

$(\mathbf{MixVD_0})$ Maximize $\int_I \left\{ f(t,u,\dot{u}) + \sum_{i\in I_0} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2}p(t)^T H^0(t) p(t) \right\} dt$

subject to

$$f_u(t,u,\dot{u}) + y(t)^T g_u(t,u,\dot{u}) - D(f_{\dot{u}} + y(t)^T g_{\dot{u}}) + Hp(t) = 0, \; t \in I$$

$$\int_I \left\{ \sum_{i\in I_0} y^i(t) g^i(t,u,\dot{u}) - \frac{1}{2}p(t)^T G(\alpha,t) p(t) \right\} dt \ge 0$$

$$\left(f_{\dot{u}}(t,u,\dot{u})+\sum_{i\in I_0}y^i(t)g_u^i(t,u,\dot{u})\right)\Bigg|_{t=a}^{t=b}=0$$

$$\sum_{i\in I_\alpha}y^i(t)g_u^i(t,u,\dot{u})\Bigg|_{t=a}^{t=b}=0,\quad \alpha=1,2,\ldots$$

6. Mixed Type Nonlinear Programming Problems

If all the functions are independent of t, then we have following pair of problems treated in Zhang and Mond [11] except that square root of a quadratic form is to be omitted from the expression of the problems.

($\mathbf{VP_0}$) Minimize $f(x)$

subject to $g(x)\leq 0$,

($\mathbf{MixCD_0}$) Maximize $f(u)+\sum_{i\in I_0}y^ig^i(u)-\frac{1}{2}p(t)^T\left[\nabla^2 f(u)+\sum_{i\in I_0}y^ig^i(u)\right]p$

subject to $\nabla\left(f(u)+\sum_{i\in I_0}y^ig^i(u)\right)+\nabla^2\left(f(u)+\sum_{i\in I_0}y^ig^i(u)\right)p=0$

$$\sum_{i\in I_\alpha}y^ig^i(u)-\frac{1}{2}p^T\left[\sum_{i\in I_\alpha}y^i(u)g^i(u)\right]p\geq 0$$

$y\geq 0$.

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