



Exact Solutions for Generalized Klein-Gordon Equation

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Abstract. The Modified $(\frac{G}{G'})$ -expansion method is used to solve the Generalized Klein-Gordon equation, by means of the method, three types of exact traveling wave solutions are obtained, including the hyperbolic functions, trigonometric and rational function solutions. In this method, G is general solution of a second order linear ODE, so the method is direct, simple; more importantly, this method can be used in many other nonlinear evolution equations to obtain traveling wave solutions. This will have a good sense to promote the broad application of Klein-Gordon equation.

1. Introduction

The progress of physics is heavily dependent on non-linear mathematics and methods for solving nonlinear equations [1]. In recent years, searching for explicit solutions of nonlinear evolution equations by using various different methods is the main goal for many researchers, and several powerful methods have been proposed to construct exact solutions for nonlinear partial differential equations, such as inverse scattering method [2], the Hirota's bilinear operators [3], homogeneous balance method [4], the hyperbolic tanh-function expansion and its various extension [5], Jacobian elliptic functions expansion method [6–8, 11], the F-expansion method [9, 10], $(\frac{G}{G'})$ -expansion method [12–14] and so on. At present, we found that many important nonlinear evolution equations have solitary wave solutions, such as Sin-Gordon equation, KdV equation, Schrodinger equation and so on. Therefore, the search for new forms of exact solutions is still a very meaningful work.

In this paper, we consider the Generalized Klein-Gordon equation in the form

$$u_{tt} - u_{xx} + \beta_1 u + \beta_3 u^3 = 0, \quad (1)$$

where β_1 and β_3 are constants.

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2. Description of the $(\frac{G}{G'})$ -expansion method

Consider a given nonlinear evolution equation, with variables x and t ,

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \dots) = 0, \quad (2)$$

where $u = u(x, t)$ is an unknown function, $P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \dots)$ is a polynomial with the variables u, u_x, u_t, \dots .

Step 1. We make the gauge transformation

$$u(x, t) = u(\xi), \quad \xi = x - kt, \quad (3)$$

where k is a nonzero constant to be determined later. Substituting eq. (3) into eq. (2) yields a complex ordinary differential equation of $u(\xi)$, namely

$$O(u, u', u'', \dots) = 0, \quad (4)$$

where $u' = \frac{du}{d\xi}$, $u'' = \frac{d^2u}{d\xi^2}$, $O(u, u', u'', \dots)$ is a polynomial with the variables u and u' .

Step 2. We assume the equation (4) has the solutions in the following form:

$$u(\xi) = a_0 + \sum_{i=1}^n a_i \left(\frac{G'}{G}\right)^i, \quad (5)$$

where a_i ($i = 0, 1, 2, \dots, n$) are constants to be determined later, the positive integer n can be determined by considering the homogeneous balance between governing nonlinear terms and the highest order derivatives of u in eq. (4) where $G = G(\xi)$ satisfies the second order LODE

$$G'' + \lambda G' + \mu G = 0, \quad (6)$$

where $G' = \frac{dG}{d\xi}$, $G'' = \frac{d^2G}{d\xi^2}$, and λ, μ are constants to be determined later solutions of eq. (6) as following:

Case 1. When $\lambda^2 - 4\mu > 0$, we have

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right), \quad (7)$$

where $\tanh \xi_0 = \frac{C_2}{C_1}$, $\left|\frac{C_2}{C_1}\right| > 1$.

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi + \xi_0\right), \quad (8)$$

where $\coth \xi_0 = \frac{C_2}{C_1}$, $\left|\frac{C_2}{C_1}\right| < 1$.

Case 2. When $\lambda^2 - 4\mu < 0$, we have

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi + \xi_0\right), \quad (9)$$

where $\tan \xi_0 = \frac{C_2}{C_1}$.

Case 3. When $\lambda^2 - 4\mu = 0$, we have

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}, \tag{10}$$

where C_1 and C_2 are constants.

Step 3. Substituting (5) into eq. (4), the left-hand side of eq. (4) is converted into another polynomial in $(\frac{G}{G'})$, collecting all terms with the same order of $(\frac{G}{G'})$ together, setting the coefficients of $(\frac{G'}{G})^i$, $(i = 1, 2, \dots, n)$ to zero, yields a set of nonlinear algebraic equations (NAEs) in $a_0, a_i, (i = 1, 2, \dots, n), k, \lambda, \mu$, solving the NAEs, we obtain the solution of NAEs.

Step 4. Substituting these results into (5), we can obtain several exact solutions of eq. (2).

3. The Exact Solutions to Generalized Klein-Gordon Equation

We introduce a gauge transformation for eq. (1), set

$$u(x, t) = u(\xi), \quad \xi = x - kt, \tag{11}$$

where k is a nonzero constant to be determined later. Substituting (11) into eq. (1), we have

$$(k^2 + 1)u'' + \beta_1 u + \beta_3 u^3 = 0. \tag{12}$$

Obviously, the balance constant $n = 1$. Therefore, we assume eq. (1) has the following solutions

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G} \right). \tag{13}$$

Substituting (13) and (6) into eq. (12), we have a polynomial equation in $(\frac{G}{G'})$. Setting the coefficients of $(\frac{G'}{G})^i$, $(i = 1, 2, \dots, n)$ to zero, yields a set of nonlinear algebraic equations (NAEs) in $a_0, a_i, (i = 1, 2, \dots, n), k, \lambda, \mu$, solving the NAEs by mathematic. we could determine the following solutions:

Case 1.

$$a_0 = 0, \quad a_1 = \sqrt{\frac{-2(k^2 + 1)}{\beta_3}}, \quad \mu = -\frac{\beta_1}{2(k^2 + 1)}, \quad \lambda = 0. \tag{14}$$

Substituting (14) into (13), we have

$$u_1 = \sqrt{\frac{-2(k^2 + 1)}{\beta_3}} \frac{G'}{G}(\xi). \tag{15}$$

Using (7)-(10), we have the following solutions of eq. (1).

(1) When $\lambda^2 - 4\mu > 0$, we have

$$u_{111} = \sqrt{\frac{-\beta_1}{\beta_3}} \tanh \left(\sqrt{\frac{\beta_1}{2(k^2+1)}} \xi + \xi_0 \right), \quad (16)$$

$$u_{112} = \sqrt{\frac{-\beta_1}{\beta_3}} \coth \left(\sqrt{\frac{\beta_1}{2(k^2+1)}} \xi + \xi_0 \right). \quad (17)$$

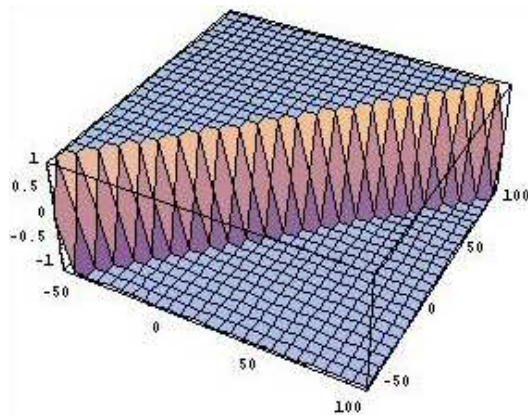


Figure 1. Digital simulation of u_{111}

(2) When $\lambda^2 - 4\mu < 0$, we have

$$u_{12} = \sqrt{\frac{\beta_1}{\beta_3}} \cot \left(\sqrt{\frac{-\beta_1}{2(k^2+1)}} \xi + \xi_0 \right). \quad (18)$$

(3) When $\lambda^2 - 4\mu = 0$, we have

$$u_{13} = \sqrt{\frac{-2(k^2+1)}{\beta_3}} \frac{C_2}{C_1 + C_2 \xi}. \quad (19)$$

Case 2.

$$a_0 = 0, \quad a_1 = -\sqrt{\frac{-2(k^2+1)}{\beta_3}}, \quad \mu = -\frac{\beta_1}{2(k^2+1)}, \quad \lambda = 0. \quad (20)$$

Substituting (20) into (13), we have

$$u_2 = -\sqrt{\frac{-2(k^2+1)}{\beta_3}} \frac{G'}{G}(\xi). \quad (21)$$

Using (7)-(10), we have the following solutions of eq. (1).

(1) When $\lambda^2 - 4\mu > 0$, we have

$$u_{211} = \sqrt{\frac{\beta_1}{\beta_3}} \tanh \left(\sqrt{\frac{\beta_1}{2(k^2+1)}} \xi + \xi_0 \right), \quad (22)$$

$$u_{212} = \sqrt{\frac{\beta_1}{\beta_3}} \coth \left(\sqrt{\frac{\beta_1}{2(k^2+1)}} \xi + \xi_0 \right). \tag{23}$$

(2) When $\lambda^2 - 4\mu < 0$, we have

$$u_{22} = -\sqrt{\frac{\beta_1}{\beta_3}} \cot \left(\sqrt{\frac{-\beta_1}{2(k^2+1)}} \xi + \xi_0 \right). \tag{24}$$

(3) When $\lambda^2 - 4\mu = 0$, we have

$$u_{23} = -\sqrt{\frac{-2(k^2+1)}{\beta_3}} \frac{C_2}{C_1 + C_2 \xi}. \tag{25}$$

Case 3.

$$a_0 = \sqrt{\frac{-(k^2+1)\lambda^2}{2\beta_3}}, \quad a_1 = \frac{1}{\lambda} a_0, \quad \mu = \frac{\lambda^2}{4} - \frac{\beta_1}{2(k^2+1)}. \tag{26}$$

Substituting (26) into (13), we have

$$u_3 = \sqrt{\frac{-(k^2+1)\lambda^2}{2\beta_3}} + \frac{1}{\lambda} \sqrt{\frac{-2(k^2+1)\lambda^2}{\beta_3}} \frac{G'}{G}(\xi). \tag{27}$$

Using (7)-(10), we have the following solutions of eq. (1).

(1) When $\lambda^2 - 4\mu > 0$, we have

$$u_{311} = \sqrt{\frac{(k^2+1)(-\lambda^2+4\mu)}{2\beta_3}} \tanh \left(\frac{\sqrt{\lambda^2-4\mu}}{2} \xi + \xi_0 \right), \tag{28}$$

$$u_{312} = \sqrt{\frac{(k^2+1)(-\lambda^2+4\mu)}{2\beta_3}} \coth \left(\frac{\sqrt{\lambda^2-4\mu}}{2} \xi + \xi_0 \right). \tag{29}$$

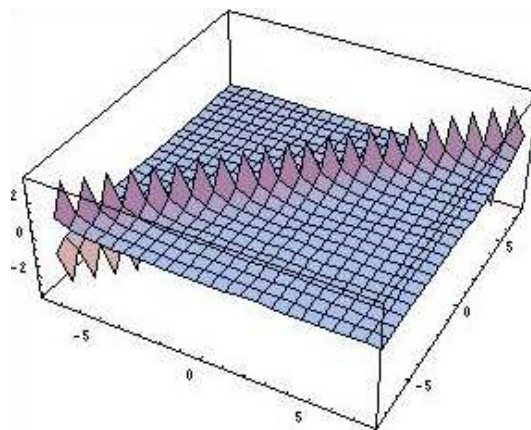


Figure 2. Digital simulation of u_{312}

(2) When $\lambda^2 - 4\mu < 0$, we have

$$u_{32} = \sqrt{\frac{(k^2 + 1)(\lambda^2 - 4\mu)}{2\beta_3}} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi + \xi_0\right). \tag{30}$$

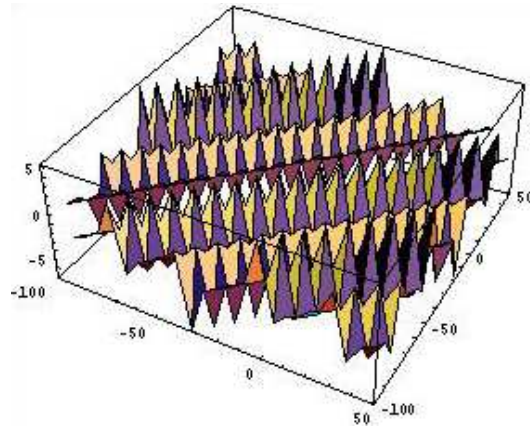


Figure 3. Digital simulation of u_{32}

(3) When $\lambda^2 - 4\mu = 0$, we have

$$u_{33} = -\sqrt{\frac{-2(k^2 + 1)}{\beta_3}} \frac{C_2}{C_1 + C_2 \xi}. \tag{31}$$

Case 4.

$$a_0 = -\sqrt{\frac{(k^2 + 1)\lambda^2}{2\beta_3}}, \quad a_1 = -\frac{1}{\lambda} a_0, \quad \mu = \frac{\lambda^2}{4} - \frac{\beta_1}{2(k^2 + 1)}. \tag{32}$$

Substituting (32) into (13), we have

$$u_4 = -\sqrt{\frac{-(k^2 + 1)\lambda^2}{2\beta_3}} - \frac{1}{\lambda} \sqrt{\frac{-2(k^2 + 1)\lambda^2}{\beta_3}} \frac{G'}{G}(\xi). \tag{33}$$

Using (7)-(10), we have the following solutions of eq. (1).

(1) When $\lambda^2 - 4\mu > 0$, we have

$$u_{411} = -\sqrt{\frac{(k^2 + 1)(-\lambda^2 + 4\mu)}{2\beta_3}} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \xi_0\right) - 2\sqrt{\frac{-(k^2 + 1)\lambda^2}{2\beta_3}}, \tag{34}$$

$$u_{412} = \sqrt{\frac{(k^2 + 1)(-\lambda^2 + 4\mu)}{2\beta_3}} \coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \xi_0\right) - \sqrt{\frac{-2(k^2 + 1)\lambda^2}{\beta_3}}. \tag{35}$$

(2) When $\lambda^2 - 4\mu < 0$, we have

$$u_{42} = \sqrt{\frac{(k^2 + 1)(\lambda^2 - 4\mu)}{2\beta_3}} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi + \xi_0\right) - \sqrt{\frac{-2(k^2 + 1)\lambda^2}{\beta_3}}. \tag{36}$$

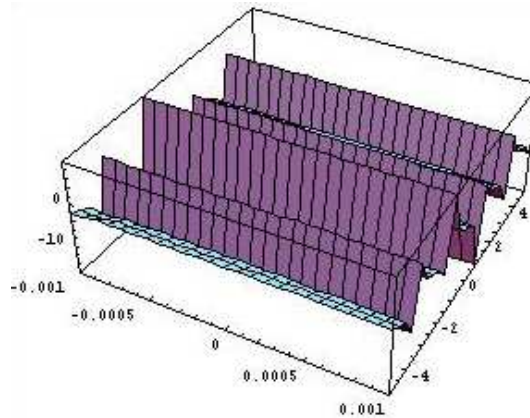


Figure 4. Digital simulation of u_{42}

(3) When $\lambda^2 - 4\mu = 0$, we have

$$u_{43} = -\sqrt{\frac{-2(k^2 + 1)}{\beta_3}} \frac{C_2}{C_1 + C_2\xi} - \sqrt{\frac{-2(k^2 + 1)\lambda^2}{\beta_3}}. \quad (37)$$

4. Conclusion

By using this modified $(\frac{G}{G'})$ -expansion method we obtained several exact solutions of Generalized Klein-Gordon equation, including the hyperbolic functions, trigonometric and rational function solutions. And we carry out numerical simulations for some solutions by math software. This method is concise and easy understand. Besides, the method can be used to other nonlinear evolution equations.

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