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Asymptotic Solutions of Fourth Order Near Critically Damped Nonlinear Systems

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Abstract. A perturbation technique is developed in this article to solve asymptotic solutions of fourth order near critically damped nonlinear systems based on the Krylov-Bogoliubov-Mitroplskii method. The method is illustrated by an example. The results obtained by the presented technique show excellent coincidence with those obtained by numerical method

1. **Introduction**

The Krylov-Bogoliulov-Mitropolskii (KBM) [9, 10] method is particularly convenient and extensively used technique for obtaining approximate solutions of weakly nonlinear differential systems. The method was primarily developed for periodic solutions of second order nonlinear systems. Later, the method was extended by Popov [13] to damped oscillatory systems. Murty and Deekshatulu [11] and Alam [5] extended the method to nonlinear over-damped systems. Sattar [14] investigated an asymptotic solution of a second order critically damped nonlinear system. Alam and Sattar [4] extended the KBM method for third order critically damped nonlinear systems. Alam [7] also examined the solution of third order nonlinear systems when two of the eigenvalues are almost equal (near critically damped) and the other is small. Later Alam [8] extended the technique presented in [7] for integral multiple roots. Akbar *et al.* [1] presented an asymptotic method for fourth order over-damped nonlinear systems, which is easier than the method presented by Murty *et al.* [12]. Later, Akbar *et al.* [2] extended the method presented in [1] for fourth order damped oscillatory systems. Akbar *et al.* [3] also extended the KBM method for solving fourth order more critically damped nonlinear systems.

In this article, we have investigated asymptotic solutions of fourth order near critically damped nonlinear systems with small nonlinearities based on the KBM

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method. The results obtained by the presented method show good agreement with the numerical results.

2. **The Method**

Let us consider the fourth order weakly nonlinear ordinary differential systems

$$\frac{d^4x}{dt^4} + c_1\frac{d^3x}{dt^3} + c_2\frac{d^2x}{dt^2} + c_3\frac{dx}{dt} + c_4x = -\varepsilon f(x), \tag{2.1}$$

where ε is a positive small parameter, $f(x)$ is the given nonlinear function and c_1, c_2, c_3, c_4 are constants, defined in terms of the eigen-values $-\lambda_i$, $i = 1,2,3,4$ of the unperturbed equation of (2.1) as

$$c_1 = \sum_{i=1}^{4}\lambda_i, \quad c_2 = \sum_{\substack{i,j=1\\ i\neq j}}^{4}\lambda_i\lambda_j, \quad c_3 = \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}\lambda_i\lambda_j\lambda_k \text{ and } c_4 = \prod_{i=1}^{4}\lambda_i .$$

When $\varepsilon = 0$, the equation (2.1) becomes linear and suppose the eigenvalues $-\lambda_1$ and $-\lambda_2$ are almost equal ($\lambda_1 \approx \lambda_2$) and other two eigenvalues $-\lambda_3$ and $-\lambda_4$ are distinct. Therefore, the unperturbed solution is

$$\begin{aligned} x(t,0) \;=\; & \frac{1}{2}a_{1,0}(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_{2,0}\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) \\ & + a_{3,0}e^{-\lambda_3 t} + a_{4,0}e^{-\lambda_4 t}, \end{aligned} \tag{2.2}$$

where $a_{i,0}$ $(i = 1,2,3,4)$ are arbitrary constants.

When $\varepsilon \neq 0$, following Alam's [7, 8] technique we choose the solution of (2.1) in the form

$$\begin{aligned} x(t,\varepsilon) \;=\; & \frac{1}{2}a_1(t)(e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2(t)\left(\frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + a_3(t)\,e^{-\lambda_3 t} \\ & + a_4(t)\,e^{-\lambda_4 t} + \varepsilon\, u_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \ldots, \end{aligned} \tag{2.3}$$

where a_i $(i = 1,2,3,4)$ satisfy the first order differential equation

$$\frac{da_i(t)}{dt} = \varepsilon A_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots, \quad (i = 1,2,3,4). \tag{2.4}$$

Confining only to a first few terms $1,2,3,\ldots,n$ in the series expansions (2.3) and (2.4), we calculate the functions u_i and A_i, $i = 1,2,3,4$ such that $a_i(t)$, $i = 1,2,3,4$ appearing in (2.3) and (2.4) satisfy the given differential equation (2.1) with an accuracy of order ε^{n+1}. To determine the unknown functions u_1, A_1, A_2, A_3, A_4 it is assumed (as customary in the KBM method) that the correction term, u_1 does not contain secular-type terms $te^{-\lambda_1 t}$, which make them large.

Differentiating equation (2.3) four times with respect t, substituting the derivatives $\frac{d^4x}{dt^4}$, $\frac{d^3x}{dt^3}$, $\frac{d^2x}{dt^2}$, $\frac{dx}{dt}$ and x in the original equation (2.1), utilizing the relations presented in (2.4) and finally equating the coefficients of ε, we obtain

$$\begin{aligned}
&\frac{1}{2}\begin{pmatrix} e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4) \\ +e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_4) \end{pmatrix} A_1 \\
&\quad +(D+\lambda_4)\left(e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D)+e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\right)A_2 \\
&\quad +e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3 \\
&\quad +e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4 \\
&\quad +\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)(D+\lambda_4)D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2 \\
&\quad -\left(\frac{\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2 \\
&\quad +(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1=-f^{(0)}, \qquad (2.5)
\end{aligned}$$

where

$$\begin{aligned}
&f^{(0)}=f(x_0) \quad \text{and} \\
&x_0=\frac{1}{2}a_1(t)(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2(t)\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right) \\
&\qquad +a_3(t)e^{-\lambda_3 t}+a_4(t)e^{-\lambda_4 t}.
\end{aligned}$$

It is assumed, in this article that the functional $f^{(0)}$ can be expanded in power series (Taylor's series) in the form (see also [7, 8] for details)

$$\begin{aligned}
f^{(0)} &= \sum_{r=0}^{n} F_r(a_3 e^{-\lambda_3 t}, a_4 e^{-\lambda_4 t}) \\
&\quad \times\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r, \qquad (2.6)
\end{aligned}$$

where n is the order of polynomial of the nonlinear function f. This assumption is certainly valid when f is a polynomial function of x. Such polynomial functions cover some special and important systems in mechanics. Following Alam's [6, 7, 8], in this article we assume that u_1 does not contain the terms F_0 and F_1 of $f^{(0)}$, since the system is considered to near critically damped. Substituting the value of $f^{(0)}$ from (2.6) into (2.5) and equating the coefficients of like powers of $\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)$, we obtain

$$\begin{aligned}
&e^{-\lambda_3 t}\left(D-\lambda_3+\lambda_1\right)\left(D-\lambda_3+\lambda_2\right)\left(D-\lambda_3+\lambda_4\right)A_3 \\
&\quad +e^{-\lambda_4 t}\left(D-\lambda_4+\lambda_1\right)\left(D-\lambda_4+\lambda_2\right)\left(D-\lambda_4+\lambda_3\right)A_4 \\
&\quad +\frac{1}{2}\left\{\begin{matrix} e^{-\lambda_1 t}\left(D-\lambda_1+\lambda_2\right)\left(D-\lambda_1+\lambda_3\right)\left(D-\lambda_1+\lambda_4\right) \\ +e^{-\lambda_2 t}\left(D-\lambda_2+\lambda_1\right)\left(D-\lambda_2+\lambda_3\right)\left(D-\lambda_2+\lambda_4\right) \end{matrix}\right\}A_1 \\
&\quad +(D+\lambda_4)\left\{e^{-\lambda_1 t}\left(\lambda_1-\lambda_3-\frac{3}{2}D\right)+e^{-\lambda_2 t}\left(\lambda_2-\lambda_3-\frac{3}{2}D\right)\right\}A_2
\end{aligned}$$

$$-\left(\frac{\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\times D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2$$
$$=-F_0-\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})F_1\,, \tag{2.7}$$

$$(D+\lambda_4)\times D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2=a_2F_1 \tag{2.8}$$

and

$$(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1$$
$$=-\sum_{r=2}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})$$
$$\times\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^r. \tag{2.9}$$

KBM [9, 10], Sattar [14], Alam [5, 6, 7, 8] imposed the condition that u_1 does not contain the fundamental terms (the solution presented in equation (2.2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. The solution of (2.8) gives value of the unknown function A_2. It is not easy to solve the equation (2.7) for the unknown functions A_1, A_3 and A_4, if the nonlinear function f and the eigenvalues $-\lambda_1$, $-\lambda_2$, $-\lambda_3$, $-\lambda_4$ of the corresponding linear equation of (2.1) are not specified. When these are specified the values of A_1, A_3 and A_4 can be found subject to the condition that the coefficients in the solutions of A_1, A_3 and A_4 do not become large (see Akbar *et al.* [3], Alam [7, 8] for details), as well as A_1, A_3 and A_4 do not contain terms involving te^{-t}. In the article, we have imposed the condition that the relation $\lambda_3 \approx 3\lambda_4$ exists between the eigenvalues λ_3 and λ_4 (and $\lambda_1 \to \lambda_2$ since the system is near critically damped). These relations are important, because under these relations the coefficients in the solution of A_1, A_3 and A_4 do not become large. Under these imposed conditions we obtain the values of A_1, A_3 and A_4 from equation (2.7). Substituting the values of A_1, A_2, A_3 and A_4 in the equation (2.4), we obtain the results of $\frac{da_i}{dt}$ $(i = 1,2,3,4)$, which are proportional to the small parameter so they are slowly varying functions of time t, that is, they are almost constants and by integrating, we obtain the values of a_i $(i = 1,2,3,4)$. It is laborious to solve (2.9) for u_1. However, as $\lambda_1 \to \lambda_2$ it takes simple form

$$(D+\lambda_1)^2(D+\lambda_3)(D+\lambda_4)u_1$$
$$=-\sum_{r=2}^{n}F_r(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\{e^{-\lambda_1 t}(a_1-a_2t)\}^r\,. \tag{2.10}$$

Solving equation (2.10), we obtain the value of u_1. Finally, substituting the values of a_i $(i = 1,2,3,4)$ and u_1 in the equation (2.3), we obtain the complete solution of (2.1).

3. **Example**

As an example of the above method, we consider the fourth order nonlinear differential equation

$$\frac{d^4x}{dt^4}+c_1\frac{d^3x}{dt^3}+c_2\frac{d^2x}{dt^2}+c_3\frac{dx}{dt}+c_4x=-\varepsilon x^3\,. \tag{3.1}$$

Here

$$f(x) = x^3 \quad \text{and}$$
$$x_0 = \frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)+a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t}.$$

Therefore,

$$f^{(0)}=\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)+a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t}\right\}^3. \tag{3.2}$$

Therefore, for example (3.1), the equations (2.7)-(2.9) respectively become

$$\begin{aligned}
&e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3\\
&\quad+e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4\\
&\quad+\frac{1}{2}\left\{\begin{matrix} e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)\\ +e^{-\lambda_2 t}(D-\lambda_2+\lambda_1)(D-\lambda_2+\lambda_3)(D-\lambda_2+\lambda_3)\end{matrix}\right\}A_1\\
&\quad+(D+\lambda_4)\left\{e^{-\lambda_1 t}(\lambda_1-\lambda_3-\frac{3}{2}D)+e^{-\lambda_2 t}(\lambda_2-\lambda_3-\frac{3}{2}D)\right\}A_2\\
&\quad-\left(\frac{(\lambda_1e^{-\lambda_1 t}-\lambda_2e^{-\lambda_2 t})}{(\lambda_1-\lambda_2)}\right)\times D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2\\
&\qquad=-\left[(a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t})^3+3(a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t})^2\right.\\
&\qquad\qquad\left.\cdot\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})\right],
\end{aligned} \tag{3.3}$$

$$(D+\lambda_4)\times D\left(D+\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)A_2=-3a_2(a_3e^{-\lambda_3 t}+a_4e^{-\lambda_4 t})^2 \tag{3.4}$$

and

$$\begin{aligned}
&(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4)u_1\\
&\quad=-\left[3(a_3e^{-\lambda_3 t},a_4e^{-\lambda_4 t})\left\{\frac{1}{2}a_1(e^{-\lambda_t t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}\right.\\
&\qquad\left.+\left\{\frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\right\}^3\right].
\end{aligned} \tag{3.5}$$

Solving equation (3.4), we obtain

$$A_2=a_2[n_1a_3^2e^{-2\lambda_3 t}+n_2a_3a_4e^{-(\lambda_3+\lambda_4)t}+n_3a_4^2e^{-2\lambda_4 t}], \tag{3.6}$$

where $n_1=\frac{3}{\lambda_3(\lambda_1+\lambda_2+2\lambda_3)(2\lambda_3-\lambda_4)}$, $n_2=\frac{12}{\lambda_3(\lambda_3+\lambda_4)(\lambda_1+\lambda_2+2\lambda_4)}$, $n_3=\frac{3}{\lambda_4^2(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)}$.

Now substituting the value of A_2 from (3.6) into (3.3) and in order to separate the equation (3.3) for determining the unknown functions A_1,A_3 and A_4, we use the conditions as discussed in the method (see also Akbar *et al.* [3], Alam [6, 7, 8]. It is interesting to note that our solution approaches toward critically damped solution (see Alam [8]) if $\lambda_1 \to \lambda_2$). However, equation (3.3) has not an exact solution unless $\lambda_1 \to \lambda_2$. Under these imposed conditions and by equating like terms on both sides of the equation (3.3), we obtain

$$\begin{aligned}
&e^{-\lambda_1 t}(D-\lambda_1+\lambda_2)(D-\lambda_1+\lambda_3)(D-\lambda_1+\lambda_4)A_1\\
&\quad = -a_2 a_3^2 n_1 \lambda_2\lambda_3(\lambda_1+\lambda_2+2\lambda_3)te^{-(\lambda_1+2\lambda_3)t}\\
&\quad\quad -\frac{1}{2}a_2 a_3 a_4 n_2 \lambda_2(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4)te^{-(\lambda_1+\lambda_3+\lambda_4)t}\\
&\quad\quad -a_2 a_4^2 n_3 \lambda_2\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)te^{-(\lambda_1+2\lambda_4)t}
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&e^{-\lambda_3 t}(D-\lambda_3+\lambda_1)(D-\lambda_3+\lambda_2)(D-\lambda_3+\lambda_4)A_3\\
&\quad = \left[a_2 n_1\{(\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)+\lambda_3(\lambda_1+\lambda_2+2\lambda_3)\}-\frac{3}{2}a_1\right]\\
&\quad\quad \times a_3^2 e^{-(\lambda_1+2\lambda_3)t}+\left[\frac{1}{2}a_2 n_2\{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3+3\lambda_4)\right.\\
&\quad\quad \left.+(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4)\}-3a_1\right]a_3\,a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t}\\
&\quad\quad +\left[a_2 n_3\{(\lambda_1+\lambda_4)(\lambda_1-\lambda_3+3\lambda_4)+\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)\}-\frac{3}{2}a_1\right]\\
&\quad\quad \times a_4^2 e^{-(\lambda_1+2\lambda_4)t}+\left[a_2 n_1(\lambda_2+2\lambda_3)(\lambda_2+2\lambda_3-\lambda_4)-\frac{3}{2}a_1\right]a_3^2 e^{-(\lambda_2+2\lambda_3)t}\\
&\quad\quad +\left[\frac{1}{2}a_2 n_2(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3+3\lambda_4)-3a_1\right]a_3\,a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t}\\
&\quad\quad +\left[a_2 n_3(\lambda_2+\lambda_4)(\lambda_2-\lambda_3+3\lambda_4)-\frac{3}{2}a_1\right]a_4^2 e^{-(\lambda_2+2\lambda_4)t}\\
&\quad\quad -[a_3^3 e^{-3\lambda_3 t}+3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t}+3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t}+a_4^3 e^{-3\lambda_4 t}]
\end{aligned} \tag{3.8}$$

and

$$e^{-\lambda_4 t}(D-\lambda_4+\lambda_1)(D-\lambda_4+\lambda_2)(D-\lambda_4+\lambda_3)A_4=0\,. \tag{3.9}$$

The particular solutions of equations (3.7)-(3.9) yield respectively

$$\begin{aligned}
A_1 =\;& I_1 a_2 a_3^2 te^{-(\lambda_1-\lambda_2+2\lambda_3)t}+I_2 a_2 a_3^2 te^{-(\lambda_1-\lambda_2+2\lambda_3)t}\\
&+I_3 a_2 a_3 a_4 te^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t}+I_4 a_2 a_3 a_4 te^{-(\lambda_1-\lambda_2+\lambda_3+\lambda_4)t}\\
&+I_5 a_2 a_4^2 te^{-(\lambda_1-\lambda_2+2\lambda_4)t}+I_6 a_2 a_4^2 te^{-(\lambda_1-\lambda_2+2\lambda_4)t}\,,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
A_3 =\;& (M_1 a_2+M_2 a_1)a_3^2 e^{-(\lambda_1+\lambda_3)t}+(M_3 a_2+M_4 a_1)a_3 a_4 e^{-(\lambda_1+\lambda_4)t}\\
&+(M_5 a_2+M_6 a_1)a_4^2 e^{-(\lambda_1+2\lambda_4-\lambda_3)t}+(M_7 a_2+M_8 a_1)a_3^2 e^{-(\lambda_2+\lambda_3)t}\\
&+(M_9 a_2+M_{10} a_1)a_3 a_4 e^{-(\lambda_2+\lambda_4)t}+(M_{11} a_2+M_{12} a_1)a_4^2 e^{-(\lambda_2+2\lambda_4-\lambda_3)t}\\
&+M_{13} a_3^3 e^{-2\lambda_3 t}+M_{14} a_3^2 a_4 e^{-(\lambda_3+\lambda_4)t}+M_{15} a_3 a_4^2 e^{-2\lambda_4 t}\\
&+M_{16} a_4^3 e^{-(3\lambda_4-\lambda_3)t}
\end{aligned} \tag{3.11}$$

and

$$A_4 = 0 \tag{3.12}$$

where

$$\begin{aligned}
r_1 &= -n_1\lambda_2\lambda_3(\lambda_1+\lambda_2+2\lambda_3),\\
r_2 &= -\frac{1}{2}n_2\lambda_2(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4),\\
r_3 &= -n_3\lambda_2\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4),\\
I_1 &= -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)},\\
I_2 &= -\frac{r_1}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)}\left(\frac{1}{2\lambda_3}+\frac{1}{(\lambda_1+\lambda_3)}+\frac{1}{(\lambda_1+2\lambda_3-\lambda_4)}\right),\\
I_3 &= -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)},\\
I_4 &= -\frac{r_2}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_3+\lambda_4)}\left(\frac{1}{(\lambda_1+\lambda_3)}+\frac{1}{(\lambda_1+\lambda_4)}+\frac{1}{(\lambda_3+\lambda_4)}\right),\\
I_5 &= -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)},\\
I_6 &= -\frac{r_3}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_3)}\left(\frac{1}{2\lambda_4}+\frac{1}{(\lambda_1+\lambda_4)}+\frac{1}{(\lambda_1+2\lambda_4-\lambda_3)}\right).\\
m_1 &= n_1\left\{(\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3-\lambda_4)+\lambda_3(\lambda_1+\lambda_2+2\lambda_3)\right\}, \qquad m_2=-\frac{3}{2},\\
l_1 &= \frac{1}{2}n_2\begin{cases}(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3+3\lambda_4)\\ +(2\lambda_4^2+2\lambda_3\lambda_4+\lambda_1\lambda_3+\lambda_2\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_4)\end{cases}, \qquad l_2=-3,\\
p_1 &= n_3\left\{(\lambda_1+\lambda_4)(\lambda_1-\lambda_3+3\lambda_4)+\lambda_4(\lambda_1+\lambda_2-2\lambda_3+4\lambda_4)\right\}, \qquad p_2=-\frac{3}{2},\\
q_1 &= n_1(\lambda_2+2\lambda_3)(\lambda_2+2\lambda_3-\lambda_4), \qquad q_2=-\frac{3}{2},\\
h_1 &= \frac{1}{2}n_2(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3+3\lambda_4), \qquad h_2=-3,\\
s_1 &= n_3(\lambda_2+\lambda_4)(\lambda_2-\lambda_3+3\lambda_4), \qquad s_2=-\frac{3}{2},\\
M_1 &= -\frac{m_1}{2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)},\\
M_2 &= -\frac{m_2}{2\lambda_3(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)},\\
M_3 &= -\frac{l_1}{(\lambda_3+\lambda_4)(\lambda_1+\lambda_3+\lambda_4-\lambda_2)(\lambda_1+\lambda_3)},\\
M_4 &= -\frac{l_2}{(\lambda_3+\lambda_4)(\lambda_1+\lambda_3+\lambda_4-\lambda_2)(\lambda_1+\lambda_3)},\\
M_5 &= -\frac{p_1}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_2)},
\end{aligned}$$

$$
\begin{aligned}
M_6 &= -\frac{p_2}{2\lambda_4(\lambda_1+\lambda_4)(\lambda_1+2\lambda_4-\lambda_2)},\\
M_7 &= -\frac{q_1}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)},\\
M_8 &= -\frac{q_2}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)},\\
M_9 &= -\frac{h_1}{(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)(\lambda_2+\lambda_3+\lambda_4-\lambda_1)},\\
M_{10} &= -\frac{h_2}{(\lambda_2+\lambda_3)(\lambda_3+\lambda_4)(\lambda_2+\lambda_3+\lambda_4-\lambda_1)},\\
M_{11} &= -\frac{s_1}{2\lambda_4(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4-\lambda_1)},\\
M_{12} &= -\frac{s_2}{2\lambda_4(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4-\lambda_1)},\\
M_{13} &= \frac{1}{(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(3\lambda_3-\lambda_4)},\\
M_{14} &= \frac{3}{2\lambda_3(3\lambda_3+\lambda_4-\lambda_1)(2\lambda_3+\lambda_4-\lambda_2)},\\
M_{15} &= \frac{3}{(\lambda_3+\lambda_4)(2\lambda_4+\lambda_3-\lambda_1)(2\lambda_4+\lambda_3-\lambda_2)},\\
M_{16} &= \frac{1}{2\lambda_4(3\lambda_4-\lambda_1)(3\lambda_4-\lambda_2)}.
\end{aligned}
$$

The solution of the equation (3.5) is

$$
\begin{aligned}
u_1 &= -3a_3e^{-(2\lambda_1+\lambda_3)t}[d_0a_1^2+2d_1a_1a_2+d_3a_2^2+(d_2a_2^2-2d_0a_1a_2)\,t+d_0a_2^2t^2]\\
&\quad -3a_4e^{-(2\lambda_1+\lambda_4)t}[d_4a_1^2+2d_5a_1a_2+d_7a_2^2+(d_6a_2^2-2d_4a_1a_2)\,t+d_4a_2^2t^2]\\
&\quad -e^{-3\lambda_1 t}[d_8a_1^3+3a_1a_2(d_9a_1+d_{10}a_2)+d_{12}a_2^3+a_2(d_{11}a_2^2-3\,d_8a_1^2\\
&\quad -6d_9a_1a_2)\,t+3a_2^2(d_8a_1+d_9a_2)\,t^2-d_8\,a_2^3t^3] \qquad (3.13)
\end{aligned}
$$

where

$$
\begin{aligned}
d_0 &= \frac{1}{2\lambda_1(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_4)},\\
d_1 &= -\frac{1}{2\lambda_1(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_4)}\left(\frac{1}{2\lambda_1}+\frac{2}{(\lambda_1+\lambda_3)}+\frac{1}{(2\lambda_1+\lambda_3-\lambda_4)}\right),\\
d_2 &= \frac{1}{2\lambda_1(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_3)}+\frac{2}{(2\lambda_1+\lambda_3-\lambda_4)}\right),\\
d_3 &= \frac{1}{2\lambda_1(\lambda_1+\lambda_3)^2(2\lambda_1+\lambda_3-\lambda_4)}\times\left[\frac{1}{2\lambda_1^2}+\frac{2}{\lambda_1(\lambda_1+\lambda_3)}+\frac{6}{(\lambda_1+\lambda_3)^2}\right.\\
&\quad \left.+\frac{2}{(2\lambda_1+\lambda_3-\lambda_4)^2}+\frac{1}{(2\lambda_1+\lambda_3-\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_3)}\right)\right],
\end{aligned}
$$

$$d_4 = \frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)},$$

$$d_5 = -\frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)}\left(\frac{1}{2\lambda_1}+\frac{2}{(\lambda_1+\lambda_4)}+\frac{1}{(2\lambda_1-\lambda_3+\lambda_4)}\right),$$

$$d_6 = \frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_4)}+\frac{2}{(2\lambda_1-\lambda_3+\lambda_4)}\right),$$

$$d_7 = \frac{1}{2\lambda_1(\lambda_1+\lambda_4)^2(2\lambda_1-\lambda_3+\lambda_4)}\times\left[\frac{1}{2\lambda_1^2}+\frac{2}{\lambda_1(\lambda_1+\lambda_4)}+\frac{6}{(\lambda_1+\lambda_4)^2}\right.$$
$$\left.+\frac{2}{(2\lambda_1-\lambda_3+\lambda_4)^2}+\frac{1}{(2\lambda_1-\lambda_3+\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{4}{(\lambda_1+\lambda_4)}\right)\right],$$

$$d_8 = \frac{1}{4\lambda_1^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)},$$

$$d_9 = -\frac{1}{4\lambda_1^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\left(\frac{1}{\lambda_1}+\frac{1}{3\lambda_1-\lambda_3}+\frac{1}{3\lambda_1-\lambda_4}\right),$$

$$d_{10} = \frac{1}{4\lambda_1^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\left[\frac{2}{(3\lambda_1-\lambda_3)^2}+\frac{2}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\right.$$
$$\left.+\frac{2}{(3\lambda_1-\lambda_4)^2}+\frac{2}{(3\lambda_1-\lambda_3)}+\frac{2}{(3\lambda_1-\lambda_4)}+\frac{3}{2\lambda_1^2}\right],$$

$$d_{11} = -\frac{1}{4\lambda_1^2(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\left[\frac{6}{(3\lambda_1-\lambda_3)^2}+\frac{6}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\right.$$
$$\left.+\frac{9}{2\lambda_1^2}+\frac{6}{(3\lambda_1-\lambda_4)^2}+\frac{6}{\lambda_1(3\lambda_1-\lambda_3)}+\frac{6}{\lambda_1(3\lambda_1-\lambda_4)}\right],$$

$$d_{12} = -\left[\frac{6}{(3\lambda_1-\lambda_3)^3}+\frac{6}{(3\lambda_1-\lambda_3)^2(3\lambda_1-\lambda_4)}+\frac{6}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)^2}\right.$$
$$+\frac{6}{(3\lambda_1-\lambda_4)^3}+\frac{1}{\lambda_1}\left\{\frac{6}{(3\lambda_1-\lambda_3)^2}+\frac{6}{(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)}\right.$$
$$\left.\left.+\frac{6}{(3\lambda_1-\lambda_4)^2}\right\}+\frac{3}{2\lambda_1^2}\left(\frac{3}{3\lambda_1-\lambda_3}+\frac{3}{3\lambda_1-\lambda_4}\right)+\frac{3}{\lambda_1^3}\right].$$

Putting the values of A_1,A_2,A_3 and A_4 from equations (3.10), (3.6), (3.11), (3.12) into equation (2.4) and integrating, we obtain

$$a_1(t) = a_{1,0}+\varepsilon\left[a_{2,0}a_{3,0}^2\frac{\left\{\begin{array}{l}I_2(1-e^{(-\lambda_1+\lambda_2-2\lambda_3)t})\\ -I_1\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_3)t}+\frac{e^{(-\lambda_1+\lambda_2-2\lambda_3)t}-1}{\lambda_1-\lambda_2+2\lambda_3}\right)\end{array}\right\}}{(\lambda_1-\lambda_2+2\lambda_3)}\right.$$

$$+a_{2,0}\,a_{3,0}\,a_{4,0}\frac{\left\{\begin{array}{l} I_4(1-e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}) \\ -I_3\left(t\,e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}+\frac{e^{(-\lambda_1+\lambda_2-\lambda_3-\lambda_4)t}-1}{\lambda_1-\lambda_2+\lambda_3+\lambda_4}\right)\end{array}\right\}}{(\lambda_1-\lambda_2+\lambda_3+\lambda_4)}$$

$$+a_{2,0}\,a_{4,0}^2\frac{\left\{\begin{array}{l} I_6(1-e^{(-\lambda_1+\lambda_2-2\lambda_4)t}) \\ -I_5\left(t\,e^{(-\lambda_1+\lambda_2-2\lambda_4)t}+\frac{e^{(-\lambda_1+\lambda_2-2\lambda_4)t}-1}{\lambda_1-\lambda_2+2\lambda_4}\right)\end{array}\right\}}{(\lambda_1-\lambda_2+2\lambda_4)}\Bigg]$$

$$\begin{aligned}
a_2(t) &= a_{2,0}+\varepsilon\,a_{2,0}\left[n_1 a_{3,0}^2\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right)\right. \\
&\left.+n_2\,a_{3,0}\,a_{4,0}\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right)+n_3 a_{4,0}^2\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right)\right],\\
a_3(t) &= a_{3,0}+\varepsilon\left[a_{3,0}^2\{M_1 a_{2,0}+M_2\,a_{1,0}\}\left(\frac{1-e^{-(\lambda_1+\lambda_3)t}}{\lambda_1+\lambda_3}\right)\right.\\
&+a_{3,0}\,a_{4,0}\{M_3 a_{2,0}+M_4 a_{1,0}\}\left(\frac{1-e^{-(\lambda_1+\lambda_4)t}}{\lambda_1+\lambda_4}\right)\\
&+a_{4,0}^2\{M_5\,a_{2,0}+M_6\,a_{1,0}\}\left(\frac{1-e^{-(\lambda_1-\lambda_3+2\lambda_4)t}}{\lambda_1-\lambda_3+2\lambda_4}\right)\\
&+a_{3,0}^2\{M_7\,a_{2,0}+M_8\,a_{1,0}\}\left(\frac{1-e^{-(\lambda_2+\lambda_3)t}}{\lambda_2+\lambda_3}\right)\\
&+a_{3,0}\,a_{4,0}\{M_9\,a_{2,0}+M_{10}\,a_{1,0}\}\left(\frac{1-e^{-(\lambda_2+\lambda_4)t}}{\lambda_2+\lambda_4}\right)\\
&+a_{4,0}^2\{M_{11}a_{2,0}+M_{12}a_{1,0}\}\left(\frac{1-e^{-(\lambda_2-\lambda_3+2\lambda_4)t}}{\lambda_2-\lambda_3+2\lambda_4}\right)\\
&+a_{3,0}^3 M_{13}\left(\frac{1-e^{-2\lambda_3 t}}{2\lambda_3}\right)+a_{3,0}^2\,a_{4,0}M_{14}\left(\frac{1-e^{-(\lambda_3+\lambda_4)t}}{\lambda_3+\lambda_4}\right)\\
&\left.+a_{3,0}\,a_{4,0}^2 M_{15}\left(\frac{1-e^{-2\lambda_4 t}}{2\lambda_4}\right)+a_{4,0}^3 M_{16}\left(\frac{1-e^{-(3\lambda_4-\lambda_3)t}}{3\lambda_4-\lambda_3}\right)\right]
\end{aligned} \tag{3.14}$$

and

$$a_4(t)=a_{4,0}\,.$$

Therefore, we obtain the first approximate solution of the equation (3.1) is

$$\begin{aligned}
x(t,\varepsilon) &= \frac{1}{2}a_1(e^{-\lambda_1 t}+e^{-\lambda_2 t})+a_2\left(\frac{e^{-\lambda_1 t}-e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right)\\
&+a_3\,e^{-\lambda_3 t}+a_4\,e^{-\lambda_4 t}+\varepsilon\,u_1\,,
\end{aligned} \tag{3.15}$$

where a_1, a_2, a_3, a_4 are given by the equation (3.14) and u_1 is given by the equation (3.13).

4. **Results and Discussion**

In this article, an analytical approximate solution of fourth order non-oscillatory nonlinear systems has been found based on the KBM method. In order to test the accuracy of an approximate analytical solution obtained by a certain perturbation technique, we compared the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this article, we refer the work of Murty *et al.* [12]. First, $x(t,\varepsilon)$ is calculated by (3.15) by using the imposed conditions that $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$ in which a_1, a_2, a_3, a_4 are calculated by the equation (3.14) and u_1 is calculated by the equation (3.13) for different sets of initial conditions and for various values of t. The corresponding numerical solution of (3.1) is also computed by fourth order Runge-Kutta method. The approximate analytic solution and numerical solutions are plotted in the figure Figure 1 and Figure 2. From the figures we observe that the analytical solution and the numerical solution show excellent agreement.

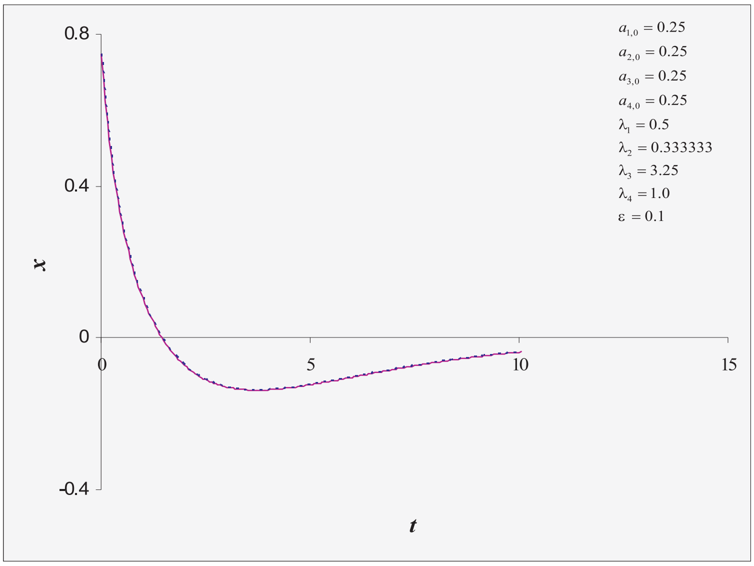

Figure 1. Comparison between analytical solution and numerical solution for chosen values of eigenvalues and small parameter. The analytical solution is denoted by doted line - - - and the numerical solution is denoted by solid line —-

5. **Conclusion**

An asymptotic method, based on the theory of Krylov-Bogoliubov-Mitropolskii, is developed for solving the fourth order near critically damped nonlinear systems

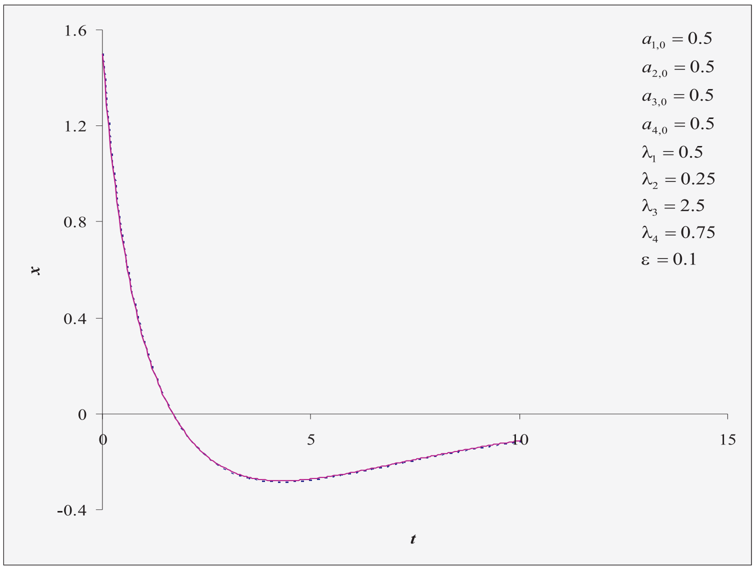

Figure 2. Comparison between analytical solution and numerical solution for chosen values of eigenvalues and small parameter. The analytical solution is denoted by doted line --- and the numerical solution is denoted by solid line —-

under some conditions with small nonlinearities, when the four eigenvalues of the corresponding linear equation are real and negative numbers. The relations $\lambda_1 \to \lambda_2$ and $\lambda_3 \approx 3\lambda_4$ among the eigenvalues are imposed to solve the systems. The results obtained by this method agree with those obtained by the numerical method.

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