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Pedagogical Article

Bloch Equations Versus Balance Equation: From F. Bloch to A. Einstein

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Abstract. We investigate the correspondence between balance equation for population inversion of *Two-Level System* (TLS) and equations for optical Bloch vector which describe the interaction of electromagnetic field and TLS. Based on an analytical treatment, the condition for such a correspondence is established and an expression is obtained for the rate of the radiative transition. Particularly, the problem of applicability of the perturbation theory when writing the balance equation, which also describes a strong change in the population inversion, is resolved.

Keywords. Bloch equations, Balance equation, Population inversion, Optical Bloch vector, Two-level system

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1. Introduction

The purpose of this article is to identify the correspondence between two widely used methods for describing the interaction of electromagnetic radiation and matter: *balance equations* and *Bloch equations*. In addition, we will determine the condition for such a correspondence and, along the way, obtain an expression for the probability per unit time of a radiative transition induced by an electromagnetic field.

2. Two-Level System and Balance Equation

Two-Level System (TLS) model is the basic one when considering a wide range of electromagnetic interactions. Figure 1 shows a diagram of a two-level system with eigenfrequency ω_0 and dipole moment $d_0 \neq 0$ (dipole-allowed transition). Strictly speaking TLS model is valid for resonance case when the frequency of electromagnetic field ω is close to eigenfrequency: $|\omega - \omega_0| \ll \omega_0$. However, it also applies in a more general case, if perturbation theory is valid.

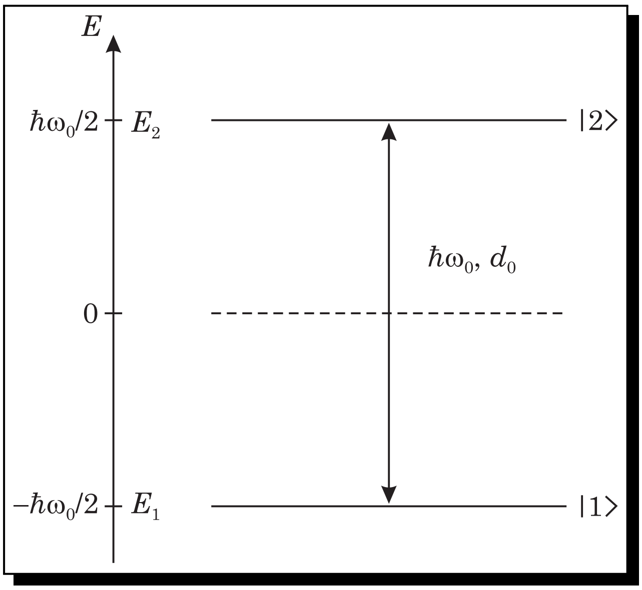

Figure 1. Two-level system with dipole-allowed transition: $|1\rangle$ and $|2\rangle$ are stationary states, ω_0, $d_0 \neq 0$ are eigenfrequency and dipole moment of TSL

The interaction of a two-level system with thermal radiation was first considered by A. Einstein in 1916 in his classical article [1], in which the concept of stimulated radiation was introduced into quantum physics and the relationships between the Einstein coefficients for spontaneous and stimulated radiation were determined. In the cited paper, a balance equation was written for the populations of two levels of a quantum system ($N_{1,2}$), transitions between which were caused by thermal and spontaneous radiation using phenomenological approach.

Subsequently, this equation was adapted to describe the interaction of laser radiation with a dipole-allowed transition (or TLS). It can be written in the form

$$\dot{N} + \frac{N - N_e}{T_1} = -2W_i N, \tag{2.1}$$

here $N = N_2 - N_1$ is population inversion at the time instant t, $N_e = \text{const}$ is equilibrium population inversion, T_1 is the time of population (or energy) relaxation, W_i is the rate of transitions (up and down along the energy scale – see Figure 1) induced by laser radiation. Particularly, equation (2.1) together with the second balance equation for the number of photons (which we do not consider here) is widely used for description of laser generation.

At the same time, it must be emphasized that the balance equation (2.1) has a limited range of applicability, since it does not take into account the phase relations between radiation and matter (TLS). Thus, it cannot be used to describe a number of coherent radiative processes such as free induction decay, self-induced transparency, photon echo in which phase relations play an important role.

3. Optical Bloch Vector

To describe coherent optical phenomena, a very convenient and physically clear way is the formalism of the optical Bloch vector. The Bloch vector was first used when considering magnetic resonance [2], when this vector is a vector of three-dimensional physical space, and its Cartesian projections correspond to the projections of the magnetic field. In the case of an electric dipole transition (what corresponds to the optical frequency range), the components of the *optical* Bloch vector are a bilinear form composed of the coefficients $a_{1,2}$ of the expansion of the wave function of the two-level system $|t\rangle$ in terms of the basis wave functions of the stationary states $|1\rangle$, $|2\rangle$ (we use the Dirac notation for the state of a quantum system so called ket-vector $|\ldots\rangle$):

$$|t\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle\,. \tag{3.1}$$

The Cartesian components of the optical Bloch vector are defined by the equalities

$$R_1 = a_1 a_2^* + a_1^* a_2 = 2Re\{a_1 a_2^*\}, \tag{3.2a}$$

$$R_2 = i(a_1 a_2^* - a_1^* a_2) = -2Im\{a_1 a_2^*\}, \tag{3.2b}$$

$$R_3 = |a_1|^2 - |a_2|^2\,. \tag{3.2c}$$

Let's find out the physical meaning of the Bloch vector components.

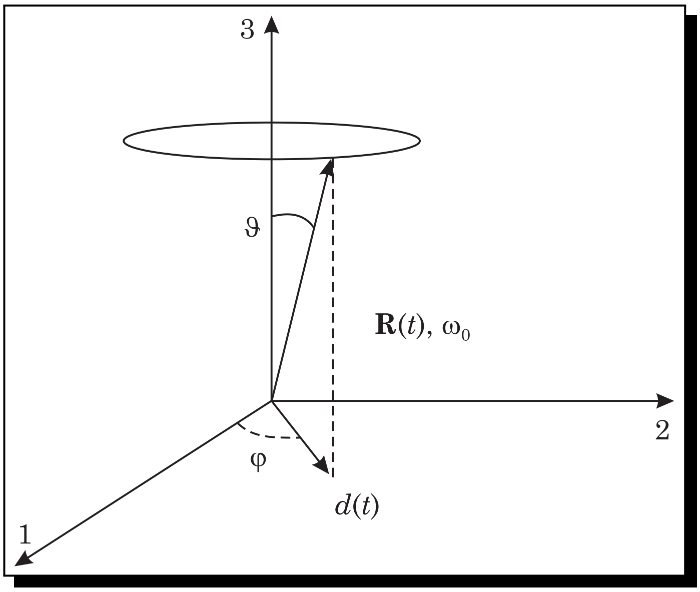

Figure 2. Optical Bloch vector

Since population of TLS is equal to $N_j = |a_j|^2$ we have from (3.2c)

$$R_3 = -N \equiv N_1 - N_2, \tag{3.3}$$

i.e., the third component of the optical Bloch vector is equal to the population inversion up to sign.

The balance equation (2.1), thus, describes the time evolution of the third projection of the optical Bloch vector.

Let us determine the average value of the dipole moment of a two-level system ($\widehat{d}$ is operator of electric dipole moment):

$$d(t) \equiv \langle t|\widehat{d}|t\rangle = (a_1 a_2^* + a_1^* a_2) d_0 = R_1 d_0 . \tag{3.4}$$

Thus, the first component of the optical Bloch vector determines the time dependence of the average dipole moment of a two-level system.

To find out the physical meaning of the second component of the Bloch vector, we introduce the complex dipole moment according to the inequality:

$$\widetilde{d} \equiv \widetilde{d}_1 + i\widetilde{d}_2 = 2d_0 a_2 a_1^* \tag{3.5}$$

then

$$d(t) = Re\{\widetilde{d}(t)\} = \widetilde{d}_1(t) = d_0 R_1 . \tag{3.6}$$

Imaginary part of dipole moment $\widetilde{d}_2$ allows us to determine the phase of the average dipole moment, which coincides with the angle φ in the complex plane of the vector $\widetilde{d}$ (see Figure 2):

$$\varphi = \mathrm{arctg}(\widetilde{d}_2/\widetilde{d}_1) = \arg(\widetilde{d}). \tag{3.7}$$

At the same time $\widetilde{d}_2$ is expressed in terms of the second component of the Bloch vector:

$$\widetilde{d}_2 = d_0 R_2 . \tag{3.8}$$

So, formulas (3.3)-(3.8) give a physical interpretation of the optical Bloch vector. We note that the complex vector of the average dipole moment (3.5) makes it possible to visualize the phase of the dipole moment as the angle of rotation of the vector in the complex plane.

As it follows from the definition (3.2), the Bloch vector is a real value, moreover, it is normalized to unity:

$$\sqrt{R_1^2 + R_2^2 + R_3^2} = |a_1|^2 + |a_2|^2 = 1 . \tag{3.9}$$

4. Bloch Equations

Based on the Schrödinger equation for the amplitudes of the state vector of TLS (3.1), we can obtain the following equation for the temporal evolution of the Bloch vector:

$$\frac{d\mathbf{R}}{dt} = \mathbf{R} \times \mathbf{W}, \tag{4.1}$$

here $\mathbf{W} = (2\Omega(t), 0, -\omega_0)$ is generalized angular velocity vector, $\Omega(t) = d_0 F(t)/\hbar$ is instant Rabi frequency, $F(t)$ is electric field strength in the laser radiation acting on TLS.

Equation (4.1) implies that there is a mechanical analogy for the radiative transition in an external monochromatic field: the optical Bloch vector is analogous to the angular momentum of a gyroscope: it precesses around the instantaneous direction of the generalized angular velocity $\mathbf{W}$. This angular velocity is determined by the eigenfrequency of TLS and the parameters of the laser field.

Since the generalized angular velocity varies with time with an optical frequency, the motion of the Bloch vector is difficult to imagine, and there is no analytical solution to the system. To simplify the consideration and obtain an analytical solution, it is necessary to switch to a coordinate system rotating around axis 3 with an angular velocity equal to the laser radiation frequency ω. Then we obtain the following equation for the Bloch vector $\mathbf{R}_0$ in the rotating coordinate system

$$\frac{d\mathbf{R}_0}{dt} = \mathbf{R}_0 \times \mathbf{W}_0\,, \quad \mathbf{W}_0 = (\Omega_0, 0, \Delta)\,, \tag{4.2}$$

here $\Omega_0 = d_0 F_0/\hbar$ is resonant Rabi frequency, F_0 is amplitude of electric field strength, $\Delta = \omega - \omega_0$ — frequency detuning of the electromagnetic field from the eigenfrequency of TLS.

When writing (4.2), the rapidly oscillating terms at the sum frequency $\omega_s = \omega + \omega_0$ were discarded on the right side of the equation. This simplification is called the rotating wave approximation. It is important that in this approximation the generalized angular velocity W_0 is a constant value, which greatly simplifies the description of the evolution of the Bloch vector.

We emphasize that the third projection of the Bloch vector R_3 does not change upon transition to a rotating coordinate system; therefore, equation (4.2) can be used to describe the population inversion of the TLS, since it coincides up to sign with R_3 according to (3.3).

In reality, a two-level system is always part of some macroscopic system called a thermostat. Interaction with it significantly affects the dynamics of TLS, causing various relaxation processes. At long times it is necessary to take into account the relaxation of TLS under the action of a thermostat. This can be done, for example, within the formalism of the density matrix, which in this case plays the role of the state vector (or wave function) of the system. The elements of the density matrix can be defined using the inequality

$$\rho_{ij} = \langle a_i a_j^* \rangle_{\text{Therm}}\,. \tag{4.3}$$

The angle brackets in equation (4.3) denote the thermostat averaging. Using definition (4.3), it is possible to redefine the Bloch vector components (3.2) in terms of the density matrix and write the corresponding equations of motion for them with account to relaxation. These equations have the form:

$$\frac{dR_{01}}{dt} = \Delta R_{02} - \frac{R_{01}}{T_2}\,, \tag{4.4}$$

$$\frac{dR_{02}}{dt} = -\Delta R_{01} - \frac{R_{02}}{T_2} + \Omega_0 R_3\,, \tag{4.5}$$

$$\frac{dR_3}{dt} = -\Omega_0 R_{02} + \frac{R_3^e - R_3}{T_1}\,, \tag{4.6}$$

here T_2 is the phase relaxation time, $R_{01,02}$ are Bloch vector components in rotating coordinate system. We assume that the laser electromagnetic field is, in the first approximation, monochromatic with a coherence time exceeding all other characteristic times, so that $\Omega_0 = \text{const}$.

Note that in equations (4.4)-(4.5) there appears a phase relaxation time T_2, which is absent in the balance equation (2.1).

5. Derivation of Balance Equation for Population Inversion

Let's now derive the balance equation (2.1) from Bloch equations. To do this, we write out the integral equation for the third component of the Bloch vector, which follows from the differential equations (4.4)-(4.6). After some algebra and insignificant approximation we obtain instead of (4.6) the following relation

$$R_3(t) - R_3^e = -\exp\left(-\frac{t}{T_1}\right)\Omega_0^2 \int_0^t d\tau \exp\left(\frac{\tau}{T_1}\right)\int_0^\infty dt' R_3(\tau - t')\exp\left(-\frac{t'}{T_2}\right)\cos(\Delta t'). \tag{5.1}$$

It follows from (5.1) that if the value R_3 changes slightly during the phase relaxation time T_2, then it can be taken out of the sign of the integral with respect to dt' at $t' = 0$. Then, we get

$$R_3(t) - R_3^e = -\exp\left(-\frac{t}{T_1}\right)\Omega_0^2 \int_0^t d\tau \exp\left(\frac{\tau}{T_1}\right) R_3(\tau)\int_0^\infty dt' \exp\left(-\frac{t'}{T_2}\right)\cos(\Delta t'). \tag{5.2}$$

In order to reduce equation (5.2) to the form corresponding to the balance equation for population inversion (2.1), it is necessary to multiply (5.2) by $\exp(t/T_1)$ and differentiate with respect to time. As a result (after reduction by the factor $\exp(t/T_1)$) we obtain

$$\frac{dR_3}{dt} + \frac{R_3 - R_3^e}{T_1} = -\Omega_0^2 R_3 \int_0^\infty dt' \exp\left(-\frac{t'}{T_2}\right)\cos(\Delta t'). \tag{5.3}$$

The integral remaining in (5.3) is equal to the Lorentzian $g_L(\omega)$, which describes the homogeneous broadening of the TLS spectral line:

$$\int_0^\infty dt' \exp\left(-\frac{t'}{T_2}\right)\cos(\Delta t') = \pi g_L(\omega), \tag{5.4}$$

here

$$g_L(\omega) = \frac{1}{\pi}\frac{1/T_2}{(\omega - \omega_0)^2 + T_2^{-2}}. \tag{5.5}$$

From inequalities (5.3)-(5.4) we finally obtain the balance equation for the population inversion (because of definition (3.3)):

$$\frac{dN}{dt} + \frac{N - N_e}{T_1} = -2\{(\pi/2) g_L(\omega)\Omega_0^2\} N. \tag{5.6}$$

Here the expression in the curly bracket in the right side of inequality (5.6) is the probability per unit time (rate) of radiative transition induced by electromagnetic field:

$$W_i = (\pi/2) g_L(\omega)\Omega_0^2. \tag{5.7}$$

It follows from the inequalities (5.5)-(5.7) that the phase relaxation time is "hidden" in the spectral linewidth (5.5) in contrast with Bloch equations (4.1)-(4.2). This, in turn, means that the phase relation between the electromagnetic field and the two-level system is lost in the balance equation (5.6).

In order to obtain the balance equation used by A. Einstein in his original work [1], it is necessary to make the following substitutions in inequality (5.6):

$$\frac{dN}{dt} = 0, \quad N_e \to -N_1, \quad \frac{1}{2T_1} \to A, \quad W_i \to B\rho, \tag{5.8}$$

here A and B are Einstein coefficients for spontaneous and stimulated radiation, ρ is spectral energy density of thermal radiation. Replacements (5.8) correspond to the problem considered in article [1] of a two-level system in statistical equilibrium due to interaction with thermal radiation.

Finally, using (5.6)-(5.8) we obtain the Einstein balance equation from paper [1]:

$$AN_2 + B\rho N_2 = B\rho N_1 . \tag{5.9}$$

The left side of inequality (5.9) describes the rate of transitions from the upper energy level to the lower one of TLS as a result of spontaneous (first term) and stimulated (second term) radiation. The right side of (5.9) is the rate of transitions from the lower level to the upper one due to absorption of thermal radiation by TLS. In a state of statistical equilibrium, these rates are equal.

Thus, we have demonstrated how the balance equation for the TLS population can be derived starting from the Bloch equations and also obtained the Einstein balance equation for the TLS in a state of equilibrium with thermal radiation. The condition for such a correspondence is the smallness of the population change during phase relaxation.

6. Conclusion

It follows from the above derivation of inequality (5.6) that the Bloch equations turn into balance equation if the change in the corresponding variables is negligibly small on times of the order of the phase relaxation time. This, in particular, implies the adequacy of the application of perturbation theory. All information about the phase remains in this case in spectral line shape $g_L(\omega)$.

Our consideration, in particular, explains why the balance equation for population inversion which used the rate of radiative transitions also describes processes outside the framework of perturbation theory.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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