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Research Article

Solving Nonlinear Integro-Differential Equations Using the Combined Homotopy Analysis Transform Method With Adomian Polynomials

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Abstract. In this paper, we propose a reliable combination of the homotopy analysis method (HAM) and laplace transform-Adomian method to find the analytic approximate solution for nonlinear integro-differential equations. In this technique, the nonlinear term is replaced by its Adomian polynomials for the index k, and hence the dependent variable components are replaced in the recurrence relation by their corresponding homotopy analysis transforms components of the same index. Thus, the nonlinear integro-differential equation can be easily solved with less computational work for any analytic nonlinearity due to the properties and available algorithms of the Adomian polynomials. The results show that the method is very simple and effective.

Keywords. Nonlinear integro-differential equations; Homotopy analysis method; Laplace transform method; Adomian polynomials

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1. Introduction

Integral and integro-differential equations are known to play an important role in characterizing many social, biological, physical and engineering problems. Nonlinear integral and integro-differential equations are usually hard to solve analytically while exact solutions are rather difficult to be obtained. Some different valid methods have been developed to solving nonlinear integral equations in the last few years [8, 9, 11]. The homotopy analysis method (HAM) is a general analytic approach to get series solutions of various types of nonlinear equations [2, 3, 7]. The HAM provides us a simple way to ensure the convergence of solution series, by introducing the auxiliary parameter $\hbar$ [31,34]. By properly choosing the basis functions of initial approximations, auxiliary linear operators, and auxiliary parameter $\hbar$, HAM gives rapidly convergent successive approximations of the exact solution. The main aim of this article is to present analytical and approximate solution of nonlinear integro-differential equations by using the combined homotopy analysis method (HAM) and Laplace transform-Adomian. In this work, we introduce a comprehensive and more efficient approach to use the homotopy analysis transform method to solve nonlinear integro-differential equations. The nonlinear function is replaced by its Adomian polynomials and then the dependent variable components are replaced by their corresponding differential transform component of the same index. The first step is to consider the following nonlinear intergro-differential equations of the second kind:

$$u^{(n)}(x) = f(x) + \int_a^x k(x,t)F(u(t))dt, \tag{1}$$

where the upper limit may be either variable or fixed, $u^n(x) = \frac{d^n u}{dx}$, the kernel of the integral $k(x,t)$ and $f(x)$ are known function and $F(u(t))$ is a nonlinear function of $u(x)$, with initial conditions $u(a) = \alpha_0$, $u^{(1)}(a) = \alpha_1, \cdots, u^{(n-1)}(a) = \alpha_{n-1}$.

2. Preliminaries and Notations

To begin with, review of the Adomian decomposition method is presented here.

2.1 Adimian's Decomposition Method

Let us define the nonlinear equation

$$x = c + N(x), \tag{2}$$

where N is a nonlinear function and c is a constant. The Adomian method consists of representing the solution of (2) as a series

$$x = \sum_{n=0}^{\infty} x_n, \tag{3}$$

and the nonlinear function as the decomposed form

$$N(x) = \sum_{n=0}^{\infty} A_n, \tag{4}$$

where A_n $(n = 0, 1, 2, \ldots)$ are the Adomian polynomials of $x_0, x_1, \ldots$ given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n}\lambda^i x_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots,$$

substituting (3) and (4) into (2) yields

$$\sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n. \tag{5}$$

The convergence of the series in (5) gives the desired relation

$$x_0 = c,$$

$$x_{n+1} = A_n, \quad n = 0,1,2,\ldots,$$

The polynomials A_n have been generated for all kinds of nonlinearity by Wazwaz [36]. Therefore Adomian polynomials are given by

$$\begin{aligned}
A_0 &= N(x_0), \\
A_1 &= x_1 N'(x_0), \\
A_2 &= x_2 N'(x_0) + \frac{1}{2!} x_1^2 N''(x_0), \\
A_3 &= x_3 N'(x_0) + x_1 x_2 N''(x_0) + \frac{1}{3!} x_1^3 N'''(x_0), \\
A_4 &= x_4 N'(x_0) + \left(\frac{1}{2!} x_2^2 + x_1 x_3\right) N''(x_0) + \frac{1}{2!} x_1^2 x_2 N'''(x_0) + \frac{1}{4!} x_1^4 N^{(iv)}(x_0).
\end{aligned}$$

It should be pointed out that A_0 depends only on x_0, A_1 depends only on x_0 and x_1, A_2 depends only on x_0, x_1 and x_2, and so on. Hence, we may also write A_n as $A_n(x_0, x_1, \ldots, x_n)$. Suppose $s_m = x_0 + x_1 + x_2 + \ldots + x_m$. Then, $s_m = c + A_0 + A_1 + A_2 + \ldots + A_{m-1}$ is the $(m+1)$-term approximation of x. Such s_m can serve as a practical solution in each iteration.

2.2 Hypothesis and Generalities

Let us consider the general nonlinear functional equation:

$$u - N(u) = f \tag{6}$$

where N and f are, respectively, operator and function given in convenient spaces. It is necessary to find a function u satisfying equation (6). N is supposed to be such that (6) assumes a unique solution in some well-adapted spaces.

Adomian technique allows us to find the solution of (6) as an infinite series $u = \sum_{i=1}^{\infty} u_i$ using the recurrent scheme written below:

$$\begin{aligned}
u_0 &= f, \\
u_1 &= A_0(u_0), \\
&\vdots \\
u_n &= A_{n-1}(u_0, \cdots, u_{n-1}), \\
&\vdots
\end{aligned}$$

where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0,1,2,\cdots. \tag{7}$$

For the present work, we shall suppose that

(i) the solution u of (6) can be found as a series of functions u_i, i.e., $u = \sum_{i=0}^{\infty} u_i$. Furthermore, this series is supposed to be absolutely convergent, i.e., $\sum_{i=0}^{\infty} |u_i| < \infty$.

(ii) the nonlinear function $N(u)$ is developable in the entire series with a convergence radius equal to infinity. In other words, we may write

$$N(u) = \sum_{i=0}^{\infty} N_{(0)}^{n} \frac{u^n}{n!}, \quad |u| < \infty.$$

This last hypothesis is almost always satisfied in concrete physical problems.

Theorem 2.1. *With the previous hypothesis* (i) *and* (ii)*, the Adomian series* $u = \sum_{i=0}^{\infty} u_i$ *is a solution of equation* (6)*, when the* u_i*'s satisfy relationship* (7)*.*

Proof of this theorem is given in [10].

3. Description of the Method

Let us consider integro-differential equations of the second kind

$$u^{(n)}(x) = f(x) + \int_a^x K(x,t)F(u(t))dt.$$

To solve the nonlinear integro-differential equations, the Laplace transform can be applied on both sides of eq. (1), to result in

$$\mathscr{L}[u^{(n)}(x)]] = \mathscr{L}[f(x)] + \mathscr{L}\left[\int_a^x K(x,t)F(u(t))dt\right], \tag{8}$$

$$s^n L[u(x)] - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - u^{(n-1)}(0) = \mathscr{L}[f(x)] + \mathscr{L}[K(x,t)]\mathscr{L}[F(u(t))], \tag{9}$$

or equivalently

$$\mathscr{L}[u(x)] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^n}u^{(n-1)}(0) + \frac{1}{s^n}\mathscr{L}[f(x)] + \frac{1}{s^n}\mathscr{L}[K(x,t)]\mathscr{L}[F(u(t))]. \tag{10}$$

To overcome the difficulty of the nonlinear term $F(u(x))$, we apply the Adomian decomposition method for handling (10).

To achieve this goal, we first represent the linear term $u(x)$ on the left side by an infinite series of components given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{11}$$

where the components $u_n(x)$, $n \geq 0$ will be recursively determined. However, the nonlinear term $F(u(x))$ on the right side of (10) will be represented by an infinite series of the Adomian polynomials A_n in the form

$$F(u(x)) = \sum_{n=0}^{\infty} A_n, \quad A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\cdots, \tag{12}$$

where A_n, $n \geq 0$ can be obtained for all forms of nonlinearity.

Substituting (11) and (12) into (10) leads to

$$\mathscr{L}\left[\sum_{n=0}^{\infty} u_n(x)\right] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^n}u^{(n-1)}(0) + \frac{1}{s^n}\mathscr{L}\left[f(x) + \frac{1}{s^n}\mathscr{L}[K(x,t)\right]\mathscr{L}\left[\sum_{n=0}^{\infty} A_n(t)\right].$$

We define the nonlinear operator

$$N[\varphi(x;q)] = \mathscr{L}[\varphi^{(n)}(x;q)] - \mathscr{L}[f(x)] - \mathscr{L}\left[\int_a^x k(x,t)\sum_{n=0}^{\infty} A_n(t)dt\right], \tag{13}$$

where $q \in [0,1]$ is an embedding parameter and $\varphi(x;q)$ is the real function of x and q. Now, we can construct the zero order deformation equation

$$(1-q)\mathscr{L}[\varphi(x;q) - u_0(x)] = q\hbar H(x)N[\varphi(x;q)], \tag{14}$$

Obviously, when $q = 0$ and $q = 1$, it holds that $\varphi(x;0) = u_0(x)$, $\varphi(x;1) = u(x)$ respectively, where the initial guess of the exact solution $u(x)$ is $u_0(x)$, $H(x) \neq 0$ an auxiliary function, $\hbar \neq 0$ is an auxiliary parameter $\varphi(x;q)$ is an unknown function and $\mathscr{L}$ is an auxiliary linear operator. Thus, as q increases from 0 to 1, $\varphi(x;q)$ varies from the guess $u_0(x)$ to the exact solution $u(x)$. Expanding $\varphi(x;q)$ Taylor's series with respect to q, we have

$$\varphi(x;q) = \varphi(x;0) + \sum_{m=1}^{\infty} u_m(x) q^m, \quad m = 1,2,\cdots,$$

where

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \varphi(x;q)}{\partial q^m}\Big|_{q=0}, \quad m = 1,2,\cdots.$$

The previous relation can be written in the following form

$$\varphi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m, \quad m = 1,2,\cdots. \tag{15}$$

The initial guess $u_0(x)$ is chosen such that satisfies the initial conditions of problem. If the auxiliary linear operator, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (15) converges at $q = 1$, and we get the solution

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x), \quad m = 1,2,\cdots, \tag{16}$$

for brevity define the vector

$$\vec{u}_{n-1}(x) = \{u_0(x), u_1(x), \cdots, u_{n-1}(x)\}.$$

According to the definition (16), the governing $u_m(x)$ can be derived from the zero-order deformation equation (14) m times with respective to q and then dividing it by $m!$ and finally setting $q = 0$, we obtain the mth-order deformation equation

$$\mathscr{L}[u_m(x) - \mathscr{X}_m u_{m-1}(x)] = \hbar q H(x) R_m(u_{m-1}(x)), \tag{17}$$

subject to initial conditions

$$u(a) = \alpha_0,\ u^{(1)}(a) = \alpha_1, \cdots, u^{(n-1)}(a) = \alpha_{n-1}, \tag{18}$$

where

$$R_m(u_{m-1}, x) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x;q)]}{\partial q^{m-1}}\Big|_{q=0} \tag{19}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m \ge 2. \end{cases}$$

From (19) and using (13), we have

$$R_m[\varphi_{m-1}(x)] = \frac{\partial^n \varphi_{m-1}}{\partial x^n} - (1 - \mathscr{X}_m) f(x) - \int_a^x K(x,t) \left[\sum_{n=0}^{m-1} A_{m-1-n}(t)\right] dt. \tag{20}$$

In this way, it is easy to obtain $u_m(x)$ for $m \ge 1$ at mth-order. We have

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x),$$

when $n \to \infty$ and we get an accurate approximation of the original equation.

Definition 3.1. The relative errors τ_n of the n terms approximation is

$$\tau(x_i) = \frac{\left|u_{\text{exact}} - u_{\text{mpp}}(x_i)\right|}{|u_{\text{exact}}(x_i)|}.$$

4. Convergence Analysis

In this section, one theorem and conditions in the framework of convergence of the homotopy analysis method are stated.

Theorem 4.1. *The series*

$$\varphi_0(x) + \sum_{m=1}^{\infty} \varphi_m(x)$$

converges, where $\varphi_m(x)$'s are resulted from (17), (18) *and* (20), *the limit of the series is an exact solution of* (1).

Proof. Since, by hypothesis, the series is convergent, it holds

$$S(x) = \sum_{m=0}^{\infty} \varphi_m(x)\cdot$$

So, the necessary condition for the convergence of the series is valid, that is,

$$\lim_{m\to\infty} \varphi_m(x) = 0\cdot \tag{21}$$

using (17) and (21), we have

$$\hbar H(x) \sum_{m=1}^{+\infty} R_m\left[\vec{\varphi}_{m-1}(x)\right] = \lim_{n\to\infty} \sum_{m=1}^{n} \mathscr{L}\left[\varphi_m(x) - \mathscr{X}_m \varphi_{m-1}(x)\right]$$

$$= \mathscr{L}\left\{\lim_{n\to\infty} \sum_{m=1}^{+\infty} \left[\varphi_m(x) - \mathscr{X}_m \varphi_{m-1}(x)\right]\right\} = \mathscr{L}\left\{\lim_{n\to\infty} \varphi_n(x)\right\} = 0\cdot$$

Since $\hbar \neq 0$, we must have

$$\sum_{m=1}^{+\infty} R_m\left[\vec{\varphi}_{m-1}(x)\right] = 0,$$

on the other hand, we have

$$\sum_{m=1}^{\infty} R_m[\varphi_{m-1}(x)] = \sum_{m=1}^{\infty} \left[\frac{\partial^n \varphi_{m-1}}{\partial x^n} - (1-\mathscr{X}_m) f(x) - \int_a^x K(x,t) \left[\sum_{n=0}^{m-1} A_{m-1-n}(t)\right] dt\right] = 0,$$

$$= \sum_{m=0}^{\infty} \varphi_m^{(n)}(x) - f(x) - \sum_{m=1}^{\infty} \left\{\int_a^x K(x,t) \times \left[\sum_{n=0}^{m-1} A_{m-1-n}(t) dt\right]\right\} = 0,$$

$$= \sum_{m=0}^{\infty} \varphi_m^{(n)}(x) - f(x) - \int_a^x K(x,t) \left[\sum_{m=1}^{\infty} \left(\sum_{n=0}^{m-1} A_{m-1-n}(t)\right)\right] dt = 0,$$

$$= \sum_{m=0}^{\infty} \varphi_m^{(n)}(x) - f(x) - \int_a^x K(x,t) \sum_{r=0}^{\infty} A_r(t) dt = 0,$$

$$s^{(n)}(x) - f(x) - \int_a^x K(x,t) F(s(t)) = 0, \tag{22}$$

$$s^{(n)}(x) = f(x) - \int_a^x K(x,t) F(s(t))\cdot \tag{23}$$

Also, from the initial conditions (18), the following holds:

$$S(0) = \sum_{i=0}^{\infty} \varphi_i(0) = \varphi_0(0) = u_0(0) = u_0,$$

since $s(x)$ satisfies (22), we conclude that it is an exact solution of (1). □

5. Applications and Numerical Results

In this section, we implement the proposed method on some different examples with different types of nonlinearity. All algebraic computations are executed using MATHEMATICA software package. We report absolute error which is defined by $E_u = |u_{\text{exact}} - u_{\text{app}}|$, where $u_{\text{app}} = \sum_{i=0}^{m} u_i(x)$.

Example 5.1. Consider the nonlinear Volterra integro-differential equation

$$\begin{cases} u'(x) = \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x} + \int_0^x (x-t)u^2(t)dt, \\ u(0) = 2, \end{cases}$$

which has the exact solution $u(x) = 1 + e^{-x}$.

The nonlinear term $F(u(x)) = u^2(x)$, can be expressed by Adomian Polynomials A_n, where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\cdots.$$

Applying Laplace transform, we have

$$su(s) - u(0) = \frac{9}{4s} - \frac{5}{2s^2} - \frac{1}{s^3} - \frac{3}{s+1} - \frac{1}{4(s+2)} + \frac{1}{s^2}\mathscr{L}\{A_n(x)\},$$

which satisfies

$$u(s) = \frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} + \frac{1}{s^3}\mathscr{L}\{A_n(x)\}.$$

A nonlinear operator can be defined as:

$$N[\varphi(x;q)] = \mathscr{L}[\varphi(x;q)] + \left(-\frac{2}{s} - \frac{9}{4s^2} + \frac{5}{2s^3} + \frac{1}{s^4} + \frac{3}{s(s+1)} + \frac{1}{4s(s+2)}\right) - \frac{1}{s^3}\mathscr{L}\{A_n(x)\}.$$

The m-th order deformation equation is:

$$u_m(x) = \mathscr{X}_m u_{m-1}(x) + \hbar H(x)\mathscr{L}^{-1}R_m(u_{m-1}(x)),$$

for which $\hbar = -1$, $H(x) = 1$,

$$R_m(u_{m-1}(x)) = \mathscr{L}[u_{m-1}] - \left(\frac{2}{s} + \frac{9}{4s^2} - \frac{5}{2s^3} - \frac{1}{s^4} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)}\right)(1 - \mathscr{X}m) - \frac{1}{s^3}\{A_n(x)\},$$

where $A_n, n \geq 0$, $n = 0,1,2,\cdots$, we consider

$$u_0(x) = 2 - x + \frac{1}{2}x^2 - \frac{5}{6}x^3 + \frac{5}{24}x^4 - \frac{7}{120}x^5 + \cdots,$$

$$\xrightarrow{m=1} u_1(x) = \frac{x^2}{2}\cdot A_0 \text{ where } A_0 = F(u_0) = u_0^2(x),$$

$$\xrightarrow{m=2} u_2(x) = \frac{x^2}{2}\cdot A_1 \text{ where } A_1 = u_1 F'(u_0) = 2u_0(x)u_1(x),$$

$$\xrightarrow{m=3} u_3(x) = \frac{x^2}{2}\cdot A_2 \text{ where } A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0) = 2u_0(x)u_2(x) + u_1^2(x),$$

using the MATHEMATICA package, we obtain

$$u_1(x) = \frac{2}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{20}x^5 + \cdots,$$

the solution is given by

$$\varphi_m(x) = u_0(x) + \sum_{i=1}^{m} u_i(x), \quad m = 1,2,3,\cdots,$$

$$u(x) = \lim_{m\to\infty} \varphi_m(x) = \lim_{m\to\infty} u_0(x) + \sum_{i=1}^{m} u_i(x) = 1 + e^{-x}.$$

Some numerical results of this solution are presented in Table 1 and Figure 1.

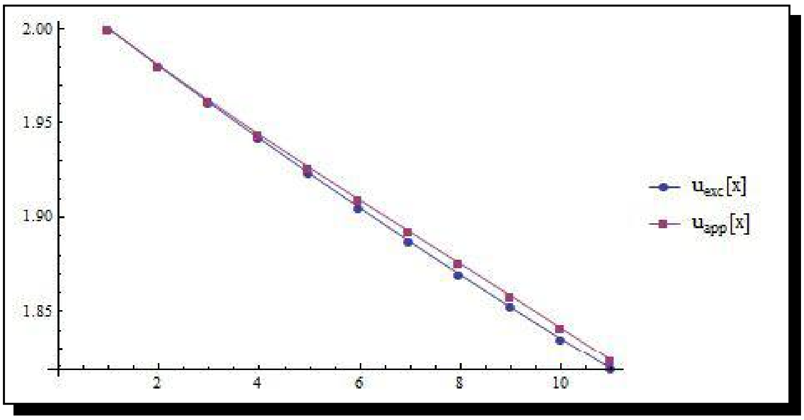

Figure 1

Table 1. Numerical results of Example 5.1

x	$u_{\text{exc}}(x)$	$u_{\text{app}}(x)$	$\lvert u_{\text{exc}}(x) - u_{\text{app}}(x)\rvert$
0.00	2.00000	2.00000	0.00000
0.02	1.98020	1.98047	2.72480E-4
0.04	1.96079	1.96177	9.83263E-4
0.06	1.94176	1.94374	1.97877E-4
0.08	1.92312	1.92623	3.1118E-3
0.10	1.90484	1.90908	4.23904E-3
0.12	1.88692	1.89214	5.21869E-3
0.14	1.86936	1.87527	5.90821E-3
0.16	1.85214	1.85831	6.16192E-3
0.18	1.83527	1.84110	5.82864E-3
0.20	1.81873	1.82348	4.74895E-3

Example 5.2. Consider the nonlinear Volterra integro differential equation

$$\begin{cases} u''(x) = 2 + 2x + x^2 - x^2e^x - e^{2x} + \int_0^x e^{x-t}u^2(t)dy, \\ u(0) = 1, \quad u'(0) = 2, \end{cases}$$

which has the exact solution $u(x) = x + e^{-x}$. The nonlinear term $F(u(x)) = u^2(x)$, can be expressed by Adomian polynomials A_n, where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i)\right)\right]_{\lambda=0}, \quad n = 0,1,2,\cdots.$$

Applying Laplace transform, we have

$$u(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} + \frac{2}{s^4} + \frac{2}{s^5} - \frac{2}{s^2(s-1)^3} - \frac{1}{s^2(s-2)} + \frac{1}{s^2(s-1)}\mathscr{L}\{A_n(x)\}.$$

A nonlinear operator can be defined as:

$$N[\varphi(x;q)] = \mathscr{L}[\varphi(x;q)] + \left(-\frac{1}{s} - \frac{2}{s} - \frac{2}{s^3} - \frac{2}{s^4} - \frac{2}{s^5} + \frac{2}{s^2(s-1)^3} + \frac{1}{s^2(s-2)}\right) - \frac{1}{s^2(s-1)}\mathscr{L}\{A_n(x)\}.$$

The m-th order deformation equation is written as

$$u_m(x) = \mathscr{X}_m u_{m-1}(x) + \hbar H(x)\mathscr{L}^{-1}R_m(u_{m-1}(x)),$$

for which $\hbar = -1$, $H(x) = 1$,

$$R_m(u_{m-1}(x)) = \mathscr{L}[u_{m-1}] - \left(\frac{1}{s} + \frac{2}{s} + \frac{2}{s^3} + \frac{2}{s^4} + \frac{2}{s^5} - \frac{2}{s^2(s-1)^3} - \frac{1}{s^2(s-2)}\right)(1-\mathscr{X}_m)$$
$$- \frac{1}{s^2(s-1)}\mathscr{L}\{A_n(x)\}, \quad n = 0,1,2\cdots,$$

where $A_n, n \geq 0$, we consider

$$u_0(x) = 1 + 2x + \frac{x^2}{2} - \frac{x^4}{6} - \frac{7}{60}x^5 + \cdots,$$

$\xrightarrow{m=1}$ $u_1(x) = \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\mathscr{L}\{A_0\}\right]$, where $A_0 = F(u_0) = u_0^2(x)$,

$\xrightarrow{m=2}$ $u_2(x) = \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\right]\mathscr{L}\{A_1\}]$, where $A_1 = u_1F'(u_0) = 2u_0(x)u_1(x)$,

$\xrightarrow{m=3}$ $u_3(x) = \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\mathscr{L}\{A_2\}\right]$, where $A_2 = u_2F''(u_0) + \frac{1}{2!}u_1^2F''(u_0) = 2u_0(x)u_2(x) + u_1^2(x)$,

$\xrightarrow{m=4}$ $u_4(x) = \mathscr{L}^{-1}\left[\frac{1}{s^2(s-1)}\mathscr{L}\{A_3\}\right]$, where

$$A_3 = u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^3F'''(u_0) = 2u_0(x)u_3(x) + 2u_1(x)u_2(x),$$

$$\vdots$$

Using the MATHEMATICA package, we obtain

$$u_1(x) = \frac{1}{6}x^3 + \frac{5}{24}x^4 + \frac{1}{8}x^5 + \frac{3}{80}x^6 + \cdots,$$
$$u_2(x) = \frac{1}{360}x^6 + \cdots.$$

The solution is given by

$$\varphi_m(x) = u_0(x) + \sum_{i=1}^{m} u_i(x), \quad m = 1,2,\cdots$$

$$u(x) = \lim_{m\to\infty}\varphi_m(x) = \lim_{m\to\infty} u_0(x) + \sum_{i=1}^{m} u_i(x) = x + e^x.$$

Some numerical results of this solution are presented in Table 2 and Figure 2.

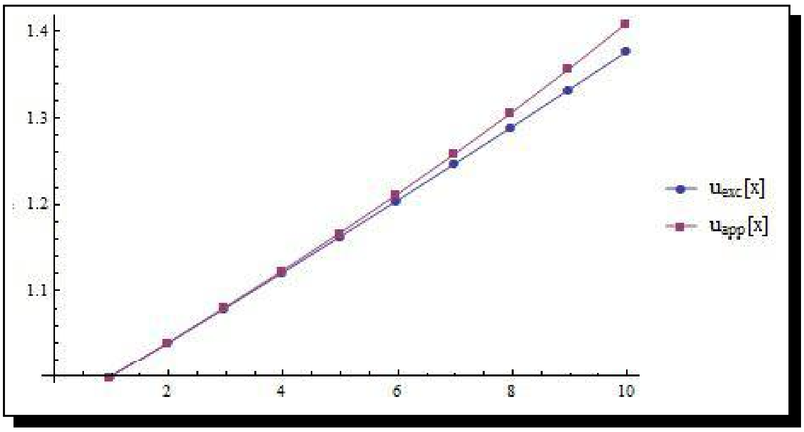

Figure 2

Table 2. Numerical results of Example 5.2

x	$u_{\text{exc}}(x)$	$u_{\text{app}}(x)$	$\|u_{\text{exc}}(x)-u_{\text{app}}(x)\|$
0.00	1.00000	1.00000	0.00000
0.02	1.0402	1.04042	2.16577E-4
0.04	1.08081	1.08175	9.37538E-4
0.06	1.12184	1.12412	2.28194E-3
0.08	1.16329	1.16767	4.38752E-3
0.10	1.20517	1.21259	7.41437E-3
0.12	1.24750	1.25905	1.15497E-3
0.14	1.29027	1.30729	1.70138E-2
0.16	1.33351	1.35758	2.40683E-2
0.18	1.37722	1.41024	3.30265E-2
0.20	1.42140	1.46567	4.42678E-2

Example 5.3. We used the proposed method to find the approximate solution of the following nonlinear integro-differential equation

$$\begin{cases} u'''(x) = -\frac{2}{3} - \frac{5}{3}\cos x + \frac{4}{3}\cos^2 x + \int_0^x \cos(x-t)u^2(t)dt, \\ u(0) = u'(0) = 1, \quad u''(0) = -1, \end{cases}$$

which has the exact solution $u(x) = \sin x + \cos x$. The nonlinear term $F(u(x)) = u^2(x)$, can be expressed by Adomian polynomials A_n, where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^2 u_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\cdots.$$

Applying Laplace transform, we have

$$u(s) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{3s^4} - \frac{5}{3s^2(1+s^2)} + \frac{4(2+s^2)}{3s^4(4+s^2)} + \frac{1}{s^2(1+s^2)}\mathscr{L}\{A_n(x)\}, \quad n = 0,1,2,\cdots.$$

A nonlinear operator can be defined as:

$$N[\varphi(x;q)] = \mathscr{L}[\varphi(x;q)] + \left(-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} + \frac{2}{3s^4} + \frac{5}{3s^2(1+s^2)} - \frac{4(2+s^2)}{3s^4(4+s^2)}\right) - \frac{1}{s^2(1+s^2)}\mathscr{L}\{A_n(x)\}, \quad n = 0,1,2,\cdots.$$

The m-th order deformation equation is:

$$u_m(x) = \chi_m u_{m-1}(x) + hH(x)\mathscr{L}^{-1}R_m(u_{m-1}(x)),$$

in which $h = -1$, $H(x) = 1$,

$$R_m(u_{m-1}(x)) = \mathscr{L}[u_{m-1}] - \left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3} - \frac{2}{3s^4} - \frac{5}{3s^2(1+s^2)} + \frac{4(2+s^2)}{3s^4(4+s^2)}\right)(1-\mathscr{X}_m) - \frac{1}{s^2(1+s^2)}\mathscr{L}\{A_n(x)\}, \quad n = 0,1,2\cdots,$$

where $n, A_n \geq 0$ we consider $u_0(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^5}{120} + \cdots$,

$$\xrightarrow{m=1} \quad u_1(x) = \mathscr{L}^{-1}\left[\frac{1}{s^2(1+s^2)} \times \mathscr{L}\{A_0\}\right], \text{ where } A_0 = F(u_0) = u_0^2(x),$$

$$\Rightarrow \quad u_1(x) = \frac{x^3}{6} + \frac{x^4}{3} - \frac{x^5}{120} - \frac{43x^6}{180} - \frac{23x^7}{1680} + \cdots,$$

and for $m = 2,3,4,\cdots$

$$u_2(x) = \frac{x^6}{18} + \frac{x^3}{6} + \frac{7x^8}{90} - \cdots,$$

using the MATHEMATICA package, the solution is given by

$$\varphi_m(x) = u_0(x) + \sum_{i=1}^{m} u_i(x), \quad m = 1,2,3,\cdots$$

$$u(x) = \lim_{m\to\infty} \varphi_m(x) = \lim_{m\to\infty} u_0(x) + \sum_{i=1}^{m} u_i(x) = \sin x + \cos x.$$

Some numerical results of this solution are presented in Table 3 and Figure 3.

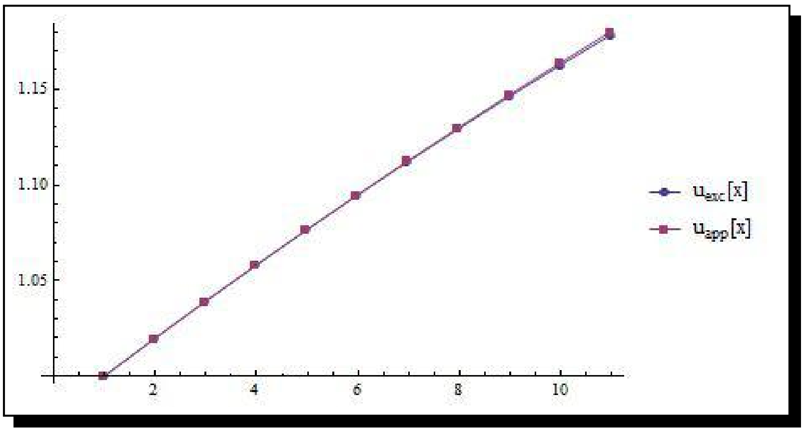

Figure 3

Table 3. Numerical results of Example 5.3

x	$u_{\text{exc}}(x)$	$u_{\text{app}}(x)$	$\lvert u_{\text{exc}}(x) - u_{\text{app}}(x)\rvert$
0.00	1.00000	1.00000	0.00000
0.02	1.0198	1.01980	1.37991E-6
0.04	1.03919	1.03920	1.14101E-5
0.06	1.05816	1.05820	3.97525E-5
0.08	1.07672	1.07681	9.71538E-5
0.10	1.09484	1.09503	1.95418E-4
0.12	1.11252	1.11287	3.47374E-4
0.14	1.12976	1.13033	5.66843E-4
0.16	1.14655	1.14741	8.68596E-4
0.18	1.16287	1.16414	1.26832E-3
0.20	1.17874	1.18052	1.78257E-3

6. Conclusion

The homotopy analysis transform with Adomian polynomials is applied to solve nonlinear integro-differential equations. This method is clearly a very powerful and efficient technique to find the analytical solutions for the wide class of differential and integro-differential equations. In this way, we have great freedom to choose the auxiliary linear operator $\mathscr{L}$, and the auxiliary function $H(x)$ and initial function $u_0(x)$, but in other methods we do n't have this advantages. The convergence accuracy of this method was examined in several numerical examples.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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