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Common Fixed Point Results for Multivalued Mappings in Hausdorff Intuitionistic Fuzzy Metric Spaces

Research Article

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Abstract. The main objective of this paper is to derive common fixed points on a sequence contained in a closed ball for a family of multivalued mapping in a complete intuitionistic fuzzy metric space. Simple and different technique has been used. To give the strength of our result, an illustrative example is constructed.

Keywords. Fuzzy metric spaces; Intuitionistic fuzzy metric spaces; Fixed points; Common fixed points; Hausdorff metric spaces; Multivalued map

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1. Introduction

Fuzzy set theory was first introduced and studied by Zadeh [17] in 1965. The idea of fuzzy metric spaces in different ways has been introduced by some authors [4, 5, 8]. In 1975 Kramosil and Michalek [10] have introduced and studied the notion of fuzzy metric space with the help of continuous t-norm, which is modified by George and Veeramani [6] in 1994 in order to generate a Hausdorff Topology induced by fuzzy metric. In 2004, Park [12], using the idea of intuitionistic fuzzy sets [2], defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to George and Veeramani [6, 7].

In 2004, López and Romaguera [13] introduced the Hausdorff fuzzy metric on a collection of nonempty compact subsets of a given fuzzy metric spaces. Kiany and Amini-Harandi [9] proved fixed point and endpoint theorems for multivalued contraction mappings in fuzzy metric spaces. Recently Shoaib et al. [14] proved the existence of a common fixed point of a family of multivalued mappings which are contractions on a sequence contained in a closed ball instead of the whole space by using the notion of Hausdorff fuzzy metric spaces. In 2012, Arshad and Shoaib [1] obtained the necessary and sufficient conditions for the existence of fixed point of multivalued map in fuzzy metric spaces. In (2016), Shoaib [15] have established and proved fixed point theorems for locally and globally contractive mappings in ordered spaces.

In 2014, Shojaei [16] introduced the concept of Hausdorff *Intuitionistic Fuzzy Metric Space* (HIFMS). In this paper, we prove the existence of common fixed point of a family of multivalued maps in a closed ball of HIFMS. An interesting example is also presented to support our result.

2. Preliminaries

We start this section by recalling some pertinent concepts.

Definition 2.1. Let (X,d) be a metric space. The set of nonempty closed and bounded subsets of X is denoted by $CB(X)$. The function H_d (see [3]) defined on $CB(X)\times CB(X)$ by

$$H_d(A,B)=\max\left(\sup_{a\in A} d(a,B),\sup_{b\in B} d(A,b)\right)$$

for all $A,B\in CB(X)$, is a metric on $CB(X)$ called the Hausdorff metric of d.

Definition 2.2 ([17])**.** Let X be an arbitrary non-empty set. A fuzzy set in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x\in X$, then the function-value $A(x)$ is called the grade of membership of x in A. $F(X)$ stands for the collection of all fuzzy sets in X unless and until stated otherwise.

Definition 2.3 ([2])**.** Let X be a non-empty set. An intuitionistic fuzzy set is defined as:

$$A=\{x\in X:\langle\mu_A(x),\nu_A(x)\rangle\},$$

where $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$ denote the degree of membership and degree of non-membership of each element x to the set A respectively such that

$$0\le\mu_A(x)+\nu_A(x)\le 1,\quad \text{for all } x,y\in X.$$

Definition 2.4 ([12])**.** A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called continuous triangular norm (t-norm) if it satisfies the following conditions:

(1) $*$ is associative and commutative;

(2) $*$ is continuous;

(3) $a*1=a$, for all $a\in[0,1]$;

(4) if $a\le c$ and $b\le d$ with $a,b,c,d\in[0,1]$, then $a*b\le c*d$.

Example 2.1. Three basic t-norms are defined as follows:

(1) The minimum t-norm, $a*_1 b=\min(a,b)$,

(2) The product t-norm, $a*_2 b=a\cdot b$,

(3) The Lukasiewicz t-norm, $a*_3 b=\max(a+b-1,0)$.

Definition 2.5 ([12])**.** A binary operation $\diamondsuit:[0,1]\times[0,1]\to[0,1]$ is called continuous triangular conorm (t-conorm) if it satisfies the following conditions:

(1) $\diamondsuit$ is associative and commutative;

(2) $\diamondsuit$ is continuous;

(3) $a\diamondsuit 0=a$, for all $a\in[0,1]$;

(4) $a\diamondsuit b\le c\diamondsuit d$, whenever $a\le c$ and $b\le d$ $\forall a,b,c,d\in[0,1]$.

Example 2.2. Some examples of basic t-conorms are given below:

(1) $a\diamondsuit_1 b=\min(a+b,1)$;

(2) $a\diamondsuit_2 b=a+b-ab$;

(3) $a\diamondsuit_3 b=\max(a,b)$.

Definition 2.6 (Kramosil and Michalek [10])**.** The triple $(X,M,*)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ such that for all $x,y,z\in X$ we have:

(M1) $M(x,y,0)=0$;

(M2) $M(x,y,t)=1$, for all $t>0$ iff $x=y$;

(M3) $M(x,y,t)=M(y,x,t)$ for all $t\ge 0$;

(M4) $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$, for all $t,s\ge 0$;

(M5) $M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous.

Lemma 2.1 ([11])**.** *Let $(X,M,*)$ be a fuzzy metric space. Then $M(x,y,\cdot)$ is non-decreasing with respect to t, for all $x,y\in X$.*

Definition 2.7 (George and Veeramani [6])**.** The triple $(X,M,*)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ such that for all $x,y,z\in X$ we have:

(M1) $M(x,y,t) > 0$;

(M2) $M(x,y,t) = 1$, for all $t > 0$ iff $x = y$;

(M3) $M(x,y,t) = M(y,x,t)$ for all $t \geq 0$;

(M4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$, for all $t,s \geq 0$;

(M5) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is continuous.

Example 2.3 ([6])**.** Let (X,d) be a metric space and $a * b = ab$ (or $a * b = \min(a,b)$), for all $a,b \in [0,1]$ and let M_d be fuzzy set on $X^2 \times (0,\infty)$, defined as follows:

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}.$$

This metric is called standard fuzzy metric induced by a metric d.

Definition 2.8 ([12])**.** A 5-tuple $(X,M,N,*,\diamondsuit)$ is said to be an intuitionistic fuzzy metric space (IFbMS), if X is an arbitrary set, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, M and N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions: for all $x,y,z \in X$,

(a) $M(x,y,t) + N(x,y,t) \leq 1$;

(b) $M(x,y,t) > 0$;

(c) $M(x,y,t) = 1$, for all $t > 0$ iff $x = y$;

(d) $M(x,y,t) = M(y,x,t)$, for all $t > 0$;

(e) $M(x,z,(t+u)) \geq M(x,y,t) * M(y,z,u)$, for all $t,u > 0$;

(f) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is continuous;

(g) $N(x,y,t) > 0$;

(h) $N(x,y,t) = 0$, for all $t > 0$ iff $x = y$;

(i) $N(x,y,t) = N(y,x,t)$, for all $t > 0$;

(j) $N(x,z,(t+u)) \leq N(x,y,t) \diamondsuit N(y,z,u)$, for all $t,u > 0$;

(k) $N(x,y,\cdot) : [0,\infty) \to [0,1]$ is continuous.

Here $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Note 2.1. $\lim_{t\to\infty} M(x,y,t) = 1$ and $\lim_{t\to\infty} N(x,y,t) = 0$.

Example 2.4 ([12])**.** Let (X,d) be a metric space and $a * b = \min(a,b)$, $a \diamondsuit b = \max(a,b)$ for all $a,b \in [0,1]$ and let M_d, N_d be fuzzy sets on $X^2 \times (0,\infty)$, defined as follows:

$$M_d(x,y,t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if } t > 0, \\ 0, & \text{if } t = 0 \end{cases}$$

and

$$N_d(x,y,t) = \begin{cases} \frac{d(x,y)}{t+d(x,y)}, & \text{if } t > 0, \\ 1, & \text{if } t = 0. \end{cases}$$

then $(X,M_d,N_d,*,\diamondsuit)$ is an intuitionistic fuzzy metric space.

Proposition 2.1 ([12]). *In an intuitionistic fuzzy metric space* $(X,M,N,*,\diamondsuit)$, $M(x,y,\cdot):[0,\infty)\to[0,1]$ *is non-decreasing and* $N(x,y,\cdot):[0,\infty)\to[0,1]$ *is non-increasing for all* $x,y\in X$.

Definition 2.9 ([12]). Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space.

(a) A sequence $\{x_n\}$ in X is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty} M(x_n,x,t)=1$ and $\lim_{n\to\infty} N(x_n,x,t)=0$, for all $t>0$. In this case x is called the limit of the sequence x_n and we write $\lim_{n\to\infty} x_n = x$, or $x_n\to x$.

(b) A sequence $\{x_n\}$ in $(X,M,N,*,\diamondsuit)$ is said to be Cauchy sequence if $\lim_{n\to\infty} M(x_n,x_{n+p},t)=1$ and $\lim_{n\to\infty} N(x_n,x_{n+p},t)=0$ for each $t>0$ and $p>0$.

(c) The space X is said to be complete if and only if every Cauchy sequence in X is convergent and it is called compact if every sequence has a convergent subsequence.

Definition 2.10 ([12]). Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space. An open ball $B(x,r,t)$ with center $x\in X$ and radius r, $0<r<1$, $t>0$ is defined as

$$B(x,r,t)=\{y\in X : M(x,y,t)>1-r, N(x,y,t)<r\}.$$

Definition 2.11. Let B be a nonempty subset of an IFMS $(X,M,N,*,\diamondsuit)$. For $a\in X$ and $t>0$,

$$M(a,B,t)=\sup\{M(a,b,t): b\in B\},$$

$$N(a,B,t)=\inf\{N(a,b,t): b\in B\}.$$

Definition 2.12 (Hausdorff intuitionistic fuzzy metric space [16]). Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space. H_M and H_N on $C(X)\times C(X)\times(0,\infty)$ defined by:

$$H_M(A,B,t)=\min\left(\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\right),$$

$$H_N(A,B,t)=\max\left(\sup_{a\in A} N(a,B,t), \sup_{b\in B} N(A,b,t)\right),$$

for all $A,B\in C(X)$ and for all $t>0$, is an intuitionistic fuzzy metric on $C(X)$ called the Hausdorff intuitionistic fuzzy metric of $(H_M,H_N,*,\diamondsuit)$, where $C(X)$ is the collection of all nonempty compact subsets of X.

Proposition 2.2 ([13]). *Let* $(X,M,N,*,\diamondsuit)$ *be an intuitionistic fuzzy metric space. Then* M *and* N *are continuous functions on* $X\times X\times(0,\infty)$.

3. Main Results

Lemma 3.1. *Let* $(X,M,N,*,\diamondsuit)$ *be an intuitionistic fuzzy metric space. Then for each* $a\in X$, $B\in C(X)$ *and* $t>0$, *there is* $b_0\in B$ *such that*

$$M(a,B,t)=M(a,b_0,t),$$

$$N(a,B,t)=N(a,b_0,t).$$

Proof. Let $a \in X$, $B \in C(X)$ and $t > 0$. By above proposition, the functions $y \mapsto M(a,y,t)$ and $y \mapsto N(a,y,t)$ are continuous. Thus by the compactness of B, there exists $b_0 \in B$ such that $\sup_{b\in B} M(a,b,t) = M(a,b_0,t)$ and $\inf_{b\in B} N(a,b,t) = N(a,b_0,t)$, i.e. $M(a,B,t) = M(a,b_0,t)$ and $N(a,B,t) = N(a,b_0,t)$. □

Lemma 3.2. *Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space. $(C(X),H_M,H_N,*,\diamondsuit)$ is a hausdorff intuitionistic fuzzy metric space on $C(X)$. Then for all $A,B \in C(X)$, for each $a \in A$ and for all $t > 0$ there exists $b_a \in B$, satisfies $M(a,B,t) = M(a,b_a,t)$ and $N(a,B,t) = N(a,b_a,t)$ then*

$$H_M(A,B,t) \le M(a,b_a,t),$$

$$H_N(A,B,t) \ge N(a,b_a,t).$$

Proof.

$$M(a,B,t) \ge \inf_{a\in A} M(a,B,t) \ge \min\left(\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\right),$$

$$M(a,b_a,t) \ge H_M(A,B,t). \quad \text{(by Lemma 3.1)}$$

Similarly

$$N(a,B,t) \le \sup_{a\in A} N(a,B,t) \le \max\left(\sup_{a\in A} N(a,B,t), \sup_{b\in B} N(A,b,t)\right),$$

$$N(a,b_a,t) \le H_N(A,B,t). \quad \text{(by Lemma 3.1)}$$

Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space, $x_0 \in X$ and let $\{S_\gamma : \gamma \in \Omega\}$ be a family of multivalued mappings from X to $C(X)$. Then there exists $x_1 \in S_a x_0$ for some $a \in \Omega$, such that $M(x_0,S_a x_0,t) = M(x_0,x_1,t)$ and $N(x_0,S_a x_0,t) = N(x_0,x_1,t)$, for all $t > 0$. Let $x_2 \in S_b x_1$, such that $M(x_1,S_b x_1,t) = M(x_1,x_2,t)$ and $N(x_1,S_b x_1,t) = N(x_1,x_2,t)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in S_\delta x_n$, $M(x_n,S_\delta x_n,t) = M(x_n,x_{n+1},t)$ and $N(x_n,S_\delta x_n,t) = N(x_n,x_{n+1},t)$, for all $t > 0$. We denote this iterative sequence by $\{XS_\gamma(x_n) : \gamma \in \Omega\}$ and say that $\{XS_\gamma(x_n)\}$ is a sequence in X generated by x_0. □

Theorem 3.3. *Let $(X,M,N,*,\diamondsuit)$ be a complete intuitionistic fuzzy metric space with $*$ t-norm and $\diamondsuit$ conorm defined as $a*a \ge a$ or $a*b = \min\{a,b\}$ and $a*a \le a$ or $a\diamondsuit b = \max\{a,b\}$, respectively. Let $(C(X),H_M,H_N,*,\diamondsuit)$ be a hausdorff intuitionistic fuzzy metric space on $C(X)$, $\{S_\gamma : \gamma \in \Omega\}$ be a family of multivalued mappings from X to $C(X)$ and $\{XS_\gamma(x_n) : \gamma \in \Omega\}$ be a sequence in X generated by x_0. Assume that for some $0 < \alpha_{i,j} \le k < 1$, for all $t > 0$, $x_0 \in X$, for all $x,y \in \overline{B(x_0,r,t)} \cap \{XS_\gamma(x_n) : \gamma \in \Omega\}$, with $x \ne y$ and for all $i,j \in \Omega$ with $i \ne j$, we have*

$$H_M(S_i x, S_j y, \alpha_{i,j} t) \ge M(x,y,t), \tag{3.1}$$

$$H_N(S_i x, S_j y, \alpha_{i,j} t) \le N(x,y,t) \tag{3.2}$$

also for some $t > 0$

$$M(x_0,x_1,(1-k)t) \ge 1-r, \tag{3.3}$$

$$N(x_0,x_1,(1-k)t) \le r. \tag{3.4}$$

Then, $\{XS_\gamma(x_n):\gamma\in\Omega\}$ *is a sequence in* $\overline{B(x_0,r,t)}$ *and* $\{XS_\gamma(x_n):\gamma\in\Omega\}\to z\in\overline{B(x_0,r,t)}$. *Also, if* (3.1), (3.2) *hold for* z, *then there exists a common fixed point for the family of multivalued mappings* $\{S_\gamma:\gamma\in\Omega\}$ *in* $\overline{B(x_0,r,t)}$.

Proof. It is supposed that $\{XS_\gamma(x_n):\gamma\in\Omega\}$ is a sequence in X generated by x_0. If $x_0=x_1$ then x_0 is a common fixed point of S_a for all $a\in\Omega$. Let $x_0\neq x_1$ and by lemma (3.2), we have

$$M(x_1,x_2,t)\geq H_M(S_ax_0,S_bx_1,t)\quad\text{and}\quad N(x_1,x_2,t)\leq H_N(S_ax_0,S_bx_1,t).$$

By induction, we have by Lemma 3.2

$$M(x_n,x_{n+1},t)\geq H_M(S_ix_{n-1},S_\gamma x_n,t),\tag{3.5}$$

$$N(x_n,x_{n+1},t)\leq H_N(S_ix_{n-1},S_\gamma x_n,t).\tag{3.6}$$

First we show that $x_n\in\overline{B(x_0,r,t)}$. By eq. (3.3), (3.4), we get

$$M(x_0,x_1,t)=M(x_0,S_ax_0,t)>M(x_0,x_1,(1-k)t)\geq 1-r,$$

$$N(x_0,x_1,t)=N(x_0,S_ax_0,t)<N(x_0,x_1,(1-k)t)\leq r.$$

This shows that $x_1\in\overline{B(x_0,r,t)}$. Let $x_2,x_3,\dots,x_j\in\overline{B(x_0,r,t)}$. Now, we have

$$\begin{aligned}M(x_j,x_{j+1},t)&\geq H_M(S_\delta x_{j-1},S_\eta x_j,t)\\&\geq M\left(x_{j-1},x_j,\frac{t}{\alpha_{\delta,\eta}}\right)\\&\geq H_M\left(S_\rho x_{j-2},S_\delta x_{j-1},\frac{t}{\alpha_{\delta,\eta}}\right)\\&\geq M\left(x_{j-2},x_{j-1},\frac{t}{\alpha_{\rho,m},\alpha_{\delta,\eta}}\right)\\&\geq M\left(x_{j-2},x_{j-1},\frac{t}{k^2}\right)\geq\ldots\geq M\left(x_0,x_1,\frac{t}{k^j}\right)\\&\geq M\left(x_0,x_1,\frac{t}{k^j}\right).\end{aligned}\tag{3.7}$$

Moreover,

$$\begin{aligned}N(x_j,x_{j+1},t)&\leq H_N(S_\delta x_{j-1},S_\eta x_j,t)\\&\leq N\left(x_{j-1},x_j,\frac{t}{\alpha_{\delta,\eta}}\right)\\&\leq H_N\left(S_\rho x_{j-2},S_\delta x_{j-1},\frac{t}{\alpha_{\delta,\eta}}\right)\\&\leq N\left(x_{j-2},x_{j-1},\frac{t}{\alpha_{\rho,m},\alpha_{\delta,\eta}}\right)\\&\leq N\left(x_{j-2},x_{j-1},\frac{t}{k^2}\right)\leq\ldots\leq N\left(x_0,x_1,\frac{t}{k^j}\right)\\&\leq N\left(x_0,x_1,\frac{t}{k^j}\right).\end{aligned}\tag{3.8}$$

Now,

$$\begin{aligned}
M(x_0,x_{j+1},t) &\geq M(x_0,x_{j+1},(1-k^{j+1})t) \\
&\geq M(x_0,x_1,(1-k)t) * M(x_1,x_2,(1-k)kt) * \ldots * M(x_j,x_{j+1},(1-k)k^j t) \\
&\geq M(x_0,x_1,(1-k)t) * M(x_1,x_2,(1-k)t) * \ldots * M(x_j,x_{j+1},(1-k)t) \quad \text{(by (3.7))} \\
&\geq 1-r * 1-r * \ldots * 1-r = 1-r \\
&\geq 1-r
\end{aligned}$$

and

$$\begin{aligned}
N(x_0,x_{j+1},t) &\leq N(x_0,x_{j+1},(1-k^{j+1})t) \\
&\leq N(x_0,x_1,(1-k)t)\diamondsuit N(x_1,x_2,(1-k)kt)\diamondsuit \ldots \diamondsuit N(x_j,x_{j+1},(1-k)k^j t) \\
&\leq N(x_0,x_1,(1-k)t)\diamondsuit N(x_1,x_2,(1-k)t)\diamondsuit \ldots \diamondsuit N(x_j,x_{j+1},(1-k)t) \quad \text{(by (3.8))} \\
&\leq r\diamondsuit r\diamondsuit \ldots \diamondsuit r = r \\
&\leq r.
\end{aligned}$$

This implies that $x_{j+1} \in \overline{B(x_0,r,t)}$. Now inequalities (3.7) and (3.8) can be written as

$$M(x_n,x_{n+1},t) \geq M\left(x_0,x_1,\frac{t}{k^n}\right), \tag{3.9}$$

$$N(x_n,x_{n+1},t) \leq N\left(x_0,x_1,\frac{t}{k^n}\right), \tag{3.10}$$

for all n and $t > 0$.

Now, for each $n,m \in N$; $m > n$, we have

$$\begin{aligned}
M(x_n,x_m,t) &> M(x_n,x_m,(1-k^{m-n})t) \\
&\geq M(x_n,x_{n+1},(1-k)t) * M(x_{n+1},x_{n+2},(1-k)kt) * \ldots * M(x_{m-1},x_m,(1-k)k^{m-n-1}t) \\
&\geq M\left(x_0,x_1,\frac{(1-k)t}{k^n}\right) * M\left(x_0,x_1,\frac{(1-k)kt}{k^{n+1}}\right) * \ldots * M\left(x_0,x_1,\frac{(1-k)k^{m-n-1}t}{k^{m-1}}\right) \\
&= M\left(x_0,x_1,\frac{(1-k)t}{k^n}\right) * M\left(x_0,x_1,\frac{(1-k)t}{k^n}\right) * \ldots * M\left(x_0,x_1,\frac{(1-k)t}{k^n}\right) \\
&= M\left(x_0,x_1,\frac{(1-k)t}{k^n}\right)
\end{aligned}$$

As, $\lim_{t\to\infty} M(x,y,t) = 1$, for all $x,y \in X$. In particular

$$M\left(x_0,x_1,\frac{(1-k)t}{k^n}\right) = 1 \quad \text{as} \quad n \to \infty.$$

Hence

$$M(x_n,x_m,t) = 1 \quad \text{as} \quad n \to \infty.$$

Also

$$N(x_n,x_m,t) < N(x_n,x_m,(1-k^{m-n})t)$$

$$\leq N(x_n,x_{n+1},(1-k)t)\Diamond N(n+1,x_{n+2},(1-k)kt)\Diamond\dots\Diamond N(x_{m-1},x_m,(1-k)k^{m-n-1}t)$$
$$\leq N\left(x_0,x_1,\frac{(1-k)t}{k^n}\right)\Diamond N\left(x_0,x_1,\frac{(1-k)kt}{k^{n+1}}\right)\Diamond\dots\Diamond N\left(x_0,x_1,\frac{(1-k)k^{m-n-1}t}{k^{m-1}}\right)$$
$$= N\left(x_0,x_1,\frac{(1-k)t}{k^n}\right)\Diamond N\left(x_0,x_1,\frac{(1-k)t}{k^n}\right)\Diamond\dots\Diamond N\left(x_0,x_1,\frac{(1-k)t}{k^n}\right)$$
$$= N\left(x_0,x_1,\frac{(1-k)t}{k^n}\right).$$

As, $\lim_{t\to\infty} N(x,y,t)=0$, for all $x,y\in X$. In particular

$$N\left(x_0,x_1,\frac{(1-k)t}{k^n}\right)=0 \text{ as } n\to\infty.$$

Hence

$$N(x_n,x_m,t)=0 \text{ as } n\to\infty.$$

Hence $\{XS_\gamma(x_n)\}$ is a Cauchy sequence in $\overline{B(x_0,r,t)}$. As every closed ball in a complete fuzzy metric space is complete. So, $\overline{B(x_0,r,t)}$ is complete. Then there exists a point z in $\overline{B(x_0,r,t)}$ such that

$$\lim_{n\to\infty} XS_\gamma(x_n)=z.$$

Now for some $q\in\Omega$, we have

$$M(z,S_qz,t)\geq M(z,x_n,(1-k)t)*M(x_n,S_qz,kt).$$

By Lemma 3.2, we have

$$M(z,S_qz,t)\geq M(z,x_n,(1-k)t)*H_M(S_rx_{n-1},S_qz,kt)$$
$$\geq M(z,x_n,)(1-k)t)*M(x_{n-1},z,kt/\alpha_{r,q})$$
$$\geq M(z,x_n,(1-k)t)*M(x_{n-1},z,t).$$

Letting $n\to\infty$, we have

$$M(z,S_qz,t)\geq 1*1=1$$

and

$$N(z,S_qz,t)\leq N(z,x_n,(1-k)t)\Diamond N(x_n,S_qz,kt)$$
$$N(z,S_qz,t)\leq N(z,x_n,(1-k)t)*H_N(S_rx_{n-1},S_qz,kt)$$
$$\leq N(z,x_n,)(1-k)t)\Diamond N(x_{n-1},z,kt/\alpha_{r,q})$$
$$\leq N(z,x_n,(1-k)t)\Diamond N(x_{n-1},z,t).$$

Letting $n\to\infty$, we have

$$N(z,S_qz,t)\leq 0\Diamond 0=0.$$

This implies that $z\in S_qz$. Hence, $z\in\cap\{S_qz:q\in\Omega\}$. This completes the proof. □

Let $(X,M,N,*,\diamondsuit)$ be an intuitionistic fuzzy metric space, $x_0 \in X$ and let S be a multivalued mapping from X to $C(X)$. Then there exists $x_1 \in Sx_0$, such that $M(x_0,Sx_0,t) = M(x_0,x_1,t)$ and $N(x_0,Sx_0,t) = N(x_0,x_1,t)$, for all $t > 0$. Let $x_2 \in Sx_1$, such that $M(x_1,Sx_1,t) = M(x_1,x_2,t)$ and $N(x_1,Sx_1,t) = N(x_1,x_2,t)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Sx_n$, $M(x_n,Sx_n,t) = M(x_n,x_{n+1},t)$ and $N(x_n,Sx_n,t) = N(x_n,x_{n+1},t)$, for all $t > 0$. We denote this iterative sequence by $\{XS(x_n)\}$ and say that $\{XS(x_n)\}$ is a sequence in X generated by x_0.

Corollary 3.4. *Let $(X,M,N,*,\diamondsuit)$ be a complete intuitionistic fuzzy metric space with $*$ t-norm and $\diamondsuit$ conorm defined as $a*a \geq a$ or $a*b = \min\{a,b\}$ and $a*a \leq a$ or $a\diamondsuit b = \max\{a,b\}$, respectively. Let $(C(X),H_M,H_N,*,\diamondsuit)$ be a hausdorff intuitionistic fuzzy metric space on $C(X)$, $S : X \to C(X)$ be a multivalued mapping from X to $C(X)$ and $\{XS(x_n)\}$ be a sequence in X generated by x_0. Assume that for some $0 < k < 1$, $t > 0$, $x_0 \in X$, for all $x,y \in \overline{B(x_0,r,t)} \cap \{XS(x_n)\}$, with $x \neq y$, we have*

$$H_M(Sx,Sy,kt) \geq M(x,y,t), \tag{3.11}$$

$$H_N(Sx,Sy,kt) \leq N(x,y,t) \tag{3.12}$$

and

$$M(x_0,Sx_0,(1-k)t) \geq 1-r, \tag{3.13}$$

$$N(x_0,Sx_0,(1-k)t) \leq r. \tag{3.14}$$

Then, $\{XS(x_n)\}$ is a sequence in $\overline{B(x_0,r,t)}$ and $\{XS(x_n)\} \to z \in \overline{B(x_0,r,t)}$. Also, if (3.11), (3.12) hold for z, then there exists a fixed point for S in $\overline{B(x_0,r,t)}$.

Proof. The technique which is used in above theorem can also be applied easily to prove this corollary. □

Corollary 3.5. *Let $(X,M,N,*,\diamondsuit)$ be a complete intuitionistic fuzzy metric space with $*$ t-norm and $\diamondsuit$ conorm defined as $a*a \geq a$ or $a*b = \min\{a,b\}$ and $a*a \leq a$ or $a\diamondsuit b = \max\{a,b\}$, respectively. $S : X \to X$ be a self mapping from X to X. Assume that for some $0 < k < 1$, $t > 0$, $x_0 \in X$, for all $x,y \in \overline{B(x_0,r,t)}$, with $x \neq y$, we have*

$$M(Sx,Sy,kt) \geq M(x,y,t), \tag{3.15}$$

$$N(Sx,Sy,k) \leq N(x,y,t) \tag{3.16}$$

and

$$M(x_0,Sx_0,(1-k)t) \geq 1-r, \tag{3.17}$$

$$N(x_0,Sx_0,(1-k)t) \leq r. \tag{3.18}$$

Then S has a fixed point in $\overline{B(x_0,r,t)}$.

Example 3.1. Let $X = [0,2]$ and d be a Euclidean metric on X. Denote $a*b = \min\{a,b\}$ and $a\diamondsuit b = \max\{a,b\}$ for all $a,b \in [0,1]$, $M(x,y,t) = \frac{t}{t+d(x,y)}$ and $N(x,y,t) = \frac{d(x,y)}{t+d(x,y)}$ for all $x,y \in X$ and

$t > 0$. Then we can find that $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Consider the multivalued mappings $S_\gamma : X \to C(X)$, where $\gamma = a, 1, 2, 3, \dots$ defined as

$$S_n x = \begin{cases} \left[\frac{x}{4n}, \frac{x}{3n}\right], & \text{if } x \in \left[0, \frac{3}{2}\right] \\ [3nx, 4nx], & \text{if } x \in \left(\frac{3}{2}, 2\right], \end{cases}$$

where $n = 1, 2, 3, \dots$

$$S_a x = \begin{cases} \left[\frac{x}{4}, \frac{5x}{18}\right], & \text{if } x \in \left[0, \frac{3}{2}\right] \\ [3x, 4x], & \text{if } x \in \left(\frac{3}{2}, 2\right]. \end{cases}$$

Consider $x_0 = \frac{1}{2}$ and $r = \frac{1}{2}$, then $\overline{B(x_0, r, t)} = [0, \frac{3}{2}]$. Now,

$$M(x_0, S_a(x_0), t) = M\left(\frac{1}{2}, S_a\left(\frac{1}{2}\right), t\right) = M\left(\frac{1}{2}, \frac{5}{36}, t\right),$$

$$M(x_1, S_1(x_1), t) = M\left(\frac{5}{36}, S_1\left(\frac{5}{36}\right), t\right) = M\left(\frac{5}{36}, \frac{5}{108}, t\right),$$

$$M(x_2, S_2(x_2), t) = M\left(\frac{5}{108}, S_2\left(\frac{5}{108}\right), t\right) = M\left(\frac{5}{108}, \frac{5}{648}, t\right).$$

So we obtain a sequence $\{XS_\gamma(x_n)\} = \left\{\frac{1}{2}, \frac{5}{36}, \frac{5}{108}, \frac{5}{648}, \dots\right\}$ in X generated by x_0.

Now for $x = \frac{8}{5}$, $y = \frac{9}{5}$, $k = \alpha_{1,a} = \frac{1}{4}$ and $t = 1$, we have

$$H_M\left(S_1\left(\frac{8}{5}\right), S_a\left(\frac{9}{5}\right), \frac{1}{4}\right) = \min\left\{\inf_{b \in S_1(\frac{8}{5})}\left(M\left(b, S_a\left(\frac{9}{5}\right), \frac{1}{4}\right)\right), \inf_{c \in S_a(\frac{9}{5})}\left(M\left(S_1\left(\frac{8}{5}\right), c, \frac{1}{4}\right)\right)\right\} = 0.238,$$

$$M\left(\frac{8}{5}, \frac{9}{5}, 1\right) = \frac{1}{1 + \left|\frac{8}{5} - \frac{9}{5}\right|} = \frac{5}{6} = 0.833.$$

Its clear that

$$H_M\left(S_1\left(\frac{8}{5}\right), S_a\left(\frac{9}{5}\right), \frac{1}{4}\right) < M\left(\frac{8}{5}, \frac{9}{5}, 1\right).$$

Now for all $x, y \in \overline{B(x_0, r, t)} \cap \{XS_\gamma(x_n)\}$, we have

$$\begin{aligned} H_M(S_n x, S_a y, kt) &= \min\left\{\inf_{b \in S_n x}(M(b, S_a y, kt), \inf_{c \in S_a y}(M(S_n x, c, kt)\right\} \\ &= \min\left\{\inf_{b \in S_n x}\left(Mv\left(b, \left[\frac{y}{4}, \frac{5y}{18}\right], \frac{1}{4}t\right)\right), \inf_{c \in S_a y}\left(M\left(\left[\frac{x}{4n}, \frac{x}{3n}\right], c, \frac{1}{4}t\right)\right)\right\} \\ &= \min\left\{M\left(\frac{x}{3n}, \frac{5y}{18}, \frac{1}{4}t\right), M\left(\frac{x}{4n}, \frac{y}{4}, \frac{1}{4}\right)\right\} \\ &= \min\left\{\frac{(1/4)t}{(1/4)t + |x/3n - 5y/18|}, \frac{(1/4)t}{(1/4)t + |x/4n - y/4|}\right\}, \\ H_M(Sx, Sy, kt) &= \frac{(1/4)t}{(1/4)t + |x/4 - y/4|} \\ &\geq \frac{t}{(t + |x - y|)} \end{aligned}$$

$$= M(x,y,t),$$

$$H_N(S_n x, S_a y, kt) = \max\left\{\sup_{b\in S_n x}(N(b,S_a y,kt), \sup_{c\in S_a y}(N(S_n x,c,kt)\right\}$$

$$= \max\left\{\sup_{b\in S_n x}\left(N\left(b,\left[\frac{y}{4},\frac{5y}{18}\right],\frac{1}{4}t\right)\right), \sup_{c\in S_a y}\left(N\left(\left[\frac{x}{4n},\frac{x}{3n}\right],c,\frac{1}{4}t\right)\right)\right\}$$

$$= \max\left\{\frac{|x/3n - 5y/18|}{(1/4)t + |x/3n - 5y/18|}, \frac{|x/4n - y/4|}{(1/4)t + |x/4n - y/4|}\right\},$$

$$H_N(Sx,Sy,kt) = \frac{|x/4 - y/4|}{(1/4)t + |x/4 - y/4|}$$

$$\leq \frac{|x-y|}{(t + |x-y|)}$$

$$= N(x,y,t).$$

So the contractive conditions hold on $\overline{B(x_0,r,t)} \cap \{XS_\gamma(x_n)\}$. Now for $t = 1$

$$M(x_0,x_1,(1-k)t) = M\left(\frac{1}{2},\frac{5}{36},\frac{3}{4}\right) = \frac{27}{40}$$

$$> \frac{1}{2} = 1 - r,$$

$$N(x_0,x_1,(1-k)t) = N\left(\frac{1}{2},\frac{5}{36},\frac{3}{4}\right) = 1 - M\left(\frac{1}{2},\frac{5}{36},\frac{3}{4}\right) = 1 - \frac{27}{40} = \frac{13}{40}$$

$$< \frac{1}{2} = r.$$

Hence all the conditions of above theorem are satisfied. Now we have $\{XS_\gamma(x_n)\}$ is a sequence in $\overline{B(x_0,r,t)}$ and $\{XS_\gamma(x_n)\} \to 0 \in \overline{B(x_0,r,t)}$. Moreover $\{S_\gamma : \gamma = a,1,2\ldots\}$ has a common fixed point 0.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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