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Global Best Approximate Solutions for Set Valued Contraction in b-metric Spaces with Applications

Research Article

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Abstract. The aim of this paper is to introduce the notion of multivalued Ćirić type α_*-ψ-proximal contraction and prove some best proximity point results for such contraction in b-metric spaces. We also deduce some best proximity point results for single valued mapping. Moreover, we apply our results to obtain relative best proximity point results in partially ordered metric spaces. As an application of our results we obtain fixed point results for the spaces concern. We give some examples to illustrate the obtained results. Finally, an application to nonlinear integral equation is presented. Our results extended and generalized many existing results in the literature.

Keywords. Best proximity point; Weak P-property; α-admissible mapping

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1. Introduction

The well known Banach contraction principle [11] is one of the most useful tool in analysis. Many authors generalized this classical result in many directions (see for example [16, 33]).

The result was extended by S.B. Nadler [28] to the context of set valued contraction. Recently, Samet *et al.* [33] introduced the notion of α-ψ-contraction and proved some fixed point theorems for such mappings in the context of complete metric spaces. Karapinar *et al.* [25] generalized the contractive condition of Samet *et al.* [33] and obtained fixed point results for such mappings. Some interesting multivalued generalizations of α-ψ-contractive type mappings are given in [2, 4, 7, 8, 27].

Let Y be a nonempty subset of a metric space (X,d). A mapping $T:Y\to X$ is said to have a fixed point in Y, if the fixed point equation $Tx=x$ has at least one solution. That is, $x\in Y$ is a fixed point of T if $d(x,Tx)=0$. The case when fixed point equation $Tx=x$ does not have a solution, then $d(x,Tx)>0$ for all $x\in Y$. In such circumstances, we are in searching for an element $x\in Y$ such that $d(x,Tx)$ is minimum in some sense. The best approximation theory and best proximity pair theorems are studied in this direction. Consider a pair of nonempty subsets (A,B) of a metric space (X,d). A mapping $T:A\to B$ is said to have a best proximity point if $d(x,Tx)=d(A,B)$. If $d(A,B)=0$, best proximity point is nothing but a fixed point of T. Many authors has explored the existence and convergence of best proximity points under different contractive conditions in certain distance spaces (see e.g. [1, 3, 5, 18, 22, 23, 26] and references therein).

Metric spaces have been generalized according to requirement and their applicability to solve a particular problem. The problem of convergence of measurable functions with respect to measure leaded to generalize the metric space in such a way that set considered in metric space is replaced with the space and consequently the function d is replaced with the functional d. The metric space defined in the above is called b-metric space. It was first introduced as quasi metric in 1989 by Bakhtin [10]. Formally, in 1993, Czerwik [19, 20] introduced the notion of b-metric space as a generalization of ordinary metric space and proved contraction mapping principle in b-metric spaces. Later on Samet [32] introduced the notion of α-ψ-contraction and prove some fixed point results in b-metric space. Bota *et al.* [15] established the existence of fixed point theorems for α-ψ-contractive mapping of type-(b) in the framework of b-metric spaces. Many authors showed their interest in investigating the existence and uniqueness of certain fixed point as well as best proximity point results in b-metric space (see for example [6, 9] and references therein).

The purpose of this paper is to workout for the multivalued and single valued best proximity point results for Ćirić type α_*-ψ-proximal contraction in the framework of b-metric spaces and to apply the results obtained for the same results for partially ordered b-metric spaces. We will also find some fixed point results for such mappings as an applications of our results.

The paper is arranged in the following way: In Section 2, some preliminaries and known results are presented. In Section 3, we present best proximity point theorems for multivalued mappings. In Section 4, some best proximity point results for single valued mappings are presented. Section 5 is devoted to best proximity point theorems in partially ordered b-metric spaces. In Section 6, we give some fixed point results in b-metric and partially ordered b-metric spaces. In last section we give application to nonlinear integral equation.

2. Preliminaries

Definition 2.1 ([10, 19]). Let X be a nonempty set, and let $k \geq 1$ be a given real number. A functional $d : X \times X \to [0,\infty)$ is said to be a b-metric if for all $x,y,z \in X$, the following conditions are satisfied:

(1) $d(x,y) = 0 \Leftrightarrow x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq k(d(x,z) + d(z,y))$.

In this case pair (X,d) is called b-metric space with constant k.

Example 2.1 ([13]). The space L_p $(0 < p < 1)$ of all real function $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is b-metric space if we take

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}.$$

Definition 2.2 ([10, 19]). Let (X,d) be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) the sequence $\{x_n\}$ converges to a point $x \in X$ if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$ we have $d(x_n,x) < \epsilon$,

(ii) the sequence $\{x_n\}$ is Cauchy sequence iff for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $m,n \geq n(\epsilon)$ we have $d(x_n,x_m) < \epsilon$,

(iii) (X,d) is said to be a complete b-metric space if every Cauchy sequence in X converges to some $x \in X$.

In the sequel, (X,d) a b-metric space, $CL(X)$, $CB(X)$ and $K(X)$ by the families of all nonempty closed subsets, closed and bounded subsets and compact subsets of (X,d), respectively. For any $A,B \in CB(X)$ and $x \in X$, define

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\},$$

$$A_0 = \{a \in A : \text{there exists some } b \in B \text{ such that } d(a,b) = D(A,B)\},$$

$$B_0 = \{b \in B : \text{there exists some } a \in A \text{ such that } d(a,b) = D(A,B)\},$$

$$D(x,A) = \inf\{d(x,a) : a \in A\},$$

$$H(A,B) = \max\{\delta(A,B), \delta(B,A)\}.$$

The above H is called the Pompeiu-Hausddroff metric.

Definition 2.3 ([21]). Let A and B be nonempty subsets of a b-metric space (X,d). A point x is called a best proximity point of mapping $T : A \to B$ if

$$d(x,Tx) = \mathrm{d}(A,B).$$

Lemma 2.1 ([2]). *Let (X,d) be a metric space and $B \in CB(X)$. Then for each $x \in X$ with $D(x,B) > 0$ and $q > 1$, there exists an element $b \in B$ such that*

$$d(x,b) < qD(x,B).$$

Lemma 2.2 ([14,20]). *Let (X,d) be a b-metric space with constant k. Then*

$$D(x,A) \le k[d(x,y) + D(y,A)], \quad \text{for all } x,y \in X, A \subseteq X.$$

Lemma 2.3 ([18]). *Let (X,d) be a metric space and $A,B \in CB(X)$. Let $q \ge 1$. Then for every $x \in A$, there exists $y \in B$ such that $d(x,y) \le qH(A,B)$.*

Definition 2.4 ([4]). Let (A,B) be a pair of nonempty subsets of a b-metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have weak P-property if and only if for any $x_1,x_2 \in A$ and $y_1,y_2 \in B$,

$$\left.\begin{aligned} d(x_1,y_1) = \mathrm{D}(A,B) \\ d(x_2,y_2) = \mathrm{D}(A,B) \end{aligned}\right\} \Rightarrow d(x_1,x_2) \le d(y_1,y_2).$$

Let us denote by Ψ the set of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty, \quad \text{for all } t > 0,$$

where ψ^n is the nth iterate of ψ. These functions are known are comparison functions. Also $\psi(t) < t$ for all $t > 0$.

Usman *et al.* (see [3]) introduced the notions of α-ψ-proximal contraction and α-proximal admissibility to multivalued maps and proved some best proximity point theorems for multivalued mappings.

Definition 2.5 ([3]). Let A and B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to CB(B)$ is said to be an α-ψ-proximal contraction, if there exists $\psi \in \Psi$ and $\alpha : A \times A \to [0,\infty)$ such that

$$\alpha(x,y)H(Tx,Ty) \le \psi(d(x,y)), \quad \text{for all } x,y \in A. \tag{2.1}$$

Theorem 2.1 ([3]). *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is non-empty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \to CB(B)$ is a mapping satisfying the following assertions:*

(1) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property;*

(2) *T is α-proximal admissible;*

(3) *there exists $x_0,x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1,y_1) = \mathrm{dist}(A,B), \quad \alpha(x_0,x_1) \ge 1;$$

(4) *T is continuous α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $D(x^*,Tx^*) = \mathrm{dist}(A,B)$.

3. Multivalued Best Proximity Point Results

We begin this section with the following definition:

Definition 3.1. Let A and B be two nonempty subsets of a b-metric space (X,d). A mapping $T : A \to CB(B)\setminus\emptyset$ is called modified α_*-proximal admissible if there exists a mapping $\alpha_* : A\times A \to [0,\infty)$ such that

$$\left.\begin{array}{l}\alpha_*(Tx_0,Tx_1)\geq 1\\ D(x_1,Tx_0)=D(A,B)\\ D(x_2,Tx_1)=D(A,B)\end{array}\right\}\Rightarrow \alpha_*(Tx_1,Tx_2)\geq 1,$$

where $x_0,x_1,x_2\in A$ and $\alpha_*(A,B)=\inf\{\alpha(x,y)|x\in A, y\in B\}$.

Example 3.1. Let $X=\mathbb{N}$ with metric $d(x,y)=|x-y|^2$ for all $x,y\in X$. Let $A=\{0,2,4,\dots\}$ and $B=\{1,3,5,\dots\}$ be two subsets of X, then $D(A,B)=1$. Define a mapping $T : A\to CB(B)$ by

$$Tx=\{1,3,5,\dots,x+1\} \text{ for all } x\in A. \tag{3.1}$$

Also define $\alpha_* : A\times A\to[0,\infty)$ by

$$\alpha_*(x,y)=\begin{cases}1 & \text{if } x,y\in A\\ 2 & \text{if } x,y\in B.\end{cases}$$

Now $\alpha_*(Tx,Ty)=\inf\{\alpha(a,b)|a\in Tx, b\in Ty\}$. For $x_0=4\in A$ and $x_1=6\in A$, we have $Tx_0=\{1,3,5\}$ and $Tx_1=\{1,3,5,7\}$. Then $\alpha_*(Tx_0,Tx_1)=2$, let $x_2=8\in A$ then $D(x_1,Tx_0)=1=D(A,B)$ and $D(x_2,Tx_1)=1=D(A,B)$. Now, we have

$$\left.\begin{array}{l}\alpha_*(Tx_0,Tx_1)\geq 1\\ D(x_1,Tx_0)=D(A,B)\\ D(x_2,Tx_1)=D(A,B).\end{array}\right\}$$

Now, $\alpha_*(Tx_1,Tx_2)=\inf\{d(x,y)|x\in Tx, y\in Ty\}=2$. So, we have

$$\alpha_*(Tx_1,Tx_2)\geq 1.$$

Hence T is modified α_*-proximal admissible.

Definition 3.2. Let (X,d) be a b-metric space, let A and B be two subsets of X, $\alpha_* : A\times A\to[0,\infty)$ and $T : A\to CB(B)$ be given mappings. We say T is α_*-continuous multivalued mapping on $(CB(X),H)$, if for all sequences $\{x_n\}$ with $x_n\xrightarrow{d} x\in A$ as $n\to\infty$ and $\alpha_*(Tx_n,Tx_{n+1})\geq 1$ for all $n\in\mathbb{N}$, we have $Tx_n\xrightarrow{H} Tx$ as $n\to\infty$.

Definition 3.3. Let A and B be two nonempty subsets of a b-metric space (X,d). A multivalued mapping $T : A\to CB(B)$ is said to be Ćirić type α_*-ψ-proximal contraction, if there exists $\psi\in\Psi$ and $\alpha_* : A\times A\to[0,\infty)$ such that

$$\alpha_*(Tx,Ty)H(Tx,Ty)\leq\psi(M(x,y)), \text{ for all } x,y\in A, \tag{3.2}$$

where

$$M(x,y)=\max\left\{d(x,y),\frac{1}{k}D(x,Tx)-D(A,B),\frac{1}{k}D(y,Ty)-D(A,B)\right\}.$$

Example 3.2. Consider $X = \mathbb{R}^2$ with b-metric $d(x,y) = |x_1 - x_2|^3 + |y_1 - y_2|^3$, for all (x_1,x_2), $(y_1,y_2) \in \mathbb{R}^2$ and $k = 4$. Suppose $A = \{(1,x) : 0 \le x \le 1\}$ and $B = \{(0,x) : 0 \le x \le 1\}$, then $D(A,B) = 1$. Define $T : A \to CB(B) \setminus \emptyset$ by

$$T(1,x) = \begin{cases} \{(0,1)\} & x = 1, \\ \left\{\left(0,\frac{a}{2}\right) : 0 \le a \le x\right\} & \text{otherwise,} \end{cases}$$

$\alpha_* : A \times A \to [0,\infty)$ by

$$\alpha_*((x_1,y_1),(x_2,y_2)) = \begin{cases} 1 & x_1,x_2 \in \{0,1\},\ y_1,y_2 \in \left[0,\frac{1}{2}\right] \\ 0 & \text{otherwise,} \end{cases}$$

and $\psi(t) = \frac{t}{8}$ for all t. Now

$$\alpha_*(A,B) = \inf\{\alpha(a,b), a \in A, b \in B\} = 1.$$

If $z_1 = (1,x_1)$ and $z_2 = (1,x_2)$ in A, for $x_1,x_2 \in \left[0,\frac{1}{2}\right]$. Then,

$$Tz_1 = \left\{\left(0,\frac{a}{2}\right) : 0 \le a \le x_1\right\}$$

and

$$Tz_2 = \left\{\left(0,\frac{a}{2}\right) : 0 \le a \le x_2\right\}.$$

This shows that $d(u_1,y_1) = 1 = \mathrm{D}(A,B)$ and $d(u_2,y_2) = 1 = \mathrm{D}(A,B)$ for all $y_1 \in Tx_1$ and $y_2 \in Tx_2$ if and only if $u_1,u_2 \in \left\{\left(1,\frac{x}{2}\right) : 0 \le x \le \frac{1}{2}\right\}$. Now $\alpha_*(Tz_1,Tz_2) = 1$ and

$$\begin{aligned} \alpha_*(Tz_1,Tz_2)H(Tz_1,Tz_2) &= |1-1|^3 + \left|\frac{x_1}{2} - \frac{x_2}{2}\right|^3 \\ &= 0 + \frac{1}{8}|x_1 - x_2|^3 \\ &= \frac{1}{8}|x_1 - x_2|^3. \end{aligned} \tag{3.3}$$

On the other hand

$$\begin{aligned} M(z_1,z_2) &= \max\left\{d(z_1,z_2), \frac{1}{k}D(z_1,Tz_1) - D(A,B), \frac{1}{k}D(z_2,Tz_2) - D(A,B)\right\} \\ &= \max\left\{|x_1-x_2|^3, \frac{1}{4}\left(|1-0|^3 + \left|x_1 - \frac{x_1}{2}\right|^3\right) - 1, \frac{1}{4}\left(|1-0|^3 + \left|x_2 - \frac{x_2}{2}\right|^3 - 1\right)\right\} \\ &= \max\left\{|x_1-x_2|^3, \frac{1}{4}\left(1 + \frac{x_1^3}{8}\right) - 1, \frac{1}{4}\left(1 + \frac{x_2^3}{8}\right) - 1\right\} \\ &= |x_1 - x_2|^3. \end{aligned} \tag{3.4}$$

So by definition of ψ, we have

$$\psi(M(z_1,z_2)) = \frac{1}{8}|x_1 - x_2|^3. \tag{3.5}$$

From (3.3) and (3.5), we get

$$\alpha_*(Tz_1,Tz_2)H(Tz_1,Tz_2) = \psi(M(z_1,z_2)),$$

for all $z_1,z_2 \in A$.

Theorem 3.1. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty. Let $\alpha_* : A \times A \to [0,\infty)$, $\psi \in \Psi$ be a strictly increasing map and $T : A \to CB(B)$ be continuous multivalued mapping satisfying the following assertions:*

(1) *T is Ćirić type α_*-ψ-proximal contraction;*

(2) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property;*

(3) *T is modified α_*-proximal admissible;*

(4) *there exists $x_0, x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that $d(x_1, y_0) = D(A,B)$ and $\alpha_*(Tx_0, Tx_1) \geq 1$.*

Then the mapping T has a best proximity point.

Proof. By assumption, there exists $x_0, x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that

$$D(x_1, y_0) = D(A,B) \text{ and } \alpha_*(Tx_0, Tx_1) \geq 1. \tag{3.6}$$

By Lemma 2.3 corresponding to $y_0 \in Tx_0$, there exists $y_1 \in Tx_1$ such that

$$D(y_0, y_1) \leq \alpha_*(Tx_0, Tx_1) H(Tx_0, Tx_1)$$

since $y_1 \in Tx_1 \subseteq B_0$, there exists $x_2 \in A_0$ such that $D(x_2, y_1) = D(A,B)$. Now, $x_0, x_1, x_2 \in A_0 \subseteq A$ and $y_0 \in Tx_0, y_1 \in Tx_1$ such that $\alpha_*(Tx_0, Tx_1) \geq 1$, $D(x_1, y_0) = D(A,B)$, $D(x_2, y_1) = D(A,B)$. Then it follows from condition (4) that $\alpha_*(Tx_1, Tx_2) \geq 1$. Thus, we have

$$d(x_2, y_1) = D(A,B) \text{ and } \alpha_*(Tx_1, Tx_2) \geq 1.$$

Again by Lemma 2.3, corresponding to $y_1 \in Tx_1$, there exists $y_2 \in Tx_2$ such that

$$D(y_1, y_2) \leq \alpha_*(Tx_1, Tx_2) H(Tx_1, Tx_2).$$

Continuing in this fashion we construct two sequences $\{x_n\}$ and $\{y_n\}$ respectively in $A_0 \subseteq A$ and $B_0 \subseteq B$ such that for $n = 0, 1, 2, \ldots$,

$$D(x_{n+1}, y_n) = D(A,B) \text{ and } \alpha_*(Tx_n, Tx_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\} \tag{3.7}$$

and

$$D(y_n, y_{n+1}) \leq \alpha_*(Tx_n, Tx_{n+1}) H(Tx_n, Tx_{n+1}). \tag{3.8}$$

Since, $d(x_{n+1}, y_n) = D(A,B)$ and $d(x_n, y_{n-1}) = D(A,B)$ for all $n \geq 1$, it follows by the weak P-property of the pair (A,B) that

$$d(x_n, x_{n+1}) \leq d(y_{n-1}, y_n). \tag{3.9}$$

Since T is Ćirić type α_*-ψ proximal contraction, and by using (3.7), (3.8) and (3.9), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(y_{n-1}, y_n) \\ &\leq H(Tx_{n-1}, Tx_n) \\ &\leq \alpha_*(Tx_{n-1}, Tx_n) H(Tx_{n-1}, Tx_n) \\ &\leq \psi(M(x_{n-1}, x_n)) \end{aligned} \tag{3.10}$$

where

$$M(x_{n-1},x_n)=\max\left\{d(x_{n-1},x_n),\frac{1}{k}D(x_{n-1},Tx_{n-1})-D(A,B),\frac{1}{k}D(x_n,Tx_n)-D(A,B)\right\}. \quad (3.11)$$

By Lemma 2.2, we derive

$$\begin{aligned}M(x_{n-1},x_n)\le \max&\left\{d(x_{n-1},x_n),\frac{1}{k}[k(d(x_{n-1},x_n)+D(x_n,Tx_{n-1})]-D(A,B),\right.\\&\left.\frac{1}{k}[k(d(x_n,x_{n+1})+D(x_{n+1},Tx_n)]-D(A,B)\right\}\\&=\max\{d(x_{n-1},x_n),d(x_{n-1},x_n)+D(A,B)-D(A,B),d(x_n,x_{n+1})+D(A,B)-D(A,B)\}\\&=\max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1})\}\\&=\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}.\end{aligned}$$

This together with (3.10) gives

$$d(x_n,x_{n+1})\le\psi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\quad\text{for all } n\in\mathbb{N}. \quad (3.12)$$

Suppose that

$$d(x_n,x_{n+1})>0\quad\text{for all } n\in\mathbb{N}\cup\{0\}. \quad (3.13)$$

If

$$\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}=d(x_n,x_{n+1}). \quad (3.14)$$

From (3.12), we get that

$$d(x_n,x_{n+1})\le\psi(d(x_n,x_{n+1}))<d(x_n,x_{n+1}), \quad (3.15)$$

which is not possible. Thus

$$\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}=d(x_{n-1},x_n). \quad (3.16)$$

From (3.12) we have

$$d(x_n,x_{n+1})\le\psi(d(x_{n-1},x_n)). \quad (3.17)$$

Now

$$\begin{aligned}d(x_n,x_{n+1})&\le\psi(d(x_{n-1},x_n))\\&\le\psi(\psi(d(x_{n-1},x_{n-2})))\\&=\psi^2(d(x_{n-1},x_{n-2}))\\&\;\vdots\\&\le\psi^n(d(x_0,x_1))\end{aligned}$$

for all $n\in\mathbb{N}\bigcup\{0\}$. Then by Definition of ψ, we have

$$\sum_n d(x_n,x_{n+1})\le\sum_n\psi^n(d(x_0,x_1))<\infty.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. From the completeness of X, there exist $x_*, y_* \in X$ such that

$$x_n \to x_* \text{ and } y_n \to y_* \text{ as } n \to \infty. \tag{3.18}$$

Since A and B are closed and $\{x_n\}$ and $\{y_n\}$ are sequences in A and B, respectively, we have $x_* \in A$ and $y_* \in B$.

Now,

$$d(x_{n+1}, y_n) = d(A,B) \text{ for all } n \in \mathbb{N}.$$

Taking limit as $n \to \infty$, we obtain

$$d(x_*, y_*) = d(A,B). \tag{3.19}$$

Now , we claim that $y_* \in Tx_*$.

Since $y_n \in Tx_n$, we have

$$D(y_n, Tx_*) \leq H(Tx_n, Tx_*).$$

Taking limit as $n \to \infty$ in above inequality, using (3.18) and the continuity of T, we have

$$D(y_*, Tx_*) = \lim_{n\to\infty} D(y_n, Tx_*) \leq \lim_{n\to\infty} H(Tx_n, Tx_*) = 0.$$

Now, $D(y_*, Tx_*) = 0$ implies $y_* \in Tx_*$

Now, using (3.19), we have

$$D(x_*, Tx_*) \leq d(x_*, y_*) = \text{dist}(A,B) \leq D(x_*, Tx_*),$$

which implies that $D(x_*, Tx_*) = \text{dist}(A,B)$, that is, x_* is a best proximity point of T in A. $\square$

Example 3.3. Let $X = [0,\infty) \times [0,\infty)$ with b-metric $d(x,y) = |x_1 - x_2|^2 + |y_1 - y_2|^2$ for all $x = (x_1, x_2), y = (y_1, y_2) \in X$ and $k = 2$. Suppose $A = \{(\frac{1}{3}, x) : 0 \leq x \leq \infty\}$ and $B = \{(0,x) : 0 \leq x \leq \infty\}$. Define $T : A \to CB(B)$ by

$$T(\frac{1}{3}, a) = \begin{cases} \{(0, \frac{x}{3}) : 0 \leq x \leq a\} & \text{if } a \leq 1 \\ \{(0, x^2) : 0 \leq x \leq a^2\} & \text{if } a > 1, \end{cases}$$

and $\alpha_* : A \times A \to [0,\infty)$ by

$$\alpha_*(x,y) = \begin{cases} 1 & if \;\; x,y \in \{(\frac{1}{3}, a) : 0 \leq a \leq 1\} \\ 0 & \text{otherwise}, \end{cases}$$

$\psi(t) = \frac{t}{9}$ for all $t \geq 0$. Notice that $A_0 = A, B_0 = B$, and $Tx \subseteq B_0$ for each $x \in A_0$. Also the pair (A,B) satisfies weak $P-$property. Let $x_0, x_1 \in \{(\frac{1}{3}, x) : 0 \leq x \leq 1\}$, then $Tx_0, Tx_1 \subseteq \{(0, \frac{x}{3}) : 0 \leq x \leq 1\}$. Consider $y_1 \in Tx_0, y_2 \in Tx_1$, and $u_1, u_2 \in A$ such that $d(u_1, y_1) = D(A,B)$ and $d(u_, y_2) = D(A,B)$. Then we have $u_1, u_2 \in \{(\frac{1}{3}, x) : 0 \leq x \leq 1\}$. Hence T is modified α_* admissible map. For $x_0 = (\frac{1}{3}, 1) \in A_0$ and $y_1 = (0, \frac{1}{3}) \in Tx_0$ in B_0, we have $x_1 = (\frac{1}{3}, \frac{1}{3}) \in A_0$ such that $d(x_1, y_1) = D(A,B)$ and $\alpha_*(x_0, x_1) = 1$ implies $\alpha_*(Tx_0, Tx_1) = 1$. If $x = (\frac{1}{3}, x_1), y = (\frac{1}{3}, y_1) \in A$ where $0 \leq x_1, y_1 \leq 1$, then we have

$$\alpha_*(Tx,Ty)H(Tx,Ty) = \left|\frac{1}{3}-\frac{1}{3}\right|^2 + \left|\frac{x_1}{3}-\frac{y_1}{3}\right|^2$$
$$= \frac{|x_1-y_1|^2}{9}$$

and

$$M(x,y) = \max\left\{|x_1-y_1|^2, \frac{1}{2}\left(|\frac{1}{3}-0|^2 + \left|x_1-\frac{x_1}{3}\right|^2\right) - \frac{1}{9}, \frac{1}{2}\left(\left|\frac{1}{3}-0\right|^2 + \left|y_1-\frac{y_1}{3}\right|^2\right) - \frac{1}{9}\right\}$$
$$= \max\left\{|x_1-y_1|^2, \frac{1}{2}\left(\frac{1}{9}+\frac{4x_1{}^2}{9}\right) - \frac{1}{9}, \frac{1}{2}\left(\frac{1}{9}+\frac{4y_1{}^2}{9}\right) - \frac{1}{9}\right\}$$
$$= |x_1-y_1|^2.$$

So

$$\psi(M(x,y)) = \frac{|x_1-y_1|^2}{9}.$$

This implies

$$\alpha_*(Tx,Ty)H(Tx,Ty) = \psi(M(x,y)) \tag{3.20}$$

for otherwise

$$\alpha_*(Tx,Ty)H(Tx,Ty) \le \psi(M(x,y)). \tag{3.21}$$

Hence, T is Ćirić type α_*-ψ-proximal contraction. Furthermore, T is continuous and the hypothesis (4) of the Theorem 3.1 is verified. Indeed, for $x_0 = (\frac{1}{3},1), x_1 = (\frac{1}{3},0)$ and $y_1 = (0,0)$, we obtain

$$d_b(x_1,y_1) = d_b\left(\left(\frac{1}{3},0\right),(0,0)\right) = \frac{1}{9} = D(A,B) \text{ and } \alpha_*(x_0,x_1) = 1.$$

Hence all the hypothesis of Theorem 3.1 are verified. Therefore, T has a best proximity point, which is $x^* = (\frac{1}{5},0)$.

Remark 3.1. If we remove the condition of continuity of T in Theorem 3.1 and replace it with α_*-continuity of T, then we have following result:

Theorem 3.2. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty. Let $\alpha_* : A\times A \to [0,\infty)$, $\psi \in \Psi$ be a strictly increasing map and $T : A \to CB(B)$ be multivalued mapping satisfying the following assertions:*

(1) *T is Ćirić type α_*-ψ proximal contraction;*

(2) *$Tz \subseteq B_0$ for each $z \in A_0$ and (A,B) satisfies the weak P-property;*

(3) *T is modified α_*-proximal admissible;*

(4) *T is an α_*-continuous mapping;*

(5) *there exists $x_0,x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that $d(x_1,y_0) = D(A,B)$ and $\alpha_*(Tx_0,Tx_1) \ge 1$.*

Then the mapping T has a best proximity point.

Proof. Resuming the proof of Theorem 3.1 we have a Cauchy sequence $\{x_n\}$ converges to x^* in A. Then by α_*-continuity of T and (3.7), we get $Tx_n \xrightarrow{H} Tx^*$ as $n \to \infty$, that is

$$\lim_{n\to\infty} H(Tx_n, Tx^*) = 0. \tag{3.22}$$

Then from (3.7)

$$D(A,B) = \lim_{n\to\infty} D(x_{n+1}, Tx_n) = D(x^*, Tx^*). \qquad \square$$

Remark 3.2. Note that the uniqueness of the best proximity point of multivalued mapping T is not given in Theorem 3.1 (and Theorem 3.2). Thus, we can propose the following problem: Let (X,d) be a complete b-metric space and $T: A \to CB(B)$ be continuous multivalued mapping satisfying all assertions of Theorem 3.1 (Theorem 3.2).

Does T has a unique best proximity point?

By adding the following condition

$$\mathscr{H} : \alpha_*(x_1,x_2) \geq 1 \text{ for all best proximity points } x_1,x_2 \text{ of } T$$

and taking $T: A \to K(B)$, we are able to give a partial answer to the proposed problem as follows:

Theorem 3.3. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty and $T: A \to K(B)$ be continuous multivalued mapping satisfying all the assertions of Theorem 3.1 (similarly Theorem 3.2) along with condition $\mathscr{H}$. Then the mapping T has a unique best proximity point.*

Proof. We will only prove the uniqueness part. Let x_1, x_2 be two best proximity points of T such that $x_1 \neq x_2$, then by hypothesis $\mathscr{H}$ we have $\alpha_*(x_1,x_2) \geq 1$ and $D(x_1,Tx_1) = D(A,B) = D(x_2,Tx_2)$. Since Tx_1 and Tx_2 are compact, so there exist elements $u_1 \in Tx_1$ and $u_2 \in Tx_2$ such that

$$d(x_1,u_1) = d(x_1,Tx_1)$$
$$d(x_2,u_2) = d(x_2,Tx_2).$$

Since T satisfies the weak P-property, we have

$$d(x_1,x_2) \leq d(u_1,u_2).$$

Also T is Ćirić type α_*-ψ-contraction, by Lemma 2.1 there exists $q > 1$ such that

$$\begin{aligned} d(x_1,x_2) &\leq d(u_1,u_2) \\ &< qD(u_1,Tx_2) \\ &\leq qH(Tx_1,Tx_2) \\ &\leq q\psi(M(x_1,x_2)) \\ &\leq q\psi(d(x_1,x_2)) \\ &\leq qd(x_1,x_2), \end{aligned}$$

which is a contradiction. Hence $d(x_1,x_2)=0$, consequently, T has a unique best proximity point. □

If we take $M(x,y)=d(x,y)$ in Theorem 3.1, we have the following:

Corollary 3.1. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty. Let $\alpha_*: A\times A\to[0,\infty)$, $\psi\in\Psi$ be a strictly increasing map and $T: A\to CB(B)$ be continuous multivalued mapping satisfying the following assertions:*

(1) *$\alpha_*(Tx,Ty)H(Tx,Ty)\le\psi(d(x,y))$, for all $x,y\in A$;*

(2) *$Tz\subseteq B_0$ for each $z\in A_0$ and (A,B) satisfies the weak P-property;*

(3) *T is modified α_*-proximal admissible;*

(4) *there exists $x_0,x_1\in A_0$ and $y_0\in Tx_0\subseteq B_0$ such that $d(x_1,y_0)=D(A,B)$ and $\alpha_*(Tx_0,Tx_1)\ge 1$.*

Then the mapping T has a best proximity point.

If we take $\alpha_*(Tx,Ty)=1$ for all $x,y\in A$ in Theorem 3.1, we have following corollary:

Corollary 3.2. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty, $\psi\in\Psi$ a strictly increasing map and $T: A\to CB(B)$ be continuous multivalued mapping satisfying the following assertions:*

(1) *$H(Tx,Ty)\le\psi(M(x,y))$, for all $x,y\in A$;*

(2) *$Tz\subseteq B_0$ for each $z\in A_0$ and (A,B) satisfies the weak P-property;*

(3) *there exists $x_0,x_1\in A_0$ and $y_0\in Tx_0\subseteq B_0$ such that $d(x_1,y_0)=D(A,B)$.*

Then the mapping T has a best proximity point.

If we take $M(x,y)=d(x,y)$ in Corollary 3.2, then

Corollary 3.3. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty, $\psi\in\Psi$ a strictly increasing map and $T: A\to CB(B)$ be continuous multivalued mapping satisfying the following assertions:*

(1) *$H(Tx,Ty)\le\psi(d(x,y))$, for all $x,y\in A$;*

(2) *$Tz\subseteq B_0$ for each $z\in A_0$ and (A,B) satisfies the weak P-property;*

(3) *there exists $x_0,x_1\in A_0$ and $y_0\in Tx_0\subseteq B_0$ such that $d(x_1,y_0)=D(A,B)$.*

Then the mapping T has a best proximity point.

4. Best Proximity Points for Single Valued Mapping

We begin this section with the following definition:

Definition 4.1 ([24])**.** Let A and B be two nonempty subsets of b-metric space (X,d). A mapping $T : A \to B$ is called α-proximal admissible if there exists a mapping $\alpha : A \times A \to [0,\infty)$ such that

$$\left.\begin{array}{l}\alpha(x_1,x_2) \geq 1\\ d(u_1,Tx_1) = \mathrm{d}(A,B)\\ d(u_2,Tx_2) = \mathrm{d}(A,B)\end{array}\right\} \Rightarrow \alpha(u_1,u_2) \geq 1,$$

where $x_1, x_2, u_1, u_2 \in A$.

Definition 4.2. Let A and B be two nonempty subsets of a b-metric space (X,d). A mapping $T : A \to B$ is said to be a Ćirić type α-ψ-proximal contraction, if there exists $\psi \in \Psi$ and $\alpha : A \times A \to [0,\infty)$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(m(x,y)), \quad \text{for all } x,y \in A \tag{4.1}$$

where $m(x,y) = \max\left\{d(x,y), \frac{1}{k}d(x,Tx) - D(A,B), \frac{1}{k}d(y,Ty) - D(A,B)\right\}$.

Theorem 4.1. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty. Let $\alpha : A \times A \to [0,\infty)$, $\psi \in \Psi$ be a strictly increasing map and $T : A \to B$ be continuous mapping satisfying the following assertions:*

(1) *T is Ćirić type α-ψ-proximal contraction;*

(2) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak P-property;*

(3) *T is α-proximal admissible;*

(4) *there exists $x_0, x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$.*

Then the mapping T has a best proximity point.

Theorem 4.2. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty. Let $\psi \in \Psi$ be a strictly increasing map and $T : A \to B$ be a mapping satisfying the following assertions:*

(1) *T is Ćirić type α-ψ-proximal contraction;*

(2) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the weak P-property;*

(3) *T is α-proximal admissible;*

(4) *T is an α-continuous mapping;*

(5) *there exists $x_0, x_1 \in A_0$ such that $d(x_1,Tx_0) = d(A,B)$ and $\alpha(x_0,x_1) \geq 1$.*

Then the mapping T has a best proximity point

If we take $m(x,y) = d(x,y)$ in Theorem 4.1, we have the following corollary:

Corollary 4.1. *Let A and B be two nonempty closed subsets of a complete b-metric space (X,d) such that A_0 is non-empty. Let $\alpha : A \times A \to [0,\infty)$, $\psi \in \Psi$ be a strictly increasing map and $T : A \to B$ be continuous mapping satisfying the following assertions:*

(1) *$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$, for all $x,y \in A$;*

(2) $T(A_0) \subseteq B_0$ *and* (A,B) *satisfies the weak P-property;*

(3) T *is* α*-proximal admissible;*

(4) *there exists* $x_0, x_1 \in A_0$ *such that* $d(x_1, Tx_0) = D(A,B)$ *and* $\alpha(x_0, x_1) \geq 1$.

Then the mapping T *has a best proximity point.*

If we take $\alpha(x,y) = 1$ for all $x, y \in A$ in Theorem 4.1, we have the following corollary:

Corollary 4.2. *Let* A *and* B *be two nonempty closed subsets of a complete* b*-metric space* (X,d) *such that* A_0 *is non-empty. Let* $\psi \in \Psi$ *be strictly increasing map and* $T : A \to B$ *be continuous mapping satisfying the following assertions:*

(1) $d(Tx, Ty) \leq \psi(m(x,y))$, $\forall\ x, y \in A$;

(2) $T(A_0) \subseteq B_0$ *and* (A,B) *satisfies the weak P-property;*

(3) T *is* α*-proximal admissible;*

(4) *there exists* $x_0, x_1 \in A_0$ *such that* $d(x_1, Tx_0) = d(A,B)$*;*

Then the mapping T *has a best proximity point.*

If we take $m(x,y) = d(x,y)$ in Corollary 4.2, we have the following:

Corollary 4.3. *Let* A *and* B *be two nonempty closed subsets of a complete* b*-metric space* (X,d) *such that* A_0 *is non-empty. Let* $\alpha : A \times A \to [0,\infty)$, $\psi \in \Psi$ *be a strictly increasing map and* $T : A \to B$ *be continuous mapping satisfying the following assertions:*

(1) $d(Tx, Ty) \leq \psi(d(x,y))$, *for all* $x, y \in A$;

(2) $T(A_0) \subseteq B_0$ *and* (A,B) *satisfies the weak P-property;*

(3) T *is* α*-proximal admissible;*

(4) *there exists* $x_0, x_1 \in A_0$ *such that* $d(x_1, Tx_0) = d(A,B)$*;*

Then the mapping T *has a best proximity point.*

5. Best Proximity Point Results in Partially Ordered b-metric Spaces

Let $(X,d,\preceq)$ be a partially ordered metric space, A and B be two nonempty subsets of X. Many authors proved the existence of best proximity point results in the context of partially order metric spaces (see for example [12, 29, 30]). In this section, we derive new results in partially order metric spaces, as an application of our results presented in previous section.

Definition 5.1 ([12]). A mapping $T : A \to B$ is said to proximally increasing if for $z_1, z_2, u_1, u_2 \in A$

$$\left.\begin{array}{l} z_1 \preceq z_2 \\ d(u_1, Tz_1) = \mathrm{d}(A,B) \\ d(u_2, Tz_2) = \mathrm{d}(A,B) \end{array}\right\} \Rightarrow u_1 \preceq u_2.$$

Very recently, Pragadeeswarar *et al*. [30] defined the notion of proximal relation between two subsets of X as follows:

Definition 5.2 ([30]). Let A and B be two nonempty subsets of a partially ordered b-metric space $(X,d,\preceq)$ such that $A_0 \neq \emptyset$. Let B_1 and B_2 be two nonempty subsets of B_0. The proximal relation between B_1 and B_2 is denoted and defined by $B_1 \preceq B_2$, if for every $b_1 \in B_1$ with $d(a_1,b_1)=d(A,B)$, there exists $b_2 \in B_2$ with $d(a_2,b_2)=d(A,B)$ such that $a_1 \preceq a_2$.

Theorem 5.1. *Let A and B be two nonempty closed subsets of a complete partially ordered b-metric space $(X,d,\preceq)$ such that A_0 is non-empty. Let $\psi \in \Psi$ be strictly increasing map and $T : A \to CB(B)$ be continuous multivalued mapping satisfying the following assertions:*

(1) *$H(Tx,Ty) \leq \psi(M(x,y))$, for all $x \preceq y$;*

(2) *$Tz \subseteq B_0$ for each $z \in A_0$ and (A,B) satisfies the weak P-property;*

(3) *for $x_1,x_2 \in A_0$, $x_1 \preceq x_2$ implies $Tx_1 \preceq Tx_2$;*

(4) *T is proximally increasing.*

(5) *there exists $x_0,x_1 \in A_0$ and $y_0 \in Tx_0 \subseteq B_0$ such that $d(x_1,y_0)=D(A,B)$ satisfies $x_0 \preceq x_1$ and $x_1 \preceq y_0$.*

Then the mapping T has a best proximity point.

Proof. Define $\alpha_* : A \times A \to [0,\infty)$ by

$$\alpha_*(x,y)=\begin{cases}1 & \text{if } x \preceq y\\ 0 & \text{otherwise.}\end{cases}$$

First we shaw that T is α_*-proximal admissible. For this, assume that

$$\left.\begin{array}{l}\alpha_*(Tx_0,Tx_1)\geq 1\\ D(x_1,Tx_0)=\mathrm{D}(A,B)\\ D(x_2,Tx_1)=\mathrm{D}(A,B)\end{array}\right\} \Rightarrow \alpha_*(Tx_1,Tx_2)\geq 1 \text{ for } x_1 \preceq x_2.$$

Since $Tx_1 \preceq Tx_2$, therefore for $x_1,x_2,x_3 \in X$, with

$$\left.\begin{array}{l}x_1 \preceq x_2\\ D(x_2,Tx_1)=\mathrm{D}(A,B)\\ D(x_3,Tx_2)=\mathrm{D}(A,B)\end{array}\right\}.$$

Since T is proximally increasing then $x_2 \preceq x_3$. This implies that $\alpha_*(Tx_2,Tx_3)\geq 1$ for $x_2 \preceq x_3$. Thus, all the conditions of Theorem 3.1 are satisfied and hence mapping T has a best proximity point. □

$$\mathscr{H}' : \alpha(x_1,x_2)\geq 1 \text{ with } x_1 \preceq x_2 \text{ for all best proximity points } x_1,x_2 \text{ of } T.$$

Theorem 5.2. *Let A and B be two nonempty closed subsets of a complete partially ordered b-metric space $(X,d,\preceq)$ such that A_0 is non-empty and $T : A \to K(B)$ be continuous multivalued* *mapping satisfying all assertions of Theorem* 5.1 *along with* $\mathscr{H}'$. *Then* T *has a unique best proximity point.*

For single valued mapping we obtain the following results:

Theorem 5.3. *Let* A *and* B *be two nonempty closed subsets of a partially ordered complete* b*-metric space* $(X,d,\preceq)$ *such that* A_0 *is nonempty. Let* $\psi \in \Psi$ *be strictly increasing map and* $T : A \to B$ *be continuous mapping satisfying the following assertions for all* $z_1,z_2 \in A$ *with* $z_1 \preceq z_2$*:*

(1) $d(Tx,Ty) \le \psi(M(x,y))$, *for all* $x \preceq y$*;*

(2) $Tz \subseteq B_0$ *for each* $z \in A_0$ *and* (A,B) *satisfies the weak P-property;*

(3) $z_1,z_2 \in A_0$, $z_1 \preceq z_2$ *implies* $Tz_1 \preceq Tz_2$*;*

(4) *there exists* $z_0,z_1 \in A_0$ *such that* $d(z_1,Tz_0) = d(A,B)$ *satisfies* $z_0 \preceq z_1$.

Then T *has a best proximity point.*

6. Applications to Fixed Point Results

As applications of our results, we deduce some new fixed point results for multivalued Ćirić type α_*-ψ-contraction in the frame work of b-metric and partially ordered b-metric spaces. If we take $A = B = X$ in Theorem 3.1 (respectively in 3.2, 3.3), we obtain the following fixed point results:

Theorem 6.1. *Let* (X,d) *be a complete* b*-metric space. Let* $\alpha_* : X \times X \to [0,\infty)$, $\psi \in \Psi$ *be strictly increasing map and* $T : X \to CB(X)$ *be continuous multivalued mapping satisfying the following assertions:*

(1) T *is Ćirić type* α_**-*ψ*-contraction;*

(2) T *is modified* α_**-admissible;*

(3) *there exists* $x_0,x_1 \in X$ *such that* $\alpha_*(Tx_0,Tx_1) \ge 1$.

Then the mapping T *has a fixed point.*

$$\mathscr{H}'' : \alpha(x_1,x_2) \ge 1 \text{ for all fixed points } x_1,x_2 \text{ of } T.$$

Theorem 6.2. *Let* (X,d) *be a complete* b*-metric space and* $T : X \to K(X)$ *be continuous multivalued mapping satisfying satisfying all assertions of Theorem* 6.1 *along with* $\mathscr{H}''$. *Then* T *has a unique fixed point.*

In partially ordered b-metric spaces, the corresponding fixed point results are as follows:

Theorem 6.3. *Let* $(X,d,\preceq)$ *be a complete partially ordered* b*-metric space. Let* $\psi \in \Psi$ *be strictly increasing map and* $T : X \to CB(X)$ *be continuous multivalued mapping satisfying the following assertions:*

(1) $H(Tx,Ty) \le \psi(M(x,y))$, *for all* $x \preceq y$;

(2) *for* $x_1,x_2 \in X$, $x_1 \preceq x_2$ *implies* $Tx_1 \preceq Tx_2$;

(3) *there exists* $x_0,x_1 \in X$, *such that* $x_0 \preceq x_1$.

Then T *has a fixed point.*

$$\mathscr{H}''' : x_1 \preceq x_2 \text{ for all fixed points } x_1,x_2 \text{ of } T.$$

Theorem 6.4. *Let* $(X,d,\preceq)$ *be a complete partially ordered* b*-metric space and* $T : X \to K(X)$ *be continuous multivalued mapping satisfying satisfying all assertions of Theorem* 6.3 *along with* $\mathscr{H}'''$. *Then* T *has a unique fixed point.*

Theorem 6.5. *Let* (X,d) *be a complete* b*-metric space. Suppose that* $\alpha : X \times X \to [0,\infty)$ *is a function,* $\psi \in \Psi$ *be strictly increasing map and* $T : X \to X$ *be continuous mapping satisfying the following assertions:*

(1) T *is Ćirić type* α-ψ*-proximal contraction;*

(2) T *is* α*-proximal admissible;*

(3) *there exists* $x_0,x_1 \in X$ *such that* $\alpha(Tx_0,Tx_1) \ge 1$.

Then the mapping T *has a fixed point.*

7. Application to Integral Equation

Finally, we apply Theorem 6.5 to study the existence of solution to the nonlinear integral equation.

Theorem 7.1. *Let* $C[a,b]$ *be the set of all continuous functions on* $[a,b]$, b*-metric* d *with* $k = 2^{p-1}$ *defined by*

$$d(u,v) = \sup_{t\in[a,b]} |u(t)-v(t)|^p$$

for all $u,v \in C[a,b]$ *and some* $p > 1$. *Consider the nonlinear integral equation*

$$u(t) = g(t) + \int_a^b K(t,x,u(x))dx, \tag{7.1}$$

where $t \in [a,b], g : [a,b] \to \mathbb{R}$, $K : [a,b] \times [a,b] \times u[a,b] \to \mathbb{R}$ *for each* $u \in C[a,b]$.

Suppose that the following statements hold.

(1) g *is continuous on* $[a,b]$ *and* $K(t,x,u(x))$ *is integral with respect to* x *on* $[a,b]$.

(2) $Tu \in C[a,b]$ *for all* $u \in [a,b]$, *where* $Tu(t) = g(t) + \int_a^b K(t,x,u(x))dx$ *for all* $t \in [a,b]$.

(3) *For all* $u \in C[a,b]$ *and* $u(x) \ge 0$ *for all* $x \in [a,b]$, *we have* $Tu(x) \ge 0$ *for all* $x \in [a,b]$.

(4) *For all* $x,t \in [a,b]$ *and* $u,v \in C[a,b]$ *such that* $u(x),v(x) \in [0,\infty)$ *for all* $x \in [a,b]$, *we have*

$$|K(t,x,u(x)) - K(t,x,v(x))| \le \mu(t,x)\psi\left(\max\left\{|u(x)-v(x)|, \frac{|u(x)-Tu(x)|}{2^{1-\frac{1}{p}}}, \frac{|v(x)-Tv(x)|}{2^{1-\frac{1}{p}}}\right\}\right),$$

where $\mu : [a,b]\times[a,b]\to\mathbb{R}$ *is a continuous function satisfying*

$$\sup_{t\in[a,b]}\left(\int_a^b \mu^p(t,x)dx\right) < \frac{1}{2^p(b-a)^{p-1}}.$$

(5) *There exist* $u_1\in C[a,b]$ *such that* $u_1(t)\ge 0$ *and* $Tu_1(t)\ge 0$ *for all* $t\in[a,b]$.

Then nonlinear integral equation (7.1) *has a unique solution in* $C[a,b]$.

Proof. Define a mapping $T : C[a,b]\to C[a,b]$ by

$$Tu(t) = g(t)+\int_a^b K(t,x,u(x))dx$$

for all $u\in C[a,b]$ and for all $t\in[a,b]$. It follows from hypothesis (1) and (2) that T is well-defined. Notice that the existence of solution to (7.1) is equivalent to the existence of fixed point of T. Now, we will show that all hypothesis of Theorem 6.5 are satisfied.

Define a mapping $\alpha : C[a,b]\times C[a,b]\to\mathbb{R}$ by

$$\alpha(u,v)=\begin{cases}1 & \text{if } u(x),v(x)\in[0,\infty) \text{ for all } x\in[a,b]\\ 0 & \text{otherwise.}\end{cases}$$

We shall show that T is α-proximal admissible mapping. Indeed, for $u,v\in C[a,b]$ such that $\alpha(u,v)\ge 1$, we have $u(x),v(x)\ge 0$ for all $x\in[a,b]$. It follows from condition (3) that $Tu(x),Tv(x)\ge 0$. Therefore $\alpha(Tu(x),Tv(x))\ge 1$ and hence T is α-proximal admissible mapping.

We claim that T is Ćirić type α-ψ-proximal contraction. That is, there exist $\psi\in\Psi$ such that

$$\alpha(x,y)d(Tx,Ty)\le\psi(M(x,y)),$$

for each $x,y\in C[a,b]$, where

$$M(x,y)=\max\left\{d(x,y),\frac{1}{k}d(x,Tx),\frac{1}{k}d(y,Ty)\right\}.$$

Indeed, let $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. From condition (4), for all $u,v\in C[a,b]$ such that $u(x),v(x)\in[0,\infty)$ for all $x\in[a,b]$, we have

$$\begin{aligned}2^{p-1}\alpha(u,v)|Tu(x)-Tv(x)|^p &= 2^{p-1}|Tu(x)-Tv(x)|^p\\
&\le 2^{p-1}\left|\int_a^b K(t,x,u(x))dx-\int_a^b K(t,x,v(x))dx\right|^p\\
&\le 2^{p-1}\left|\int_a^b (K(t,x,u(x))-K(t,x,v(x)))dx\right|^p\\
&\le 2^{p-1}\left(\int_a^b |K(t,x,u(x))-K(t,x,v(x))|dx\right)^p\\
&\le\left[2^{p-1}\left(\int_a^b dx\right)^{\frac{1}{q}}\left(\int_a^b |K(t,x,u(x))-K(t,x,v(x))|^p dx\right)^{\frac{1}{p}}\right]^p\\
&\le 2^{p-1}(b-a)^{p-1}\left(\int_a^b \mu^p(t,x)dx\right)\psi\Bigg(\max\Bigg\{|u(x)-v(x)|^p,\end{aligned}$$

$$
\begin{aligned}
&\left.\left.\frac{|u(x)-Tu(x)|^p}{2^{p-1}}, \frac{|v(x)-Tv(x)|^p}{2^{p-1}}\right\}\right) \\
&\le 2^{p-1}(b-a)^{p-1}\left(\int_a^b \mu^p(t,x)dx\right)\psi\left(\max\left\{\sup_{x\in[a,b]}|u(x)-v(x)|^p,\right.\right. \\
&\qquad \left.\left.\sup_{x\in[a,b]}\frac{|u(x)-Tu(x)|^p}{2^{p-1}}, \sup_{x\in[a,b]}\frac{|v(x)-Tv(x)|^p}{2^{p-1}}\right\}\right) \\
&\le 2^{p-1}(b-a)^{p-1}\left(\int_a^b \mu^P(t,x)dx\right)\psi\left(\max\left\{d(u,v), \frac{1}{k}d(u,Tu), \frac{1}{k}d(v,Tv)\right\}\right) \\
&= 2^{p-1}(b-a)^{p-1}\left(\int_a^b \mu^P(t,x)dx\right)\psi(M(u,v)) \\
&\le 2^{p-1}(b-a)^{p-1}\sup_{t\in[a,b]}\left(\int_a^b \mu^P(t,x)dx\right)\psi(M(u,v)) \\
&\le \psi(M(u,v)),
\end{aligned}
$$

where $2^{p-1}(b-a)^{p-1}\sup_{t\in[a,b]}\left(\int_a^b \mu^P(t,x)dx\right) < 1$. This implies that

$$\alpha(u,v)d(Tu,Tv) \le 2^{p-1}\alpha(u,v)|Tu(x)-Tv(x)|^p \le \psi(M(u,v)).$$

Hence, T is Ćirić type α-ψ-proximal contraction.

Let $\{u_n\} \subset C[a,b]$ such that $\alpha(u_n,u_{n+1}) \ge 1$ and $\lim_{n\to\infty} u_n = u \in C[a,b]$. Then $u(x), u_n(x) \in [0,\infty)$ for all $x \in [a,b]$ and $n \ge 0$. Therefore, $\alpha(u_n,u) \ge 1$ for all $n \ge 1$.

Therefore, we conclude all the hypothesis of Theorem 6.5 are satisfied. Thus, T has a fixed point $u \in C[a,b]$ and hence equation (7.1) has a solution $u \in C[a,b]$. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

The authors wrote, read and approved the final manuscript.

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