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Nonlinear Parabolic Operators with Perturbed Coefficients

Research Article

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Abstract. We consider the Cauchy-Dirichlet problem for second order quasilinear non-divergence form parabolic equations with discontinuous data in a bounded cylinder Q. Supposing existence of strong solution u_0 and applying the Implicit Function Theorem we show that for any small L^∞-perturbation of the coefficients there exists, locally in time, exactly one solution u close to u_0 with respect to the norm in $W_p^{2,1}(Q)$ which depends smoothly on the data. For that, no structure and growth conditions on the data are needed. Moreover, applying the Newton Iteration Procedure we obtain an approximating sequence for the solution u_0.

Keywords. Nonlinear parabolic equations; Cauchy-Dirichlet problem; *VMO*; Implicit Function Theorem; Newton Iteration Procedure

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1. Introduction

We consider the following Cauchy-Dirichlet problem

$$\begin{cases} D_t u - a^{ij}(x,u,Du)D_{ij}u = f(x,u,Du) & \text{for a.a. } x \in Q \\ u = 0 & \text{on } \partial Q \end{cases} \tag{1.1}$$

in a cylinder $Q=\Omega\times(0,T)$ with a bounded domain $\Omega\subset\mathbb{R}^n$, $\partial\Omega\in C^{1,1}$ where $x=(x',t)\in\mathbb{R}^n\times\mathbb{R}$. The data supposed to be Carathéodory maps, that is they are measurable in x and continuous with respect to the other variables. The maximal regularity theory in the Sobolev spaces for linear parabolic problems along with linearization techniques and the *Implicit Function Theorem* (IFT) give local existence, uniqueness, and smooth dependence on the data for a general class of quasilinear parabolic problems. Further, the classical *Newton Iteration Procedure* (NIP) with quadratic convergence rate permits to obtain an approximative sequence of the solution.

We start with a short survey on known optimal regularity results regarding the solutions of the Cauchy-Dirichlet problem for various parabolic operators. We stress our attention on equations with measurable coefficients having *Vanishing Mean Oscillation* (*VMO*) over small parabolic cylinders shrinking to the center. Precisely, define

$$\mathscr{C}_\rho=\{y\in\mathbb{R}^{n+1}:\ |x'-y'|<\rho,|t-\tau|<\rho^2\}.$$

Let $f\in L^1(\mathbb{R}^{n+1})$ and $f_{\mathscr{C}_\rho}=\frac{1}{|\mathscr{C}_\rho|}\int_{\mathscr{C}_\rho}f(y)\,dy=⨍_{\mathscr{C}_\rho}f(y)\,dy$ be the mean integral of f. We say that

(1) $f\in BMO$ (*bounded mean oscillation,* [6]) if

$$\|f\|_*=\sup_{\mathscr{C}_\rho}⨍_{\mathscr{C}_\rho}|f(y)-f_{\mathscr{C}_\rho}|\,dy<\infty$$

and the supremum is taken over all parabolic cylinders in $\mathbb{R}^{n+1}$. The quantity $\|\cdot\|_*$ is a norm in BMO modulo constant function.

(2) $f\in VMO$ (*Vanishing Mean Oscillation*, [14]) if

$$\lim_{r\to0}\gamma_f(r):=\lim_{r\to0}\sup_{\mathscr{C}_\rho,\rho\le r}⨍_{\mathscr{C}_\rho}|f(y)-f_{\mathscr{C}_\rho}|\,dy=0.$$

The $\gamma_f(r)$ as called *VMO*-modulus of f.

Having a *VMO* function defined in some domain with $C^{1,1}$-boundary we can extend it to the whole $\mathbb{R}^{n+1}$ preserving its *VMO*-modulus (see [7], [1, Proposition 1.3]). In what follows we shall use this fact without explicit references.

Our regularity assumptions on the coefficients of (1.1) are quite general such that the case of *VMO* functions is covered and in the same time strong enough with respect to u and Du in order to ensure the application of linearization techniques and the IFT. Along with (1.1) we consider its formal linearization obtained by derivation in the sense of Fréchet at some fixed solution u_0. For this goal we define the operator

$$\mathscr{P}(u):=D_tu-a^{ij}(x,u,Du)D_{ij}u-f(x,u,Du),\quad \mathscr{P}(u_0)=0$$

and take its derivative in u_0

$$\begin{cases} D_u\mathscr{P}(u_0)u=D_tu-a^{ij}(x,u_0,Du_0)D_{ij}u-D_{\xi_l}a^{ij}(x,u_0,Du_0)D_{ij}u_0 \\ \qquad -D_ua^{ij}(x,u_0,Du_0)D_{ij}u_0-D_{\xi_l}f(x,u_0,Du_0)D_lu \\ \qquad -D_uf(x,u_0,Du_0)u=0 & \text{for a.a. } x\in Q \\ u=0 & \text{on } \partial Q. \end{cases}\tag{1.2}$$

Denote by $W_p(Q)$, the space of solutions of (1.1)

$$W_p(Q)=\left\{u\in W_p^{2,1}(Q),\ p>n+2,\ u(x)=0 \text{ on } \partial Q\right\},$$
$$\|u\|_{W_p(Q)}=\|u\|_{W_p^{2,1}(Q)}.$$

Assuming that (1.2) has no non-trivial solutions it becomes a Fredholm operator (index zero) which is an isomorphism from $W_p(Q)$ onto $L^p(Q), p>n+2$. Then we show that for small L^∞-perturbations $\{\tilde a^{ij}\}_{i,j=1}^n$ and $\tilde f$ of the data, there exists exactly one local in time solution of the perturbed problem which is close to u_0 in the sense of $W_p^{2,1}$ and depends continuously on the perturbing functions $(\{\tilde a^{ij}\},\tilde f)$.

Further, for a given u_1 we determine a Newton Iteration $\{u_{k+1}\}_{k=1}^\infty$ where u_{k+1} is a solution of the *linearized non-homogeneous problem*

$$\begin{cases} D_t u_{k+1}-a^{ij}(x,u_k,Du_k)D_{ij}u_{k+1} \\ \quad -\sum_{l=1}^n\left[D_{\xi_l}a^{ij}(x,u_k,Du_k)D_{ij}u_k+D_{\xi_l}f(x,u_k,Du_k)\right]D_lu_{k+1} \\ \quad -\left[D_ua^{ij}(x,u_k,Du_k)D_{ij}u_k-D_uf(x,u_k,Du_k)\right]u_{k+1} \\ =D_tu_k-\sum_{l=1}^n\left[D_{\xi_l}a^{ij}(x,u_k,Du_k)D_{ij}u_k+D_{\xi_l}f(x,u_k,Du_k)\right]D_lu_k \\ \quad -\left[D_ua^{ij}(x,u_k,Du_k)D_{ij}u_k-D_uf(x,u_k,Du_k)\right]u_k & \text{for a.a. } x\in Q \\ u_{k+1}=0 & \text{on } \partial Q \end{cases}$$

for each index $k\ge 1$. We prove that if the initial iteration u_1 is close enough to u_0 in $W_p^{2,1}$ then the iteration sequence converges to u_0, i.e. $\|u_k-u_0\|_{W_p^{2,1}(Q)}\to 0$ as $k\to\infty$.

Let us note that there are no any growth assumptions imposed on $a^{ij}(x,u,\xi)$ and $f(x,u,\xi)$. However certain uniform boundedness and continuity of these functions with respect to (u,ξ) is required, in order to ensure the superposition operators

$$u\mapsto a^{ij}(\cdot,u(\cdot),Du(\cdot)) \quad \text{and} \quad u\mapsto f(\cdot,u(\cdot),Du(\cdot))$$

to be C^1-maps from $W_x^{1,\infty}(Q)$ onto $L^\infty(Q)$ and $L^p(Q)$, respectively.

Results as the presented here hold also for elliptic quasilinear equations in divergence and non-divergence form (see [5,11,12]). The corresponding parabolic divergence form equations and weakly coupled systems are studied in [4]. Let us note that in [4] the conditions on the domain are more general (it has to be a set with Lipschitz boundary) but the data of the problem depend only on u. Similar results are obtained also for operators satisfying the Campanato condition (see § 2.4). It is also possible to show an IFT (see [18]) where the hypothesis of differentiability is replaced by "nearness" in the sense of Campanato.

During the paper the following notations will be used:

- $|\cdot|$ means the Euclidean norm in $\mathbb{R}^n$.
- $D_iu=\partial u/\partial x_i$, $D_tu=\partial u/\partial t$, $Du=(D_1u,\dots,D_nu)$ and $D^2u=\{D_{ij}u\}_{i,j=1}^n$.
- For any function $f:Q\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ we write D_uf and $D_{\xi_l}f$ for the partial derivatives with respect to u and the l-th component of $\xi\in\mathbb{R}^n$.

- By $L^p(Q)$, $W_p^{2,1}(Q)$ and $W_x^{1,\infty}(Q)$ we denote the classical parabolic Lebesgue and Sobolev spaces with the corresponding norms

$$\|f\|^p_{L^p(Q)} = \|f\|^p_{p,Q} = \int_Q |f(x)|^p\,dx, \quad \|f\|_{\infty,Q} = \operatorname*{esssup}_{x\in Q}|f(x)|,$$
$$\|f\|_{W_p^{2,1}(Q)} = \|f\|_{p,Q} + \|D^2 f\|_{p,Q} + \|D_t f\|_{p,Q},$$
$$\|f\|_{W_x^{1,\infty}(Q)} = \|f\|_{\infty,Q} + \|Df\|_{\infty,Q}.$$

Through the paper the standard summation convention on repeated indexes is adopted. The letter C is used for various constants and may change from one occurrence to another.

2. Selected Existence Theorems

We give some known existence results for the Cauchy-Dirichlet problem for linear and quasilinear equations without pretending for the completeness of the survey.

2.1 Linear Equations with *VMO* Coefficients

The following is a maximal regularity result between Sobolev and Lebesgue spaces. Consider the linear problem

$$\begin{cases} \mathfrak{L}u \equiv D_t u - a^{ij}(x) D_{ij} u = f(x) & \text{for a.a. } x \in Q \\ \mathfrak{L}u \equiv u = 0 & \text{on } \partial Q \end{cases} \tag{2.1}$$

with data subject to the following conditions

(a$_1$) *Uniform parabolicity:* there exists a positive constant $\lambda > 0$ such that

$$\begin{cases} \lambda^{-1}|\eta|^2 \le a^{ij}(x)\eta_i\eta_j \le \lambda|\eta|^2 & \text{for a.a. } (x)\in Q, \text{ for all } \eta\in\mathbb{R}^n, \\ a^{ij}(x) = a^{ji}(x) & \text{for all } i,j \le 1,\dots,n. \end{cases}$$

The last condition ensures $a^{ij} \in L^\infty(Q)$.

(b$_1$) $a^{ij} \in VMO(Q)$ and $f \in L^p(Q)$, $p \in (1,\infty)$.

Theorem 2.1 ([2, Theorem 4.3]). *Let the above conditions hold true. Then the problem* (2.1) *has a unique solution* $u \in W_p(Q)$ *for each* $p \in (1,\infty)$ *that is*

$$\mathfrak{L} \in \operatorname{Iso}(W_p(Q); L^p(Q)), \quad \textit{for all } p \in (1,\infty).$$

The above result still holds true if the coefficients are BMO with small BMO-norms such that $\|a^{ij}\|_* < \varepsilon_0$ with ε_0 depending on λ and $\|a^{ij}\|_{\infty,Q}$.

2.2 Quasilinear Equations with *VMO* Coefficients

The linear result permits to study the following quasilinear problem

$$\begin{cases} \mathcal{Q}u \equiv D_t u - a^{ij}(x,u) D_{ij} u = f(x,u,Du) & \text{for a.a. } x \in Q \\ \mathcal{Q}u \equiv u(x) = 0 & \text{on } \partial Q \end{cases} \tag{2.2}$$

with data subject to the conditions

(a_2) $a^{ij}(x,u)$ and $f(x,u,\xi)$ are Carathéodory functions.

(b_2) *Strong parabolicity:* for each $\eta \in \mathbb{R}^n$, there exists a positive non-increasing function $\Lambda : (0,+\infty) \to (0,+\infty)$ such that

$$\begin{cases} a^{ij}(x,u)\eta_i\eta_j \ge \Lambda(|u|)|\eta|^2, & \text{a.a } (x)\in Q, \text{ for all } \eta \in \mathbb{R}^n \\ a^{ij} = a^{ji}, & \text{for all } i,j=1,\dots,n. \end{cases}$$

(c_2) *Local uniform continuity* of a^{ij} with respect to u : for all $M>0$ and $u,u' \in [-M,M]$

$$|a^{ij}(x,u) - a^{ij}(x,u')| \le a(x)\mu_M(|u-u'|) \quad \text{for a.a. } x \in Q,$$

where $a \in L^\infty(Q)$, $\mu_M : (0,\infty) \to (0,\infty)$ such that $\mu_M(t) \searrow 0$ as $t \to 0$ and $a^{ij}(x,0) \in L^\infty(Q)$.

(d_2) $a^{ij} \in VMO(Q)$ *locally uniformly* in $u \in \mathbb{R}$:

$$\gamma_M(r) = \sup_{0\le i,j\le n}\ \sup_{\rho \le r}\ \sup_{u\in[-M,M]} \fint_{Q_\rho} \left| a^{ij}(y,u) - \fint_{Q_\rho} a^{ij}(z,u)\,dz \right| dy$$

and $\lim_{r\to 0}\gamma_M(r) = 0$. Here M is a positive constant and $Q_\rho = Q \cap \mathscr{C}_\rho$ where $\mathscr{C}_\rho$ ranges over all parabolic cylinders centered at some $x \in Q$.

(e_2) *Quadratic gradient growth of f:* for all $(u,\xi) \in \mathbb{R}\times\mathbb{R}^n$

$$|f(x,u,\xi)| \le \nu(|u|)(f_1(x) + |\xi|^2) \quad \text{for a.a. } x \in Q,$$

where $f_1 \in L^{n+1}(Q)$ is a positive function and $\nu : [0,\infty) \to [0,\infty)$ is a non-decreasing.

(f_2) *Monotonicity of f:* for all u such that $|u| \gg 0$

$$\frac{\operatorname{sign} u \cdot f(x,u,\xi)}{\Lambda(|u|)} \le \nu_1(x)|\xi| + \nu_2(x) \quad \text{for a.a. } x \in Q$$

where $\nu_1, \nu_2 \in L^{n+1}(Q)$ are nonnegative.

Theorem 2.2 ([15, Theorem 2.4]). *Under the conditions* (a_2)-(f_2) *the problem* (2.2) *has at least one solution* $u \in W_p(Q)$. *Suppose in addition that* $a^{ij}(x)$ *are measurable functions independent of* u *and* $f(x,u,\xi)$ *be nondecreasing in* u *such that*

$$|f(x,u,\xi) - f(x,u,\xi')| \le f_2(x,u)|\xi - \xi'| \quad \textit{for a.a. } x \in Q$$

where $\sup_{|u|\le M} f_2(x,u) \in L^p(Q), p > n+2$ *then the solution of* (2.2) *is unique.*

2.3 Quasilinear Equations with Smooth Coefficients

In [9] Ladyzhenskaya and Uraltseva consider initial boundary value problems for parabolic equations in general form. Precisely

$$\begin{cases} \mathfrak{Q}u \equiv D_t u - a^{ij}(x,u,Du)D_{ij}u + a(x,u,Du) = 0 & \text{for a.a. } x \in Q \\ \mathfrak{Q}u \equiv u = 0 & \text{on } \partial Q \end{cases} \tag{2.3}$$

in $Q = \Omega \times (0,T)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega$ being a surface of class W_p^2, $p > n+2$. The data $(\{a^{ij}\}_{i,j=1}^n, a)$ are subject to the conditions

(a_3) $a^{ij} \in C^1(Q \times \mathbb{R} \times \mathbb{R}^n)$ and $a : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function.

(b_3) *Uniform parabolicity:* there exists a constant $\lambda > 0$, such that

$$\lambda^{-1}|\eta|^2 \le a^{ij}(x,u,\xi)\eta_i\eta_j \le \lambda|\eta|^2 \quad \text{for a.a. } x \in Q, \ \text{for all } \ \eta \in \mathbb{R}^n.$$

(c_3) *Quadratic growth condition:*

$$|a(x,u,\xi)| \le \mu_1|\xi|^2 + b(x)|\xi| + \Phi_1(x)$$

with $\Phi_1 \in L^p(Q)$, $p > n+2$.

(d_3) *Growth conditions for a^{ij}*: the coefficients have first-order derivatives in all their arguments satisfying the conditions

$$\sum_{k=1}^{n} \left|D_{\xi_k} a^{ij}(x,u,\xi) - D_{\xi_j} a^{ik}(x,u,\xi)\right| \le \mu_2(1+|\xi|^2)^{-1/2},$$

$$\left[|D_u a^{ij}(x,u,\xi)| + |D_k a^{ij}(x,u,\xi)|\right] \le \mu(|u|+|\xi|)\Phi_2(x),$$

$$|D_{\xi_k} a^{ij}(x,u,\xi)| \le \mu(|u|+|\xi|),$$

$$\left|\sum_{k=1}^{n}\left[D_u a^{ij}(x,u,\xi)\xi_k\xi_k - D_u a^{ij}(x,u,\xi)\xi_k\xi_i + D_k a^{ij}(x,u,\xi)\xi_k - D_k a^{kj}(x,u,\xi)\xi_i\right]\right| \le (1+|\xi|^2)^{1/2}(\mu_3|\xi| + \Phi_3(x))$$

in which μ_2 and μ_3 are positive constants, $\mu : [0,\infty) \to [0,\infty)$ is a non-decreasing function and $\Phi_3 \in L^p(Q)$, $p > n+2$.

Theorem 2.3 ([9, Theorem 7.3]). *Suppose the conditions* (a_3)-(d_3) *hold, than the problem* (2.3) *has at least one solution $u \in W_p(Q)$.*

2.4 Quasilinear Equations Satisfying the Campanato Condition

In case of one space variable we consider the class of nonlinear equations satisfying the Campanato condition. This condition is a nonlinear equivalent of the Cordes-Arena condition (see [10] and the references there). The Campanato operators can be considered as "near operators" to the heat operator so it is expected to possess similar properties. Consider the following Cauchy-Dirichlet problem in a rectangle $Q = (0,d) \times (0,T)$

$$\begin{cases} \mathscr{C}u \equiv \mathscr{A}(x,u,u_x)u_{xx} - u_t = f(x,u,u_x) & \text{for a.a. } x \in Q \\ \qquad u = 0 & \text{on } \partial Q. \end{cases} \tag{2.4}$$

The data $\mathscr{A}$ and f supposed to be Carathéodory functions and the operator $\mathscr{A}\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$ to be "near" to the *heat operator* $\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$ both considered as mappings from $W_2^{2,1}(Q)$ onto $L^2(Q)$. We study strong solvability of (2.4) under the following hypothesis

(a_4) *Campanato's condition:* there exist positive constants α and $K < 1$ such that

$$|\zeta - \alpha\mathscr{A}(x,u,\xi)\zeta| \le K|\zeta|.$$

(b_4) *Quadratic growth condition* with respect to $\xi \in \mathbb{R}^n$:

$$\begin{cases} |f(x,u,\xi)| \le f_1(|u|)[f_2(x) + |\xi|^2] \\ f_1, f_2 \ge 0, \ f_1 \in C^0(\mathbb{R}^+), \ f_2 \in L^2(Q) \end{cases}$$

and f_1 is monotone non-decreasing function.

(c_4) *Monotonicity condition:*

$$\begin{cases} 2uf(x,u,\xi) \ge -\mu_1(x)2u\xi - \mu_2(x)u^2 - \mu_3(x) \\ \mu_1,\mu_3 \in L^2(Q), \mu_2 \in L^\infty(Q), \mu_i \ge 0,\ i=1,2,3, \end{cases}$$

for each $u \in \mathbb{R}$, such that $|u| \gg 0$.

Theorem 2.4 ([16, Theorem 2])**.** *Let the conditions* (a_4)-(c_4) *hold, then the problem* (2.4) *has at least one solution* $u \in W_2(Q)$*. If in addition* $\mathscr{A} = \mathscr{A}(x)$ *is independent of* (u,ξ) *and* $f(x,u,\xi)$ *is non-decreasing in u and Lipschitz continuous with respect to* ξ*, then the solution of* (2.4) *is unique.*

More existence results are obtained in [3] where the authors make use of the version of the IFT for near operators obtained in [18] and in [10,17] where an elliptic version of this result is obtained via the near operators theory of Campanato.

3. Application of the Implicit Function Theorem

Introducing the superposition operators

$$\begin{cases} \mathscr{A}_{ij}(u) := a^{ij}(x,u,Du), \quad \mathscr{F}(u) := f(x,u,Du) \\ \mathscr{P}(u) = D_t u - \mathscr{A}_{ij}(u)D_{ij}u - \mathscr{F}(u) \end{cases} \tag{3.1}$$

we can rewrite the problem (1.1) in the form

$$\mathscr{P}(u) = 0, \quad u \in W_p(Q). \tag{3.2}$$

Fixing a function $u_0 \in W_p(Q)$ and taking the Fréchet derivative of $\mathscr{P}(u)$ at u_0 we obtain the *formally linearized problem*

$$\begin{cases} D_u\mathscr{P}(u_0)v = D_t v - \mathscr{A}_{ij}(u_0)D_{ij}v \\ \quad -\big(D_u\mathscr{A}_{ij}(u_0)D_{ij}u_0 + D_u\mathscr{F}(u_0)\big)v = 0, \ \text{ for a.a. } x \in Q \\ v \in W_p(Q), \end{cases} \tag{3.3}$$

where

$$\begin{cases} D_u\mathscr{A}_{ij}(u) = D_u a^{ij}(x,u,Du) + D_{\xi_l}a^{ij}(x,u,Du)D_l \\ D_u\mathscr{F}(u) = D_u f(x,u,Du) + D_{\xi_l} f(x,u,Du)D_l. \end{cases} \tag{3.4}$$

In order to describe the regularity of the data we need the following definition.

Definition 3.1. *Let* $\mathscr{D} \subseteq \mathbb{R} \times \mathbb{R}^n$ *and* $a(x,u,\xi): Q \times \mathscr{D} \to \mathbb{R}$ *be a Carathéodory function, then it is said to be a* $\mathfrak{C}^1$*-Carathéodory function if* $a(x,\cdot,\cdot)$ *is continuously differentiable with respect to* (u,ξ) *and for each compact* $K \subset \mathscr{D}$, a, $D_u a$ *and* $D_{\xi_l}a$ *are bounded and uniformly continuous in* $(u,\xi) \in K$ *for a.a.* $x \in Q$*. The vector space of* $\mathfrak{C}^1(Q \times K)$*-Carathéodory functions is equipped with the norm*

$$\|a\|_{\mathfrak{C}^1(Q\times K)} := \sup_{(\xi,\eta)\in K} \operatorname*{esssup}_{x\in Q} \left(|a| + |D_u a| + \sum_{l=1}^n |D_{\xi_l} a| \right).$$

The function a is called $\mathfrak{C}^{1,1}$*-Carathéodory function in* $Q \times D$ *if* $a \in \mathfrak{C}^1$ *and in addition* $a, D_u a$ *and* $D_{\xi_l}a$ *are Lipschitz continuous with respect to* (u,ξ)*, that is, for each compact* $K \subset D$ *there* *exists a constant $L_a>0$ such that for a.a. $x\in Q$.*

$$|a(x,u,\xi)-a(x,u',\xi')|+|D_u a(x,u,\xi)-D_u a(x,u',\xi')|+\sum_{l=1}^n |D_{\xi_l}a(x,u,\xi)-D_{\xi_l}a(x,u',\xi')|$$
$$\le L_a\left(|u-u'|+|\xi-\xi'|\right).$$

Let K and D be as above. The following results are analogous of Lemmata 1 and 2 in [11] and describe the regularity of the operator $a(x,u(x),Du(x))$.

Lemma 3.2. *Let $a:Q\times D\to\mathbb{R}$ be a Carathéodory function satisfying*

(1) *$a(\cdot,u,\xi)\in VMO(Q)$ locally uniformly in (u,ξ) with a VMO-modulus $\gamma_K(r)$*
$$\gamma_K(r)=\sup_{(u,\xi)\in K}\ \sup_{\mathscr{C}_\rho,\rho\le r}\fint_{Q_\rho}\Big|a(y,u,\xi)-\fint_{Q_\rho}a(z,u,\xi)\,dz\Big|\,dy$$
where $Q_\rho=Q\cap\mathscr{C}_\rho$ and $\mathscr{C}_\rho$ ranges over all parabolic cylinders centered at some $x\in Q$.

(2) *$a(x,\cdot,\cdot)$ is local uniform continuous, that is, for each compact $K\subset\mathscr{D}$ there exists $C_K>0$ and a nondecreasing, nonnegative function*
$$\mu_K:(0,\infty)\to(0,\infty),\qquad \lim_{\omega\to 0}\mu_K(\omega)=0$$
such that for all $(u,\xi),(u',\xi')\in K$ it holds
$$|a(x,u,\xi)-a(x,u',\xi')|\le\mu_K(|u-u'|)+C_K|\xi-\xi'|\quad \text{for a.a. } x\in Q.$$

(3) *$a_0=a(x,0,0)\in L^\infty(Q)$.*

Then for each $u\in W^{2,1}_p(Q)$ the superposition operator $a(x,u(x),Du(x))$ is in $VMO\cap L^\infty(Q)$ with a VMO-modulus $\gamma_a(r)$
$$\gamma_a(r)=\sup_{\mathscr{C}_\rho,\rho\le r}\fint_{Q_\rho}\Big|a(y,u(y),Du(y))-\fint_{Q_\rho}a(z,u(z),Du(z))\,dz\Big|\,dy.$$

Lemma 3.3. *Let $a\in\mathfrak{C}^1(Q\times\overline{D})$ and $A(a;u):=a(x,u,Du)$ be an evaluation map. Denote*
$$\mathscr{U}=\left\{u\in W^{1,\infty}_x(Q):(u,Du)\in K\right\},$$
then $\mathscr{U}$ is an open set in $W^{1,\infty}_x(Q)$ and
$$A(a;u)\in C^1(\mathfrak{C}^1(Q\times\overline{\mathscr{D}})\times\mathscr{U};L^\infty(Q)).$$

We study the problem (1.1) subject to the following hypothesis

($\mathbf{H_1}$) $a^{ij},f:Q\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are $\mathfrak{C}^{1,1}$-Carathéodory functions.

($\mathbf{H_2}$) Let $U\subset C([0,T],C^1(\overline{\Omega}))$ be an open set. Suppose that there exists a solution $u_0\in U\cap W_p(Q)$ of (1.1).

($\mathbf{H_3}$) There exists a positive constant λ such that
$$\begin{cases}\lambda^{-1}|\eta|^2\le a^{ij}(x,u_0,Du_0)\eta_i\eta_j\le\lambda|\eta|^2, & \text{for a.a. } x\in Q,\ \text{for all } \eta\in\mathbb{R}^n,\\ a^{ij}=a^{ji}, & \text{for all } i,j=1,\dots,n.\end{cases}$$

$a^{ij}(x,u_0,Du_0) \in VMO \cap L^\infty(Q)$ with VMO-modulus

$$\gamma_a(r) = \sum_{i,j=1}^{n} \gamma_{a^{ij}}(r)$$

and $f(x,u_0,Du_0) \in L^p(Q)$, $p > n+2$.

(H₄) There are no non-trivial solutions $v \in W_p(Q)$ of (3.3).

Remark. The hypothesis **(H₂)** has sense as it is seen by the existence theorems presented in § 2. Further, because of the embedding properties of the Sobolev spaces that solution is Hölder continuous along with its gradient (see [8, Lemma 3.3]). There exists an open set $U \subset C([0,T],C^1(\overline{\Omega}))$ such that $u_0 \in U \cap W_p(Q)$.

Remark. According to (3.1) and (3.4), the hypothesis **(H₁)** and **(H₂)** mean that $\mathscr{A}_{ij}(u)$ and $\mathscr{F}(u)$ are C^1-maps with locally Lipschitz continuous derivatives, that is

$$\mathscr{A}_{ij}(u) \in C^1(U \cap W_p(Q); L^\infty(Q)), \quad \mathscr{P}(u), \mathscr{F}(u) \in C^1(U \cap W_p(Q); L^p(Q))$$

$$\|D_u\mathscr{A}_{ij}(u') - D_u\mathscr{A}_{ij}(u'')\|_{\infty,Q} \le L_{\mathscr{A}}\|u'-u''\|_{W_p(Q)}$$

$$\|D_u\mathscr{F}(u') - \mathscr{D}_u\mathscr{F}(u'')\|_{\infty,Q} \le L_{\mathscr{F}}\|u'-u''\|_{W_p(Q)}$$

$$\|\mathscr{A}\| := \sum_{i,j=1}^{n} \|a^{ij}\|_{\mathfrak{C}^1}, \quad \|\mathscr{F}\| := \|f\|_{\mathfrak{C}^1}.$$

Remark. The hypothesis **(H₃)** means that the linear operator $D_t - \mathscr{A}_{ij}(u_0)D_{ij}$ is an isomorphism from $W_p(Q)$ onto $L^p(Q), p > n+2$ (see [2]), that is, it possesses a maximal regularity property.

Let $u_0 = 0 \in U$ be a solution of (3.2) then the linear auxiliary problem

$$D_t w - \mathscr{A}_{ij}(0)D_{ij}w = \mathscr{F}(0), \qquad w \in W_p(Q), \tag{3.5}$$

is uniquely solvable according to **(H₃)** and [2]. Let U_0 and W_0 be two neighborhoods of zero such that the inclusion $\{u+w : (u,w) \in U_0 \times W_0\} \subset U$ holds true. We are looking for solutions $(u,w) \in (U_0 \cap W_p(Q)) \times W_0$ of the nonlinear auxiliary problem

$$D_t(u+w) - \mathscr{A}_{ij}(u+w)D_{ij}(u+w) = \mathscr{F}(u+w). \tag{3.6}$$

Define the operators

$$\mathscr{A}'_{ij}(u,w) = \mathscr{A}_{ij}(u+w) = a^{ij}(x,u+w,D(u+w)) \tag{3.7}$$

$$\begin{aligned}\mathscr{F}'(u,w) &= \mathscr{F}(u+w) - \mathscr{F}(0) + (\mathscr{A}_{ij}(u+w) - \mathscr{A}_{ij}(0))D_{ij}w \\ &= f(x,u+w,D(u+w)) - f(x,0,0) + \big(a^{ij}(x,u+w,D(u+w)) - a^{ij}(x,0,0)\big)D_{ij}w\end{aligned} \tag{3.8}$$

which because of hypotheses **(H₁)** and Lemma 3.3 are C^1-maps

$$\mathscr{A}'_{ij}(u,w) \in C^1((U_0 \cap W_p(Q)) \times W_0; L^\infty(Q))$$

$$\mathscr{F}'(u,w) \in C^1((U_0 \cap W_p(Q)) \times W_0; L^p(Q)).$$

Then, making use of (3.5), we rewrite (3.6) in the form

$$D_t u - \mathscr{A}'_{ij}(u,w)D_{ij}u = \mathscr{F}'(u,w), \quad u \in W_p(Q). \tag{3.9}$$

Since $\mathscr{A}'_{ij}(0,0) = \mathscr{A}_{ij}(0)$, $\mathscr{F}'(0,0) = 0$ the pair $(u,w) = (0,0) \in U_0 \times W_0$ is a solution of (3.9).

The following result gives a smooth dependence of the solution of (3.2) from the data.

Theorem 3.4. *Let U_0 and W_0 be as above. Then there exist neighborhoods $U_1 \subset U_0$ and $W_1 \subset W_0$, $(0,0) \in U_0 \times W_0$, and a solution map $\Phi : C^1(W_1; W_p(Q))$ such that the pair $(u,w) \in U_1 \times W_1$ is a solution of* (3.9) *if and only if $u = \Phi(w)$.*

Proof. Since $\mathscr{A}'_{ij}(0,0) \in VMO \cap L^\infty(Q)$, then the operators $\mathscr{A}'_{ij}(u,w)$ have a small BMO norm for (u,w) close to $(0,0)$. In fact, according to Lemma 3.3 the superposition operator $\mathscr{A}_{ij}(u)$ belongs to $VMO \cap L^\infty(Q)$ for each $u \in W_p(Q)$. Define

$$U_1 = \{u \in U_0 : \|u\|_{C([0,T],C^1(\overline{\Omega}))} \le M\},$$
$$W_1 = \{w \in W_0 : \|w\|_{C([0,T],C^1(\overline{\Omega}))} \le \varepsilon\},$$

then

$$\begin{aligned} A'_{ij}(\rho) &= \fint_{Q_\rho} \left| a^{ij}(y,u+w,Du+Dw) - \fint_{Q_\rho} a^{ij}(z,u+w,Du+Dw)\,dz \right| dy \\ &\le 2\fint_{Q_\rho} \left| a^{ij}(y,u(y)+w(y),D(u(y)+w(y))) - a^{ij}(y,u(y),Du(y)) \right| dy \\ &\quad + \fint_{Q_\rho} \left| a^{ij}(y,u(y),Du(y)) - \fint_{Q_\rho} a^{ij}(z,u(z),Du(z))\,dz \right| dy, \end{aligned}$$

$$\gamma_{\mathscr{A}'_{ij}}(r) := \sup_{(u,w)\in U_1\times W_1} \sup_{\mathscr{C}_\rho,\rho\le r} \sup_{x\in Q} A'_{ij}(\rho) \le 2L_a\varepsilon + \gamma_a(r)$$

and the last term is less than some ε_0 for $r \le r_0(\varepsilon_0)$. Hence we can look for solutions (u,w) of (3.9) close to $(0,0)$ and belonging to $(U_1 \cap W_p(Q)) \times W_1$. To do so we apply the IFT (see [19]). The space $W_p^{2,1}(Q)$, $p > n+2$ is continuously embedded into $C([0,T], C^1(\overline{\Omega}))$ hence the set $U_1 \cap W_p(Q)$ is open in $W_p(Q)$. The operator

$$\mathscr{P}(u,w) := D_t u - \mathscr{A}'_{ij}(u,w) D_{ij} u - \mathscr{F}'(u,w) F$$

is a C^1-map from $(U_1 \cap W_p(Q)) \times W_1$ onto $L^p(Q)$. Its partial derivative with respect to u at $(0,0)$ is the linear continuous map

$$D_u \mathscr{P}(0,0)v = D_t v - \mathscr{A}'_{ij}(0,0) D_{ij} v - D_u \mathscr{F}'(0,0)v : \ W_p(Q) \to L^p(Q).$$

Hence $D_u\mathscr{P}(0,0)$ is a linear isomorphism from $W_p(Q)$ onto $L^p(Q)$ and the IFT asserts unique existence of a C^1-map $u = \Phi(w)$ verifying (3.9). □

One cannot expect that the solution to the problem (3.2) exists on arbitrarily long time interval without additional structural or growth conditions on the data. Define $Q_\tau = \Omega \times (0,T)$ and $U_\tau = \{u|_{Q_\tau} : u \in U\}$. Our next assertion deals with local in time solutions of (3.2).

Theorem 3.5. *Suppose conditions* $(\mathbf{H_1})-(\mathbf{H_4})$ *hold true and $0 \in U_\tau$, then there exists at least one solution $u_\tau \in U_\tau \cap W_p(Q_\tau)$ to* (3.2)*.*

Proof. Let $v \in W_p(Q)$ be a solution of (3.5). Being continuous it is close to the initial data $v(x,0)=0$ for some small $t>0$, that is, v is small in the norm of $C([0,t];C^1(\overline{\Omega}))$. Because of the continuous embedding (see [8, Lemma 3.3,Ch. 2])

$$W_p(Q) \hookrightarrow C^{0,\alpha}([0,T];C^1(\overline{\Omega})) \quad \alpha = 1-\frac{n+2}{p},$$

we get thet for each $t\in[0,T]$ and $s\in(0,t)$ it holds

$$\begin{aligned}\|v(s)-v(0)\|_{C^1(\overline{\Omega})} &= \sup_{\overline{\Omega}}|v(s,x)| + \sup_{\overline{\Omega}}|Dv(s,x)|\\ &= s^{\alpha/2}\left(\sup_{\overline{\Omega}}\frac{|v(s,x)|}{s^{\alpha/2}} + \sup_{\overline{\Omega}}\frac{|Dv(s,x)|}{s^{\alpha/2}}\right)\\ &\le t^{\alpha/2}\|v\|_{C^{0,\alpha/2}([0,t];C^1(\overline{\Omega}))} \le Ct^{\alpha/2}\|v\|_{W_p(Q)}\end{aligned}$$

where the constant does not depend on t. Define a cut-off function $\theta\in C^\infty(\mathbb{R})$, $0\le\theta\le 1$, such that for suitable $0<\tau<t<T$ we have

$$\theta(s)=1 \text{ for all } s\le\tau, \qquad \theta(s)=0 \text{ for all } s\ge t.$$

Thus $\theta(s)v \in C([0,T];C^1(\overline{\Omega}))$ belongs to the set W_1, defined in Theorem 3.4. Choosing $w=\theta v$ in (3.9) and $U_1\subset U$ such that for $u\in U_1$, $u+v\in U$ we get that the function $u=\Phi(\theta v)$ solves

$$D_t u - \mathscr{A}'_{ij}(u,\theta v)D_{ij}u = \mathscr{F}'(u,\theta v).$$

Because of (3.5), (3.7) and (3.8) we obtain that

$$D_t(u+v) - \mathscr{A}_{ij}(u+w)D_{ij}(u+v) = \mathscr{F}(u+w).$$

Restricting the above equation to the subinterval $(0,\tau)$ and choosing

$$u_\tau := (u+w)|_{Q_\tau}$$

we get that u_τ is a solution of

$$\begin{cases} D_t u_\tau - \mathscr{A}_{ij}(u_\tau)D_{ij}u_\tau = \mathscr{F}(u_\tau),\\ u_\tau \in U_\tau \cap W_p(Q_\tau). \end{cases} \tag{3.10}$$

□

The next result gives uniqueness of that solution.

Theorem 3.6. *Let* $(\mathbf{H_1})-(\mathbf{H_4})$ *hold true and suppose* $u,v\in U\cap W_p(Q)$ *be two solutions to* (3.2), *then* $u\equiv v$.

Proof. Because of continuity of solutions u and v we can define an interval $[0,t^*]\subset[0,T]$ where

$$t^* = \sup\{t\in[0,T] : u(s)=v(s),\ 0\le s\le t^*\}.$$

Obviously $[0,t^*]$ is not empty since $u(0)=v(0)=0$ and hence at least $0\in[0,t^*]$. We are going to prove that $t^*=T$, that means uniqueness of the solution in the whole interval where it exists. Suppose to the contrary, i.e. $t^*<T$. Consider $\tau\in(t^*,T)$ and solutions u_τ and v_τ of the restricted on Q_τ problem (3.10). We are going to show that if $\tau>t^*$ is close to t^* then $u_\tau=v_\tau$, that will be contradiction with the definition of t^*. Since U is open in $C([0,T],C^1(\overline{\Omega}))$ there exists $\varepsilon>0$

such that for all $v \in U$ and $w \in C([0,T],C^1(\overline{\Omega}))$ with $\|w\|_{C([0,T],C^1(\overline{\Omega}))} \le \varepsilon$ holds $v+w \in U$. For $t^* \le s \le \tau < T$ and because of $u(t^*)-v(t^*)=0$ we get

$$\begin{aligned}\|u(s)-v(s)\|_{C^1(\overline{\Omega})} &= \sup_{\overline{\Omega}} |u(s,x)-v(s,x)| + \sup_{\overline{\Omega}} |Du(s,x)-Dv(s,x)| \\ &\le (s-t^*)^{\alpha/2}\left(\sup_{\overline{\Omega}} \frac{|u(s,x)-v(s,x)|}{(s-t^*)^{\alpha/2}} + \sup_{\overline{\Omega}} \frac{|Du(s,x)-Dv(s,x)|}{(s-t^*)^{\alpha/2}}\right) \\ &\le (s-t^*)^{\alpha/2}\|u-v\|_{C^{0,\alpha}([0,T];C^1(\overline{\Omega}))} \le C(s-t^*)^{\alpha/2}\|u-v\|_{W_p(Q)}.\end{aligned}$$

For each $\varepsilon>0$ we can take τ close to t^* such that $\|u-v\|_{W_p(Q)} \le \varepsilon$. Define a cut-off function $\theta \in C^\infty(\mathbb{R})$, $0 \le \theta \le 1$, such that for any $0 \le t^* < \tau' < \tau < T$

$$\theta(s)=1 \text{ for all } s \le \tau', \qquad \theta(s)=0 \text{ for all } s \ge \tau.$$

Than $\theta(s)(u-v) \in C([0,T];C^1(\overline{\Omega}))$ and

$$\|\theta(u-v)\|_{C([0,T],C^1(\overline{\Omega}))} \le C(s-t^*)^{\alpha/2}\|u-v\|_{W_p(Q)} \le C\varepsilon.$$

For any $\sigma \in [0,1]$ it holds $v+\sigma\theta(u-v) \in U$, then $u_\tau + \sigma(u_\tau - v_\tau) \in U_\tau$ and by the Mean Value Theorem, we get

$$\begin{aligned} D_t(u_\tau - v_\tau) - \mathscr{A}_{ij}(u_\tau)D_{ij}(u_\tau - v_\tau) &= \mathscr{F}(u_\tau) - \mathscr{F}(v_\tau) + \big(\mathscr{A}_{ij}(u_\tau) - \mathscr{A}_{ij}(v_\tau)\big)v_\tau \\ &= (u_\tau - v_\tau)\int_0^1 D_u\mathscr{F}(v_\tau + \sigma(u_\tau - v_\tau)d\sigma + v_\tau(u_\tau - v_\tau)\int_0^1 D_u\mathscr{A}_{ij}(v_\tau + \sigma(u_\tau - v_\tau)d\sigma.\end{aligned}$$

Taking $w_\tau = u_\tau - v_\tau$ we get

$$D_t w_\tau - \mathscr{A}_{ij}(w_\tau)D_{ij}w_\tau = \mathscr{N} w_\tau, \quad W_p(Q_\tau) \tag{3.11}$$

where $\mathscr{N}$ is the functional

$$\mathscr{N} w_\tau = w_\tau \int_0^1 D_u\mathscr{F}(\sigma w_\tau + v_\tau)d\sigma + v_\tau w_\tau \int_0^1 D_u\mathscr{A}_{ij}(\sigma w_\tau + v_\tau)d\sigma.$$

Since it is a linear functional over the space of linear functionals $D_u\mathscr{A}_{ij}, D_u\mathscr{F}$ it is an isomorphism

$$\mathscr{N} \in \mathbf{Ism}\big(C([0,\tau],C^1(\overline{\Omega}));L^p(Q_\tau)\big)$$

and $w_\tau = 0$ is a solution of (3.11). Than $u_\tau = v_\tau$ which contradicts to the definition of t^*. □

We are in a position now to prove our main result.

Main Theorem 3.7. *Let $D \subset \mathbb{R}\times\mathbb{R}^n$ be an open set, $\tau \in (0,T)$ be as in Theorem* 3.5, *$u_{0\tau}$ be a local in time solution of* (3.2) *under the hypothesis* $(\mathbf{H_1})-(\mathbf{H_4})$, *and $K \subset D$ be a compact such that $(u_{0\tau}, Du_{0\tau}) \in K$ for a.a. $x \in Q_\tau$. Then there exist neighborhoods $V_\tau \subseteq \mathfrak{C}^1(Q_\tau \times \overline{D})^{n^2} \times \mathfrak{C}^1(Q_\tau \times \overline{D})$ of $(0,0)$, $W_\tau \subseteq U_\tau \cap W_p(Q_\tau)$, of $u_{0\tau}$ and a C^1-map $\Phi: V_\tau \to W_\tau$ with $\Phi(0,0) = u_{0\tau}$, such that for all $\big(\{\tilde a^{ij}\}_{i,j=1}^n, \tilde f\big) \in V_\tau$ and $u_\tau \in W_\tau$ holds*

$$\begin{cases} D_t u_\tau - \big(a^{ij}(x,u_\tau,Du_\tau) - \tilde a^{ij}(x,u_\tau,Du_\tau)\big)D_{ij}u_\tau & \\ \quad = f(x,u_\tau,Du_\tau) - \tilde f(x,u_\tau,Du_\tau) & \text{for a.a. } x \in Q_\tau \\ u_\tau = 0 & \text{on } \partial Q_\tau \end{cases} \tag{3.12}$$

if and only if $u_\tau = \Phi\big(\{\tilde a^{ij}\}_{i,j=1}^n, \tilde f\big)$.

Proof. Denote by $\tilde{\mathbf{a}}$ the perturbing matrix $\{\tilde a^{ij}\}_{i,j=1}^n \in \mathfrak{C}^1(Q_\tau \times \overline{D})^{n^2}$ and by $\mathscr{U}_\tau$ the set

$$\mathscr{U}_\tau = \{u_\tau \in U_\tau \cap W_p(Q_\tau) : (u_\tau, Du_\tau) \in K \subset D\}.$$

It is easy to see that $\mathscr{U}_\tau$ is open in $W_p(Q_\tau)$. Because of the assumption $(\mathbf{H_1})$ and Lemma 3.3 the following evaluation maps are C^1-smooth

$$A_{ij}(a+\tilde a; u_\tau) = a^{ij}(x, u_\tau, Du_\tau) + \tilde a^{ij}(x, u_\tau, Du_\tau)$$

$$F(f+\tilde f; u_\tau) = f(x, u_\tau, Du_\tau) + \tilde f(x, u_\tau, Du_\tau)$$

$$A_{ij}(a+\tilde a; u_\tau) \in C^1\big(\mathfrak{C}^1(Q_\tau \times \overline{D})^{n^2} \times \mathscr{U}_\tau; L^\infty(Q_\tau)\big)$$

$$F(f+\tilde f; u_\tau) \in C^1\big(\mathfrak{C}^1(Q_\tau \times \overline{D}) \times \mathscr{U}_\tau; L^p(Q_\tau)\big).$$

Hence the problem (3.12) is equivalent to

$$\begin{cases} \tilde{\mathscr{P}}(\tilde a, \tilde f, u_\tau) = D_t u_\tau - A_{ij}(a+\tilde a; u_\tau) D_{ij} u_\tau \\ \qquad\qquad -F(f+\tilde f; u_\tau) = 0 \\ \tilde{\mathscr{P}} \in C^1\big(\mathfrak{C}^1(Q_\tau \times \overline{D})^{n^2} \times \mathfrak{C}^1(Q_\tau \times \overline{D}) \times \mathscr{U}_\tau; L^p(Q_\tau)\big). \end{cases} \tag{3.13}$$

where

$$\begin{aligned} \tilde{\mathscr{P}}(0,0,u_{0\tau}) &= D_t u_{0\tau} - A_{ij}(a; u_{0\tau}) D_{ij} u_{0\tau} - F(f; u_{0\tau}) \\ &= D_t u_{0\tau} - \mathscr{A}_{ij}(u_{0\tau}) D_{ij} u_{0\tau} - \mathscr{F}(u_{0\tau}) = 0. \end{aligned}$$

We are going to resolve (3.13) with respect to u_τ nearby the solution $(0,0,u_{0\tau})$ by means of the IFT. For this goal we need to show that the derivative operator

$$D_u \tilde{\mathscr{P}}(0,0,u_{0\tau}) v_\tau = D_t v_\tau - A_{ij}(a; u_{0\tau}) D_{ij} v_\tau - D_u F(a; u_{0\tau}) v_\tau - D_u A_{ij}(a; u_{0\tau}) D_{ij} u_{0\tau} v_\tau$$

is an isomorphism. It is a sum of two linear operators

$$v_\tau \to D_t v_\tau - A_{ij}(a; u_{0\tau}) D_{ij} v_\tau : U_\tau \cap W_p(Q_\tau) \to L^p(Q_\tau)$$

$$v_\tau \to D_u F(f; u_{0\tau}) v_\tau + D_u A_{ij}(a; u_{0\tau}) D_{ij} u_{0\tau} v_\tau : U_\tau \cap W_p(Q_\tau) \to L^p(Q_\tau).$$

The first one is isomorphism as it is shown in Remark 3 while the second one is the compact operator

$$D_u \mathscr{A}_{ij}(u_{0\tau}) D_{ij} u_{0\tau} v_\tau + D_u \mathscr{F}(u_{0\tau}) v_\tau$$

because of the compactness of the embedding $W_p^{2,1} \hookrightarrow W_x^{1,p}$ (see [8, Lemma 3.3]). Hence $D_u \tilde{\mathscr{P}}(0,0,u_{0\tau})$ is a Fredholm operator (index zero) and in particular $(\mathbf{H_4})$ implies injectivity i.e.

$$D_u \tilde{\mathscr{P}}(0,0,u_{0\tau}) \in \mathbf{Iso}(U_\tau \cap W_p(Q_\tau); L^p(Q_\tau)).$$

The assertion of the theorem follows by the IFT applied to $\tilde{\mathscr{P}}(\tilde a, \tilde f, u_\tau)$. □

4. Application of the Newton Iteration Procedure

We consider once again (3.2) and its linearization (3.3) along with the following sequence of linear non-homogeneous boundary value problems determining the Newton Iteration u_{k+1} for a given u_k, $k = 1, 2, \dots$.

$$D_t u_{k+1} - \mathscr{A}_{ij}(u_k) D_{ij} u_{k+1} - D_u \mathscr{A}_{ij}(u_k) D_{ij}(u_{k+1} - u_k) = \mathscr{F}(u_k) + D_u \mathscr{F}(u_k)(u_{k+1} - u_k) \tag{4.1}$$

Let $\mathfrak{L} = D_t - a^{ij}D_{ij}$ be the linear uniformly parabolic operator defined in § 2.1. Introduce the set $\mathfrak{A}_p$ of symmetric matrices for which $\mathfrak{L} : \mathbf{Iso}(W_p(Q); L^p(Q))$ that is

$$\mathfrak{A}_p = \left\{ \{a^{ij}\}_{i,j=1}^n \in L^\infty(Q)^{n^2} : \exists\, \lambda > 0 : a^{ij}\eta_i\eta_j \geq \lambda|\eta|^2, \text{ for all } \eta \in \mathbb{R}^n \right\}.$$

Obviously, each of the matrices satisfying a_1) and b_1) belongs to $\mathfrak{A}_p$.

Theorem 4.1. *Suppose* $(\mathbf{H_1} - \mathbf{H_4})$ *hold true, then there exists a neighborhood* $W \subset U \cap W_p(Q)$ *of* u_0 *such that for each* $u_1 \in W$ *there is a unique sequence* $\{u_k\}_{k\in\mathbb{N}} \in W$ *of solutions to* (4.1) *converging to* u_0 *in* $W_p(Q)$, *i.e.* $\|u_k - u_0\|_{W_p(Q)} \to 0$ *as* $k \to \infty$.

Proof. Note that $(\mathbf{H_3})$ ensures $\{a^{ij}(\cdot, u_0(\cdot), Du_0(\cdot))\}_{i,j=1}^n \in \mathfrak{A}_p$ and hence the problem (3.13) with $\tilde{\mathscr{P}}(0,0,u) \equiv \mathscr{P}(u)$ is equivalent to (3.2). Because of the compact embedding $W_p^{2,1} \hookrightarrow W_x^{1,\infty}$ and Lemma 3.3, the operator $\mathscr{P}(u)$ define a C^1-map from $U \cap W_p(Q)$ onto $L^p(Q)$. Further, the assumptions $(\mathbf{H_3})$ and $(\mathbf{H_4})$ imply

$$D_u\mathscr{P}(u) \in \mathbf{Iso}\big(U \cap W_p(Q); L^p(Q)\big).$$

In order to apply the abstract NIP to the solution u_0 of the problem (3.2) we need to show that $D_u\mathscr{P}(u)$ is a Lipschitz continuous in a neighborhood of u_0. Take $w \in W_p(Q)$ and $u, v \in W$ such that $\|u - v\|_{W_p(Q)} \leq \varepsilon$. Because of the regularity properties of the operators we have

$$\begin{aligned}
&\|(D_u\mathscr{P}(u) - D_u\mathscr{P}(v))w\|_{p,Q} \\
&\quad \leq \|\mathscr{A}_{ij}(u) - \mathscr{A}_{ij}(v)\|_{\infty,Q}\|D^2w\|_{p,Q} + \|\mathscr{A}_{ij}\|_{\mathfrak{C}^1}\|w\|_{\infty,Q}\|D^2(u-v)\|_{p,Q} \\
&\qquad + \|D_u\mathscr{A}_{ij}(u) - D_u\mathscr{A}_{ij}(v)\|_{\infty,Q}\|w\|_{\infty,Q}\|D^2v\|_{p,Q} + \|D_u\mathscr{F}(u) - D_u\mathscr{F}(v)\|_{\infty,Q}\|w\|_{p,Q} \\
&\quad \leq L_{\mathscr{A}}\|u - v\|_{\infty,Q}\|D^2w\|_{p,Q} + \|\mathscr{A}_{ij}\|_{\mathfrak{C}^1}\|w\|_{\infty,Q}\|D^2(u-v)\|_{p,Q} \\
&\qquad + L_{\mathscr{A}}\|u - v\|_{W_x^{1,\infty}(Q)}\|w\|_{\infty,Q}\|D^2v\|_{p,Q} + L_{\mathscr{F}}\|u - v\|_{W_x^{1,\infty}(Q)}\|w\|_{p,Q} \\
&\quad \leq C\|u - v\|_{W_p(Q)}\|w\|_{W_p(Q)} \leq C\varepsilon
\end{aligned}$$

and the constant depends on $\|\mathscr{A}_{ij}\|_{\mathfrak{C}^1}$, $L_{\mathscr{A}}$, $L_{\mathscr{F}}$ and $\|w\|_{W_0(Q)}$. Hence $D_u\mathscr{P}(u)$ is invertible for each $u \in W$. Starting the NIP with some $u_1 \in W$ we can write $D_u\mathscr{P}(u_1)(u_2 - u_1) = -\mathscr{P}(u_1)$, that is

$$\begin{aligned}
&D_t u_2 - \mathscr{A}_{ij}(u_1)D_{ij}u_2 - \big(D_u\mathscr{A}_{ij}(u_1)D_{ij}u_1 + D_u\mathscr{F}(u_1)\big)\,u_2 \\
&\quad = \mathscr{F}(u_1) - \big(D_u\mathscr{A}_{ij}(u_1)D_{ij}u_1 + D_u\mathscr{F}(u_1)\big)\,u_1
\end{aligned}$$

where the right-hand side belongs to $L^p(Q)$, $p > n+2$ which implies $u_2 \in W_p^{2,1}(Q)$. Because of the embedding properties u_2 is Hölder continuous along with its gradient and hence $u_2 \in W$. Repeating the above procedure we define a sequence $\{u_k\}_{k=1}^\infty$, $u_k \in W_0$. Now the classical NIP works for the problem (3.3) since the norm of the map $v \to D_u\mathscr{P}(u)v$ in $\mathfrak{L}(W_p(Q); L^p(Q))$ depends even Lipschitz continuously on u in a neighborhood W of u_0. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] P. Acquistapace, On *BMO* regularity for linear elliptic systems, *Ann. Mat. Pura. Appl.* **161** (1992), 231 – 270, DOI: 10.1007BF01759640.

[2] M. Bramanti and M.C. Cerutti, $W_p^{1,2}$ solvability for the Cauchy–Dirichlet problem for parabolic equations with VMO coefficients, *Comm. Partial Diff. Eq.* **18** (1993), 1735 – 1763, DOI: 10.1080/0360530930882991.

[3] D. Cassani, L. Fattorusso and A. Tarsia, Global existence for nonlocal MEMS, *Nonl. Anal., Theory Meth. Appl., Ser. A.* **74** (16) (2011), 5722 – 5726, DOI: 10.1007/j.na.2011.05.060.

[4] J.A. Griepentrog and L. Recke, Local existence, uniqueness and smooth dependence for nonsmooth quasilinear parabolic problems, *J. Evol. Eq.* **10** (2010), 341 – 375, DOI: 10.1007/s00028-010-0052-4.

[5] K. Gröger and L. Recke, Applications of differential calculus tu quasilinear elliptic boundary value problems with non-smooth data, *Nonl. Differ. Equ. Appl.* **13** (3) (2006), 263 – 285, DOI: 10.1007/s00030-006-3017-0.

[6] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* **14** (1961), 415 – 426, DOI: 10.1002/cpa.3160140317.

[7] P.W. Jones, Extension theorems for *BMO*, *Indiana Univ. Math. J.* **29** (1980), 41 – 66.

[8] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, *Transl. Math. Monographs* **23**, Amer. Math. Soc., Providence, R.I. (1968).

[9] O.A. Ladyzhenskaya and N.N. Ural'tseva, A survey of results on the solubility of boundary-value problems for second-order uniformly elliptic and parabolic quasi-linear equations having unbounded singularities, *Russ. Math. Surv.* **41** (5) (1986), 1 – 31, DOI: 10.1070/RM1986v041n05ABEH003415.

[10] A. Maugeri, D.K. Palagachev and L.G. Softova, Elliptic and Parabolic Equations with Discontinuous Coefficients, *Math. Res.* **109**, Wiley-VCH, Berlin (2000), DOI: 10.1002/3527600868.

[11] D.K. Palagachev, L. Recke and L.G. Softova, Applications of the differential calculus to nonlinear elliptic operators with discontinuous coefficients, *Math. Ann.* **336** (2006), 617 – 637, DOI: 10.1007/s00208-006-0014-X.

[12] L. Recke, Applications of the implicit function theorem to quasilinear elliptic boundary value problem with non-smooth data, *Comm. Part. Diff. Eq.* **20** (1995), 1457 – 1479, DOI: 10.1080/03605309508821140.

[13] L. Recke and L.G. Softova, Application of the differential calculus to nonlinear parabolic operators, *C. R. Acad. Bulg. Sci.* **66** (2) (2013), 185 – 192, DOI: 10.7546/CR-2013-66-2-13101331-4.

[14] D. Sarason, Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207** (1975), 391 – 405, DOI: 10.2307/1997184.

[15] L.G. Softova, Quasilinear parabolic operators with discontinuous ingredients, *Nonl. Anal., Theory Meth. Appl., Ser. A.* **52** (2003), 1079 – 1093, DOI: 10.1016/s0362-546X(02)00128-1.

[16] L.G. Softova, Strong solvability for a class of nonlinear parabolic equations, *Le Matematiche,* **LII** (1) (1997), 59 – 70, ISSN: 0373-3505.

[17] L.G. Softova, An integral estimate for the gradient for a class of nonlinear elliptic equations in the plane, *Z. Anal. Anwend.* **17** (1) (1998), 57 – 66, DOI: 10.4171/ZAA/808.

[18] A. Tarsia, Differential equations and implicit function: a generalization of the near operators theorem, *Topol. Meth. Nonl. Anal.* **11** (1998), 115 – 133, DOI: 10.12775/TMNA.1998.007.

[19] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. I: Fixed Points Theorems*, Springer, Berlin — Heidelberg — New York (1993).