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Research Article

An Optimal Control Policy for A Discrete Model with Holling-Tanner Functional Response

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Abstract. A two dimensional discrete time prey-predator model with Holling-Tanner functional response is considered. The local behavior of all its equilibria are investigated. An optimal control problem with this model is proposed, which aims to increase the number of the prey density to prevent the risk of extinction. The Pontryagin's maximum principle for discrete system is applied to achieve the optimality. The necessary conditions and the characterization for the optimal solutions of this system is derived. Finally, we present some numerical simulations to support the theoretical conclusions.

Keywords. Holling-Tanner functional response; Local stability; Discrete optimal control

MSC. 39A30; 92D25; 49J15; 39D20

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1. Introduction

One of the fundamental population models is Lotka-Volterra, which it is introduced in an autonomous system to explain how the predator and prey population fluctuate. For more realistic model Holling [8] suggested three types of functional response to depict the predation among various species. All these types have been extensively considered in many mathematical models as well as many authors studied in details the dynamical behavior of such systems in continuous time and discrete time see [3, 6, 8, 10–12, 16]. We also refer to an excellent books reference are written by Turchin, Murray and Hasting [9, 14, 19]. On the other hand, other types of functional response are used in the literature for example Beddington-De Angelis and Crowley-Martin functional response, [2, 5, 12]. In [18] Tanner investigated a model of prey-predator system and he gave a rich numerical simulations. His results end in that the behavior of the two basic models implies to that either a stable prey-predator holds strong self-limitation or the growth rate of the predator is more than that of its prey. In [15] Rai et al. studied a prey-predator model in which the Volterra predator-prey interaction term was modified by Holling-Tanner type II functional response.

Discrete time models are commonly applied to model population, in spite of their apparent simplicity, they are interesting and exhibiting complex dynamic behavior.

The size population of a single species at generation $k+1$ is generally formulated by a first order difference equation $x_{k+1} = f(x_k)$, so that the x_{k+1} is a function of the k-th generation x_k. We refer the reader to the books see [1, 7, 20] for general theory of discrete dynamical system and difference equations.

The continuous two dimensional prey-predator system with Holling-Tanner functional response is governed by the first-order ODE's

$$\left.\begin{aligned} \frac{x_1}{dt_1} &= rx_1(1-x_1) - \frac{b_1 x_1 y_1}{x_1+a} \\ \frac{y_1}{dt_1} &= -r_1 y_1 + \frac{b_2 x_1 y_1}{x_1+a} \end{aligned}\right\} \tag{1.1}$$

where the parameters r, r_1, b_1, a, and b_2 are constants. The r and r_1 are the growth rate of prey species, and the rate death of the predator, respectively. The parameters b_1, b_2 and a represent the maximum per capita killing rate, the conversion rate of predator and the half-saturation constant, respectively.

One can reduce the number of parameters by a simple transformation $t_1 = \frac{t}{r}$, $x_1 = x$, and $y_1 = \frac{r}{b_1} y$. The non-dimensional equations are:

$$\left.\begin{aligned} \tfrac{x}{dt} &= x(1-x) - \frac{xy}{x+a} \\ \tfrac{y}{dt} &= -by + \frac{exy}{x+a} \end{aligned}\right\} \tag{1.2}$$

where $b = \frac{r_1}{r}$ and $e = \frac{b_2}{r}$. Applying the Euler scheme to system (1.2), we obtain the following discrete system

$$\left.\begin{aligned} x_{k+1} &= x_k + hx_k(1-x_k) - \frac{hx_k y_k}{x_k+a} \\ y_{k+1} &= y_k - hby_k + \frac{hex_k y_k}{x_k+a} \end{aligned}\right\} \tag{1.3}$$

In this paper, we will investigate the system (1.3) in case $a = 1$, and we will also give conditions for the existence and the local stability of all its equilibira. This model also extend to an optimal control problem, which aims to increase the number of the prey density to prevent the risk of extinction. We will use the Pontryagin's maximum principle for discrete system to achieve the optimality as well as the necessary conditions and the characterization for the optimal solutions of this system will be derived. Finally, a numerical analysis presents to support the theoretical findings.

2. Stability of Equilibria of the System

In order to get the equilirbia of system (1.3), we have to solve the following algebraic equations:

$$\left.\begin{aligned} x &= x + hx(1-x) - \frac{hxy}{x+1} \\ y &= y - hby + \frac{hexy}{x+1} \end{aligned}\right\} \tag{2.1}$$

After simple calculations we get the following theorem:

Theorem 1. (1) $e_0 = (0,0)$, *the trivial equilibrium is always exist.*

(2) $e_1 = (1,0)$, *the boundary equilibria is always exist.*

(3) $e_2 = (x^*, y^*) = \left(\frac{b}{(e-b)}, \frac{e(e-2b)}{(e-b)^2}\right)$, *the unique positive equilibrium exists when* $e > 2b$.

For studying the local stability of each equilibrium one has to find the Jacobian matrix for the system (1.3). This is given by

$$J((x,y)) = \begin{bmatrix} 1+h-2hx-\frac{hy}{x+1} & \frac{-hx}{x+1} \\ \frac{he}{x+1} - \frac{hexy}{(x+1)^2} & -bh + \frac{hex}{x+1} \end{bmatrix}.$$

Thus the characteristic polynomial of J can be written as

$$F(\lambda) = \lambda^2 + p\lambda + q \tag{2.2}$$

where $p = -\text{trac}(J)$, and $q = \det(J)$. The next theorem gives the local stability of e_0 as well as e_1.

Theorem 2. (1) *For the trivial equilibrium, we have*

(i) $e_0 = (0,0)$ *is never to be sink,*
(ii) $e_0 = (0,0)$ *is saddle point if* $0 < h < \frac{2}{b}$,
(iii) $e_0 = (0,0)$ *is source if* $\frac{2}{b} < h$,
(iv) $e_0 = (0,0)$ *is non-hyperbolic point if* $h = \frac{2}{b}$.

(2) *For the boundary equilibrium* e_1*, we have*

(i) $e_1 = (1,0)$ *is sink if* $h \in \left(0, \min\left\{2, \frac{4}{2b-e}\right\}\right)$ *and* $2b > e$,
(ii) $e_1 = (1,0)$ *is saddle if* $h \in \left(\min\left\{2, \frac{4}{2b-e}\right\}, \max\left\{2, \frac{4}{2b-e}\right\}\right)$,
(iii) $e_1 = (1,0)$ *is source if* $h \in \left(\max\left\{2, \frac{4}{2b-e}\right\}, \infty\right)$,
(iv) $e_1 = (1,0)$ *is non-hyperbolic point if* $h = \frac{2}{b}$.

Proof. **(1)(i):** It is easy to check that the Jacobian matrix at e_0 is

$$J(E_0) = \begin{bmatrix} 1+h & 0 \\ 0 & 1-bh \end{bmatrix}.$$

The roots of the equation (2.2) are $\lambda_1 = 1+h$ and $\lambda_2 = 1-bh$. So that the $|\lambda_1|$is always greater than 1 and $|\lambda_2| < 1$ if and only if $0 < h < \frac{2}{b}$. Therefore, all (i), (ii) and (iii) can be directly obtained.

(2)(i): The inequality

$$J(E_1) = \begin{bmatrix} 1-h & -\frac{h}{2} \\ 0 & -bh+\frac{he}{2} \end{bmatrix},$$

then the roots of equation (2.2) are $\lambda_1 = 1-h$ and $\lambda_2 = 1-bh+\frac{he}{2}$, therefore, $|\lambda_1| < 1$ if and only if $-1 < 1-h < 1$ this holds if and only if $0 < h < 2$ and $|\lambda_2| < 1$ if and only if $0 < bh - \frac{he}{2} < 2$. iff $0 < h < \frac{4}{2b-e}$ with $2b > e$. From the proof (2)(i), the (ii), (iii) and (iv) can directly be obtained. □

In order to investigate the local stability of the unique positive equilibrium e_2, we need the following theorem:

Theorem 3. *Let $F(\lambda) = \lambda^2 + p\lambda + q$. Suppose that $F(1) > 0$, λ_1, λ_1 are roots of $F(\lambda) = 0$, then*

(1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $q < 1$,

(2) $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ (*or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$) *if and only if* $F(-1) > 0$,

(3) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $q > 1$,

(4) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $p \neq 0, 2$.

The characteristic polynomial of J at e_2 is given by $F(\lambda) = \lambda^2 + p\lambda + q$, where $p = -2-h+bh+2hx+\frac{hy}{x+1}-\frac{hxy}{(x+1)^2}-\frac{hex}{x+1}$ and $q = 1-bh+\frac{hex}{x+1}+h-bh^2+\frac{h^2ex}{x+1}-2hx+2bh^2x-\frac{2h^2ex^2}{x+1}-\frac{hy}{x+1}+\frac{bh^2y}{(x+1)^2}$.

The next theorem gives the local stability of the positive equilibrium.

Theorem 4. (i) *The positive equilibrium point e_2 is sink if the following conditions are hold:*

(1) $e \in I_1 \cap I_2$,

(2) *b, h and e that make the value of M is real and $M > 0$.*

(ii) *The positive equilibrium point e_2 is saddle point if this condition holds:*

(1) *$e \in I_1$ with b, h and e that make the value of M is real and $M < 0$.*

(iii) *The positive equilibrium point e_2 is source if one of the following condition holds:*

(1) *$e \in I_1 \cap I_3$ with b, h and e that make the value of M is real and $M > 0$,*

(2) *$e \in I_1 \cap I_4$ with b, h and e that make the value of M is real and $M > 0$.*

(iv) *The positive equilibrium e_2 is hyperbolic point if the following conditions are hold:*

(1) *b, h and e that make the value of $M = 0$,*

(2) *b, h and e that make the value of $M_1 > 0$ or $M_1 < 0$ or $M_1 < 2(e^2-eb)$ or $M_1 > 2(e^2-eb)$.*

where $I_1 = (2b, \infty)$, $I_2 = (\max\{0, E_2\}, E_1)$, $I_3 = (\max\{E_1, E_2\}, \infty)$, $I_4 = (0, E_2)$, $E_1 = \frac{3bh+\sqrt{b^2h^2+8bh}}{2h}$, $E_2 = \frac{3bh-\sqrt{b^2h^2+8bh}}{2h}$ and $M = k_1e^2+k_2e+k_3$, $k_1 = 4+bh^2$, $k_2 = 2hb-4b-3b^2h^2$, $k_3 = -4hb^2+2b^3h$ and $M_1 = (2+h)e^2+(2b-bh)e+2b^2h-hb$.

Proof. For (i)(1), since $e \in I_1$ then $F(1) > 0$ and if $e \in I_2$, then by after some simple steps one can get that $q < 1$. It is clear that $F(-1) > 0$ if and only if $M > 0$ therefore by applying the last theorem one can get that $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Hence, the e_2 is sink. From (i) and the last theorem one can get the (ii), (iii) and (iv). □

3. An Optimal Control

This section deals with an optimal control problem to the system (1.3), which is given by

$$\left.\begin{array}{l} x_{k+1} = x_k + h x_k(1-x_k) - \frac{h x_k y_k}{x_k+a} \\ y_{k+1} = y_k - hby_k + \frac{hex_k y_k}{x_k+a} - u_k y_k \end{array}\right\}. \tag{3.1}$$

Here x_k and y_k are the state variables. They represent the prey population density, and the predator population density, respectively. All parameters h, b, a and e are constants as mention before. The variable $u_k \le M$ is our control variable, which represents the harvesting amount and M is the maximum harvesting amount. The aim of this problem is to minimize the number of predators in order to increase the prey densities at some interval of time. This problem is also to reduce the risk of extincting of prey population. The existences of such control is guarantee due to the finite dimensional structure of the problem. We will use quadratic term of control in the objective functional to penalize the amount of harvesting [13]. Therefore, the objective functional will be as follows:

$$J(u) = \max_u \sum_k^T c_1 y_k + \frac{c_2}{2} u_k^2. \tag{3.2}$$

The optimal control problem is to search for u_k which minimizes the objective functional $J(u)$, over $u \in U$, where U is the set of all controls. Here, both c_1 and c_2 are constants. The extension version of Pontryagin's maximum principle for discrete system will be used. So that the adjoint variables will be introduced, they are commonly called the shadow prices [4], as well as the Hamiltonian which is defined as follows:

$$\begin{aligned} \mathcal{H}_k = c_1 y_k + \frac{c_2}{2} u_k + \lambda_{1,k+1}\left(x_k + hx_k(1-x_k) - \frac{hx_k y_k}{x_k+a}\right) \\ + \lambda_{2,k+1}\left(y_k - hby_k + \frac{hex_k y_k}{x_k+a} - u_k y_k\right). \end{aligned} \tag{3.3}$$

The necessary conditions for the optimality is given by the following theorem:

Theorem 5. *Given an optimal control u^* with corresponding states solutions x^*, y^* that minimizes the $J(u)$, over $u \in U$, then there exists adjoint variables λ_1 and λ_2 satisfy:*

$$\left.\begin{array}{l} \lambda_{1,k} = \lambda_{1,k+1}\left(1 + h - 2hx - \frac{hy}{x+1}\right) + \lambda_{2,k+1}\left(\frac{he}{x+1} - \frac{hexy}{(x+1)^2}\right) \\ \lambda_{2,k} = \lambda_{2,k} = c_1 + \lambda_{1,k+1}\left(\frac{-hx}{x+1}\right) + \lambda_{2,k+1}\left(-bh + \frac{hex}{x+1}\right) \\ \lambda_{1,T} = \lambda_{2,T} = 0. \qquad (\textit{Transversality conditions}). \end{array}\right\} \tag{3.4}$$

Furthermore, the characterization of the optimal solution u_k^* will be as:

$$u_k^* = \begin{cases} 0 & \textit{if } \frac{\lambda_{2,k} y_k}{c_2} \le 0 \\ \frac{\lambda_{2,k} y_k}{c_2} & \textit{if } 0 < \frac{\lambda_{2,k} y_k}{c_2} < M \\ M & \textit{if } M < \frac{\lambda_{2,k} y_k}{c_2} \end{cases}$$

Proof. The Hamiltonian function is given by (3.3), for each $k = 1,2,\dots,T-1$, then, by using the Pontryagin's maximum principle [13, 17]. Hence, the necessary conditions for $k = 1,2,\dots,T-1$ are

$$\lambda_{1,k} = \frac{\partial \mathcal{H}}{\partial x_k} = \lambda_{1,k+1}\left(1+h-2hx-\frac{hy}{x+1}\right)+\lambda_{2,k+1}\left(\frac{he}{x+1}-\frac{hexy}{(x+1)^2}\right),$$

$$\lambda_{2,k} = \frac{\partial \mathcal{H}}{\partial y_k} = c_1+\lambda_{1,k+1}\frac{-hx}{x+1}+\lambda_{2,k+1}\left(-bh+\frac{hex}{x+1}\right)$$

and the optimality condition is $\frac{\partial \mathcal{H}}{\partial x_k} = c_2 u_k - \lambda_{2,k+1} y_k$ and $\frac{\partial \mathcal{H}}{\partial x_k} = 0$ at u_k^* gives the characterization of the optimal control as

$$u_k^* = \begin{cases} 0 & \text{if } \frac{\lambda_{2,k} y_k}{c_2} \le 0 \\ \frac{\lambda_{2,k} y_k}{c_2} & \text{if } 0 < \frac{\lambda_{2,k} y_k}{c_2} < M \\ M & \text{if } M < \frac{\lambda_{2,k} y_k}{c_2} \end{cases}$$

□

4. Numerical Results

This section presents the numerical results that confirms the above theoretical analysis. At different set of parameters the local stability of the boundary equilibrium and the positive equilibrium are investigated numerically. For the optimal problem numerical method is used which is introduced in [13]. It is outlined as the following: First, all parameters should be defined and an initial control is guessed, namely $u_{in} = 0$. Secondly, the state system is solved forward with u_{in}, and an initial conditions x_0, y_0, thereafter the adjoint system is solved backward with transversality conditions. The next step we use a convex combination to update the controls in the previous iteration. Finally, this strategy will be repeated until the values at the last iteration are very close to the ones at current iterations.

To conform the local stability of the boundary point the following parameters are chosen $x_0 = 0.45$, $y_0 = 0.93$, $h = 0.1$, $e = 0.1$ and $b = 0.5$ so that the condition (2)(i) in theorem is satisfied. Figure 1 illustrates the local stability of e_1.

For the positive equilibrium we choose the following values $x_0 = 0.45$, $y_0 = 0.93$, $h = 0.4$, $e = 0.2$, and $b = 0.065$ then $e_2 = (4815, 7682)$, $E_1 = 0.6685$, $E_2 = -.4735$, $M = 0.1119$ and $2b = .13$ so that $I_1 = (0, 0.6685)$ and $I_1 = (0.13, \infty)$. Hence, according to Theorem 4 the positive equilibrium is locally stable. Figure 2 shows that the local stability of e_2. In Figure 3 the trajectories of the prey population and the predator population are also illustrated as a function of time.

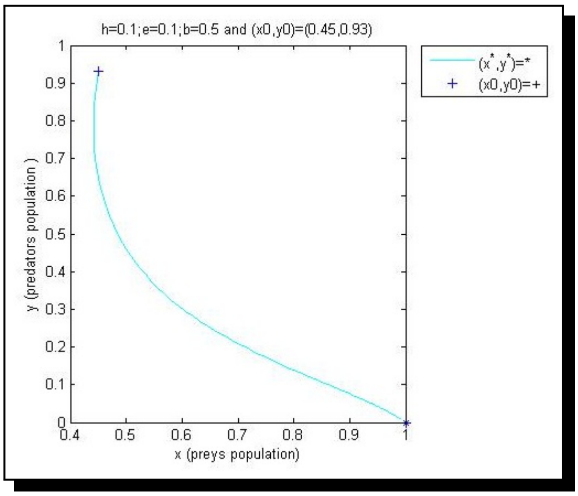

Figure 1. This figure illustrates that e_1 with all parameters above is local stability according to the condition (i) in Theorem 1

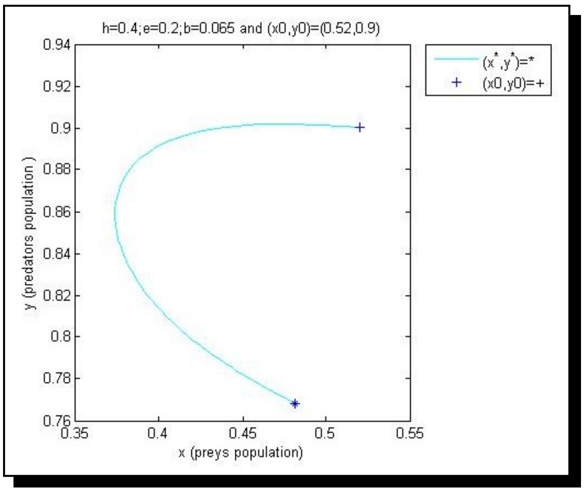

Figure 2. The plot shows the local stability of e_2

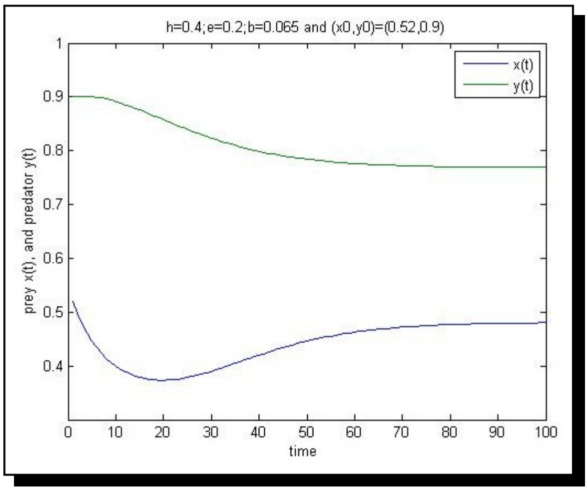

Figure 3. Time series of prey species and predator of system (2.1). Parameters are $x_0 = 0.45$, $y_0 = 0.93$, $h = 0.4$, $e = 0.2$ and $b = 0.065$

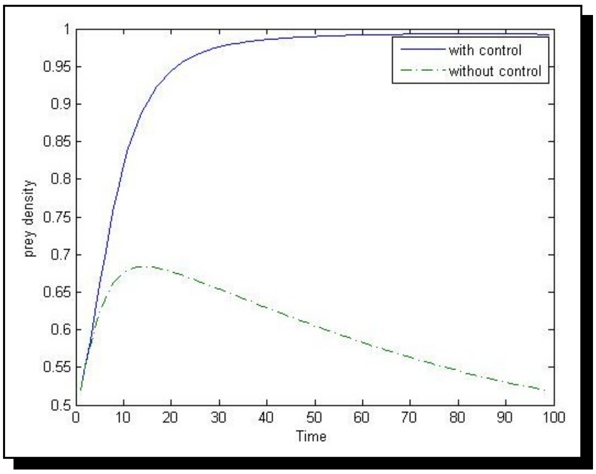

Figure 4. This plot shows the prey density with control (solid line) and without control (dotted line)

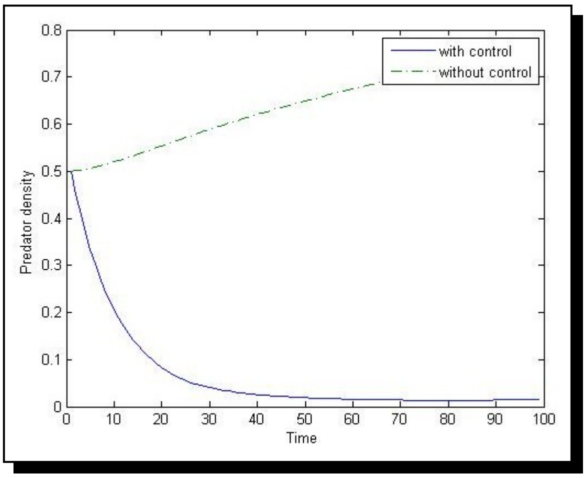

Figure 5. This plot shows the predator density with control (solid line) and without control (dotted line)

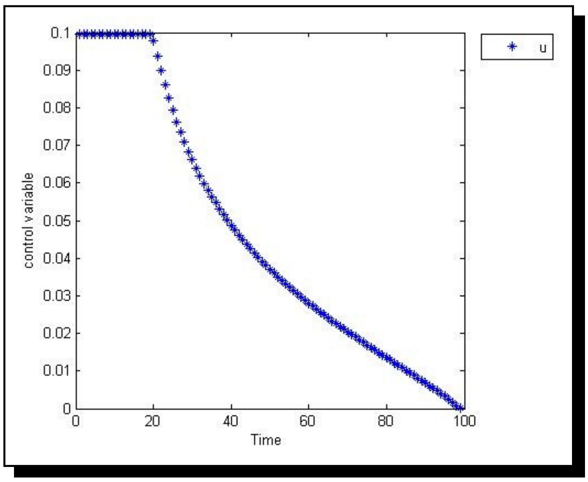

Figure 6. This plot shows the control variable as a function of time

For the optimal control approach we use the following values $x_0 = 0.52$, $y(0) = 0.5$, $h = 0.4$, $e = 0.2$ and $b = 0.065$, and Figures 4 and 5 show the effect of the harvesting on the prey density and the predator density with control and without control respectively. Finally, Figure 6 represents the control variable as a function of time.

5. Conclusions

In this paper a two dimensional discrete time prey-predator model with Holling-Tanner functional response has been investigated. This model has three equilibrium points. The trivial equilibrium point and the boundary equilibrium point are always exist, while the unique positive equilibrium is exists for some values of parameters. An optimal control theory is applied to the model. The necessary condition for optimality as been founded for linear and nonlinear objective functionals. The Ponryagins Maximum Principle is applied to determine the optimal strategy. The aim of this optimal control problem is to minimize the density of predator population. Finally, a numerical analysis shows and confirms the theoretical results for various parameters.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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